Interval Observer Composed of Observers for Nonlinear Systems

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***Abstract*— For a family of continuous-time nonlinear systems with input, output and uncertain terms, a new interval observer design is proposed. The main feature of the constructed interval observer is that it is composed of two copies of a classical observer whose corresponding error equations are, in general, not cooperative. The interval observer is globally asymptotically stable under an appropriate choice of dynamic output feedback which uses the values of the output and the bounds provided by the interval observer itself.**

Keywords Interval observer, robustness, estimation, stabilization.

I. INTRODUCTION

The guaranteed state estimation technique can be traced back to the seminal work [34], where ellipsoids containing the actual state of a system were recursively computed. This idea has been followed by many contributions, as for example [1], [27], [15] and by Gouzé et al. in [11], where a new state estimation technique based on the notion of interval observer is proposed. It relies on the design of a dynamic extension endowed with two outputs giving an upper and a lower bound for the solutions of the considered system. State estimators of this type supply certain information at any instant: if the initial condition of a solution of the system is unknown but can be bounded between two known values, then the trajectories of the interval observers starting from these bounds will enclose the solution under study. More precisely, an upper and a lower bound is provided for each component of the state and, in the absence of disturbance, the norm of the difference between the bounds converges to zero. This type of information cannot be deduced from the knowledge of a classical observer. There are two other reasons why interval observers become more and more popular. First, they make it possible to cope with large uncertainties, which is important when, for instance, one needs to tackle estimation problems of unmeasured coordinates of biological models. Second, they have been successfully applied to many real-life problems, as illustrated by the papers [4], [2], [10], [28]. Thus, during the last decade, a number of contributions presenting constructions of interval observers which apply to different types of systems have been proposed. For example [6], [19], [26] are devoted to classes of continuous-time linear systems and [28], [32], [31], [20] are devoted to various families of continuous-time nonlinear systems. Interval observers for discrete-time systems have been recently considered (see e.g. [22], [9]) and the papers [23], [25] are devoted to linear continuous-time systems with discrete measurements. For more details about the interval observer technique, the interested reader is referred to the contributions [19], [26], [32], [3], [31], to cite only a few.

In spite of the results already available in the literature, much remains to be done to complete the theory of interval observers, especially in the context of systems with input and output. The purpose of the present paper is to complement the design techniques of interval observers available for this type of systems, and in particular [31]. In [31], interval observers have been constructed for a general family of nonlinear continuous-time systems with input and output. The result relies on partial linearization and technical assumptions on the resulting nonlinear terms. In the present paper, we consider another fundamental family of systems for which classical observers can be constructed: the family of nonlinear systems affine in the unmeasured part of the state variables, which comprises the state-affine systems and the linear systems with additive output nonlinearity for which observers are proposed in [5]. More precisely, our objective is to develop interval observers for the systems

$$\left\{\begin{array}{rcl} \dot{x} & = & \alpha(y)x + \beta(y,u) + \delta(t), \\ y & = & \mathcal{C}x, \end{array}\right. \tag{1}$$

with $x \in \mathbb{R}^n$, the output $y \in \mathbb{R}^p$, the input $u \in \mathbb{R}^q$, where α and β are Lipschitz continuous nonlinear functions, where $\mathcal{C} \in \mathbb{R}^{p\times n}$ is a constant matrix and δ is a Lipschitz continuous function bounded by two known Lipschitz continuous functions δ^-, δ^+: for all $t \geq 0$, $\delta^-(t) \leq \delta(t) \leq \delta^+(t)$. For this system, under standard assumptions which ensure existence of a classical asymptotic observer, we construct interval observers without resorting to linearization or partial linearization. As in [24] for discrete-time systems of another type, we will show how two copies of classical observers associated with suitably selected initial conditions and outputs compose an interval observer. The key advantage of the new interval observer we propose is the simplicity of its dynamics. Each copy of observer, or its associated error equation, does not possess the property of being a cooperative or a nonnegative system (see, for instance, [35] for the definition of cooperative system). This fact may sound surprising since most of the designs of interval observers available in the literature are carried out for cooperative systems (sometimes after a coordinate transformation). In

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fact, we will use the notion of nonnegative and cooperative system as well, but only indirectly to select for the interval observer appropriate initial conditions and upper and lower bounds for the solutions of the studied system. This feature of our design, which is share with those of [24] and [23], is crucial: it is the reason why the equations of interval observers we propose are very simple. In addition to their simplicity, the interval observers we present offer the possibility to construct a bundle of interval observers, as done for instance in [4], without having to introduce extra dynamics, simply by proposing several choices of initial conditions and bounding outputs. Another key advantage of our technique is that, under some assumptions and an appropriate choice of output feedback, global asymptotic stability of the origin of the system with its interval observer is obtained. We establish this stability result through the construction of a strict Lyapunov function.

Finally, let us observe that the motivation for studying systems affine in the unmeasured part of the state is due to the fact that many systems belong to this family: let us mention for instance the TORA system [13], the nonlinear pendulum (see [8]), the model of electromechanical system described in [7] when, in both cases, the position variables are the measured variables and the nonholonomic system with output studied in [14]. Important results on the problem of constructing classical observers for this type of systems exist: see for instance [30], [16], [17], [29]. But to the best of our knowledge, no interval observers have been proposed.

The rest of this note is organized as follows. Notation and definitions are given in Section II. The Section III is devoted to the main result. An example borrowed from [7] illustrates the technique in Section IV. Concluding remarks are drawn in Section V and end the paper.

II. Notation, definitions, basic result

The notation will be simplified whenever no confusion can arise from the context. Any $k \times n$ matrix, whose entries are all 0 is simply denoted 0. The Euclidean norm of vectors of any dimension and the induced norm of matrices of any dimensions are denoted $|\cdot|$. All the inequalities must be understood *componentwise i.e.* $v_a = (v_{a1}, ..., v_{ar})^\top \in \mathbb{R}^r$ and $v_b = (v_{b1}, ..., v_{br})^\top \in \mathbb{R}^r$ are such that $v_a \leq v_b$ if and only if, for all $i \in \{1, ..., r\}$, $v_{ai} \leq v_{bi}$. For two matrices $A = (a_{ij}) \in \mathbb{R}^{r\times s}$ and $B = (b_{ij}) \in \mathbb{R}^{r\times s}$, $\max\{A, B\}$ is the matrix where each entry is $m_{ij} = \max\{a_{ij}, b_{ij}\}$. For a matrix $A \in \mathbb{R}^{r\times s}$, $A_p = \max\{A, 0\}$, $A_n = \max\{-A, 0\}$. A square matrix is said to be *cooperative* or Metzler if its off-diagonal entries are nonnegative. We present a notion of nonnegative system slightly less restrictive than the one given in [12]. A system $\dot{x}(t) = f(t, x(t))$, for which the solutions are unique and defined over $[0, +\infty)$, is said to be nonnegative if for any initial condition $x_0 \geq 0$ at any initial time $t_0 \geq 0$, the solution $x(t)$ is nonnegative for all $t \geq t_0$. Let $\mathcal{K}$ denote the set of all continuous functions $\rho : [0, \infty) \to [0, \infty)$ for which (i) $\rho(0) = 0$ and (ii) ρ is increasing.

For the sake of generality, we introduce a general definition of framer and interval observer for time-varying nonlinear systems. This notion has been introduced, with slightly different features, in several papers (see, for instance, [20], [26], [11] to cite only a few).

Definition 1: Consider a system:

$$\dot{x}(t) = f_1(t, x(t), \delta(t)), \tag{2}$$

with $x \in \mathbb{R}^n$, with an output $y = m(x) \in \mathbb{R}^p$, and where f_1 and m are two nonlinear locally Lipschitz functions. The uncertainty $\delta(t) \in \mathbb{R}^\ell$ is a Lipschitz continuous function such that (2) is forward complete and such that there exist two known bounds $\delta^+(t) \in \mathbb{R}^\ell$, $\delta^-(t) \in \mathbb{R}^\ell$, Lipschitz continuous, and such that, for all $t \geq 0$, $\delta^-(t) \leq \delta(t) \leq \delta^+(t)$. The initial condition at the instant t_0, $x(t_0) = x_0$, is assumed to be bounded by two known bounds:

$$x_0^- \leq x_0 \leq x_0^+. \tag{3}$$

Then, the dynamical system

$$\dot{z}(t) = f_2\left(t, z(t), y(t), \overline{\delta}(t)\right), \tag{4}$$

with $\overline{\delta} = (\delta^+, \delta^-)$, associated with the initial condition $z_0 = g(t_0, x_0^+, x_0^-) \in \mathbb{R}^{n_z}$ and bounds for the solution $x(t)$: $x^+(t) = h^+(t, z(t))$, $x^-(t) = h^-(t, z(t))$, where f_2, g, h^+ and h^- are Lipschitz continuous nonlinear functions, is called
(i) a framer for (2) if for any vectors x_0, x_0^- and x_0^+ in $\mathbb{R}^n$ satisfying (3), the solutions of (2)-(4) with respectively x_0, $z_0 = g(t_0, x_0^+, x_0^-)$ as initial condition at $t = t_0$, denoted respectively x and z, satisfy, for all $t \geq t_0$, the inequalities

$$x^-(t) = h^-(t, z(t)) \leq x(t) \leq h^+(t, z(t)) = x^+(t), \tag{5}$$

(ii) an interval observer for (2) if in addition the system (2)-(4) admits the origin as a global asymptotically stable equilibrium point when $\overline{\delta}$ is identically equal to zero.

We omit the proof of the following simple technical lemma:

Lemma 1: Consider the system

$$\dot{z} = A(t)z + \delta(t), \tag{6}$$

where $z \in \mathbb{R}^n$ and $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $\delta : \mathbb{R} \to \mathbb{R}^n$ are continuous. Assume that, for all $t \geq 0$, the matrix $A(t)$ is Metzler and $\delta(t) \geq 0$, for all $t \geq 0$. Then the system (6) is nonnegative.

The following lemma will be used to establish the main result. It is interesting for its own sake because it gives conditions ensuring that a system is nonnegative without implying that it is cooperative.

Lemma 2: Consider the system

$$\begin{array}{rcl} \dot{z}_1 &=& f(t, z_1, z_2), \\ \dot{z}_2 &=& g(z_1, z_2)z_2 + \delta(t), \end{array} \tag{7}$$

where $z_1 \in \mathbb{R}^{n_1}$, $z_2 \in \mathbb{R}^{n_2}$, the functions f, g are Lipschitz continuous and δ is continuous. Assume that this system is forward complete, $\delta(t) \geq 0$, for all $t \geq 0$ and, for all $z_1 \in \mathbb{R}^{n_1}$, $z_2 \in \mathbb{R}^{n_2}$, the matrix $g(z_1, z_2)$ is Metzler. Then, if at the instant t_0, $z_2(t_0) \geq 0$, then $z_2(t) \geq 0$ for all $t \geq t_0$.

Proof. Consider a solution (z_1, z_2) of (7) with an initial condition such that $z_2(t_0) \geq 0$. Since the system (7) is forward complete, we can define for all $t \geq 0$ the function A by $A(t) = g(z_1(t), z_2(t))$. Moreover, this matrix is Metzler for all $t \geq 0$. It follows from Lemma 1 that all the solutions of

$$\dot{\xi} = A(t)\xi + \delta(t), \tag{8}$$

with $\xi(t_0) \geq 0$ are nonnegative for all $t \geq t_0$. Therefore, in particular, the solution ξ_p of (8) with $\xi_p(t_0) = z_2(t_0)$ is nonnegative since $z_2(t_0) \geq 0$. Now observe that the solutions ξ_p and z_2 satisfy, for all $t \geq t_0$,

$$\dot{\xi}_p(t) = A(t)\xi_p(t) + \delta(t), \quad \dot{z}_2(t) = A(t)z_2(t) + \delta(t),$$

and $\xi_p(t_0) = z_2(t_0)$. Uniqueness of the solutions implies that $\xi_p(t) = z_2(t)$ for all $t \geq t_0$. Consequently, $z_2(t) \geq 0$ for all $t \geq t_0$.

III. MAIN RESULT

This section is devoted to the construction of interval observers for the system (1). We introduce some assumptions needed for our development.

Assumption 1. *There exist a Lipschitz continuous function $\lambda(y)$, a positive definite radially unbounded C^1 function V and a continuous positive definite function ω such that:*

$$\frac{\partial V}{\partial \xi}(\xi)[\alpha(y) + \lambda(y)\mathcal{C}]\xi \leq -\omega(\xi), \tag{9}$$

for all $\xi \in \mathbb{R}^n$, $y \in \mathbb{R}^p$.

Assumption 2. *There exists an invertible constant matrix $R \in \mathbb{R}^{n \times n}$ such that, for all $y \in \mathbb{R}^p$, the matrix*

$$\Gamma(y) = R[\alpha(y) + \lambda(y)\mathcal{C}]R^{-1} \tag{10}$$

is Metzler.

Assumption 3. *There exist a positive definite radially unbounded C^1 function $U : \mathbb{R}^n \to \mathbb{R}$, a Lipschitz continuous feedback $u_s : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^q$, a continuous positive definite function μ and a function γ of class $\mathcal{K}$ such that for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$, the inequality*

$$\frac{\partial U}{\partial x}(x)\left[\alpha(\mathcal{C}x)x + \beta(\mathcal{C}x, u_s(\mathcal{C}x, x + d))\right] \leq -\mu(x) + \gamma\left(|d|\right) \tag{11}$$

holds. Moreover there exists a nonnegative and increasing continuous function $\kappa : [0, +\infty) \to [0, +\infty)$ such that, for all $d \in \mathbb{R}^n$,

$$\gamma\left(|d|\right) \leq \kappa\left(V(d)\right)\omega(d). \tag{12}$$

Now, we state and prove the following result:

Theorem 1: Let the system (1) satisfy Assumptions 1 to 3. Then this system with the dynamic extension

$$\left\{\begin{array}{rcl} \dot{\hat{x}}^+ & = & \alpha(y)\hat{x}^+ + \beta(y,u) + \lambda(y)[\mathcal{C}\hat{x}^+ - y] \\ & & +S\left[R_p\delta^+ - R_n\delta^-\right], \\ \dot{\hat{x}}^- & = & \alpha(y)\hat{x}^- + \beta(y,u) + \lambda(y)[\mathcal{C}\hat{x}^- - y] \\ & & +S\left[R_p\delta^- - R_p\delta^+\right], \end{array}\right. \tag{13}$$

in closed-loop with the feedback $u_s(y, \hat{x}^+)$ is globally asymptotically stable when $\delta^+(t) = \delta^-(t) = 0$ for all $t \geq 0$. Moreover, the system (13) associated with the initial conditions

$$\hat{x}_0^+ = S\left[R_p x_0^+ - R_n x_0^-\right]\ ,\ \hat{x}_0^- = S\left[R_p x_0^- - R_n x_0^+\right], \tag{14}$$

the bounds for the solutions x

$$x^+ = S_p R\hat{x}^+ - S_n R\hat{x}^-\ ,\ x^- = S_p R\hat{x}^- - S_n R\hat{x}^+, \tag{15}$$

where $S = R^{-1}$, is an interval observer for system (1) in closed loop with $u_s(y, \hat{x}^+)$.

Discussion of Theorem 1.

• Assumptions 1 and 3 are technical assumptions which ensure the global asymptotic stabilizability by dynamic output feedback of the system (1). Assumption 1 implies that, for any constant vector $y \in \mathbb{R}^p$, the origin of $\dot{\chi} = [\alpha(y) + \lambda(y)\mathcal{C}]\chi$ is globally asymptotically stable.

• Assumption 2 ensures the existence of a time-invariant change of coordinates that transforms for all $y \in \mathbb{R}^p$ the matrix $\alpha(y) + \lambda(y)\mathcal{C}$ into a Metzler matrix. This assumption is not restrictive in the particular case where there is a scalar function $\varsigma(y)$ and a constant matrix α_o such that $\alpha(y) = \varsigma(y)\alpha_o$ and the pair $\alpha_o, \mathcal{C}$ is observable: in such a case Assumption 2 is satisfied with $\lambda(y) = \varsigma(y)\lambda_o$, where λ_o is so that all the eigenvalues of $\alpha_o + \lambda_o\mathcal{C}$ are distinct. The case mentioned about is important since it encompasses the case of the systems with a constant function α, which studied in particular in [11].

• For the system (1) in closed loop with $u_s(\mathcal{C}x, x + d)$ and no additive uncertainties δ, Assumption 3 is a robustness condition which guarantees that the system is iISS with respect to d because the function U is a iISS Lyapunov function (see for instance [18, Chapt. 2] for the definition of iISS Lyapunov function). Such a Lyapunov function is not unique and the condition (12) depends on the selected function U. We conjecture that if there is a function U such that (11) is satisfied, then there always exists another function U such that both (11) and (12) are satisfied. In many cases, the system under study is locally exponentially stabilizable, both V and ω are positive definite quadratic functions and one can select a function U lower and upper bounded by positive definite quadratic functions in a neighborhood of the origin. Then, γ can be taken equal to a quadratic function in a neighborhood of zero and then one can always determine a function κ so that (12) is satisfied by using the tools given in [18, Appendix A].

• One can check easily that if an input u is so that all the solutions of the system (1) are defined over $[0, +\infty)$ and belong to a compact set, then $\lim\limits_{t \to +\infty} |\hat{x}^+(t) - \hat{x}^-(t)| = 0$. But, even if in addition (13) is a framer for the system (1), it does not follow that it is an interval observer for this system because the asymptotic stability of the system (1) is not guaranteed. This motivates the selection of the special feedback u_s.

Proof. The proof splits up into two parts. The first is devoted to the stability analysis of the system (1)-(13) in closed-loop with the dynamic output feedback $u_s(y, \hat{x}^+)$. The second consists in showing that (13) associated to the initial conditions (14) and the bounds (15) is an interval observer for the system (1) in closed-loop with $u_s(y, \hat{x}^+)$.

1. *Stability of the systems (1) - (13) with the feedback* $u_s(y, \hat{x}^+)$ *in the absence of* δ, δ^+ *and* δ^-. This part of the proof is omitted.

2. *Property of framer.* First, consider vectors $x_0, x_0^+, x_0^-, \hat{x}_0^+, \hat{x}_0^-$ in $\mathbb{R}^n$ such that

$$x_0^- \leq x_0 \leq x_0^+ \tag{16}$$

and

$$\hat{x}_0^+ = S\left[R_p x_0^+ - R_n x_0^-\right] \ , \ \hat{x}_0^- = S\left[R_p x_0^- - R_n x_0^+\right] . \tag{17}$$

As an immediate consequence of the equalities in (17), we have

$$R\hat{x}_0^+ = R_p x_0^+ - R_n x_0^- \ , \ R\hat{x}_0^- = R_p x_0^- - R_n x_0^+ . \tag{18}$$

Since the entries of R_p and R_n are nonnegative, it follows from (16) that the inequalities

$$R_p x_0^- \leq R_p x_0 \leq R_p x_0^+ \ , \ R_n x_0^- \leq R_n x_0 \leq R_n x_0^+$$

are satisfied. We deduce that

$$-R_n x_0^+ + R_p x_0^- \leq (R_p - R_n) x_0 \leq R_p x_0^+ - R_n x_0^- .$$

These inequalities in combination with (18) give

$$R\hat{x}_0^- \leq R x_0 \leq R\hat{x}_0^+ . \tag{19}$$

Second, for an initial instant $t_0 \geq 0$, consider the solution $(x, \hat{x}^+, \hat{x}^-)$ of the system (1)-(13) in closed-loop with $u_s(y, \hat{x}^+)$ satisfying

$$x(t_0) = x_0, \ \hat{x}^+(t_0) = \hat{x}_0^+, \ \hat{x}^-(t_0) = \hat{x}_0^- . \tag{20}$$

For all $t \geq t_0$, we have

$$\left\{\begin{array}{rcl} \dot{x} & = & [\alpha(y) + \lambda(y)\mathcal{C}]\, x + \beta(y, u_s(y, \hat{x}^+)) \\ & & -\lambda(y) y + \delta, \\ \dot{\hat{x}}^+ & = & [\alpha(y) + \lambda(y)\mathcal{C}]\, \hat{x}^+ + \beta(y, u_s(y, \hat{x}^+)) \\ & & -\lambda(y) y + S\left[R_p \delta^+ - R_n \delta^-\right], \\ \dot{\hat{x}}^- & = & [\alpha(y) + \lambda(y)\mathcal{C}]\, \hat{x}^- + \beta(y, u_s(y, \hat{x}^+)) \\ & & -\lambda(y) y + S\left[R_p \delta^- - R_n \delta^+\right]. \end{array}\right. \tag{21}$$

From (10) in Assumption 2, it follows that

$$\begin{array}{rcl} R\dot{x} & = & R\left([\alpha(y) + \lambda(y)\mathcal{C}]x + \beta(y, u_s(y, \hat{x}^+))\right) \\ & & -R\lambda(y) y + R\delta \\ & = & \Gamma(y) R x + R\beta(y, u_s(y, \hat{x}^+)) - R\lambda(y) y \\ & & + (R_p - R_n)\, \delta \end{array} \tag{22}$$

and, similarly,

$$\begin{array}{rcl} R\dot{\hat{x}}^+ & = & \Gamma(y) R\hat{x}^+ + R\beta(y, u_s(y, \hat{x}^+)) \\ & & -R\lambda(y) y + R_p \delta^+ - R_n \delta^-, \\ R\dot{\hat{x}}^- & = & \Gamma(y) R\hat{x}^- + R\beta(y, u_s(y, \hat{x}^+)) \\ & & -R\lambda(y) y + R_p \delta^- - R_n \delta^+. \end{array} \tag{23}$$

Thus,

$$\begin{array}{rcl} R\dot{\hat{x}}^+ - R\dot{x} & = & \Gamma(y)(R\hat{x}^+ - Rx) + \nu_1, \\ R\dot{x} - R\dot{\hat{x}}^- & = & \Gamma(y)(Rx - R\hat{x}^-) + \nu_2, \end{array} \tag{24}$$

with $\nu_1(t) = R_p(\delta^+(t) - \delta(t)) + R_n(\delta(t) - \delta^-(t))$ and $\nu_2(t) = R_p(\delta(t) - \delta^-(t)) + R_n(\delta^+(t) - \delta(t))$.

From (19) and (20), the fact that, for all $y \in \mathbb{R}^p$, $\Gamma(y)$ is Metzler and $\nu_1(t) \geq 0$ and $\nu_2(t) \geq 0$, for all $t \geq 0$, we deduce from Lemma 2 that $R\hat{x}^-(t) \leq Rx(t) \leq R\hat{x}^+(t)$, $\forall t \geq t_0$. Since the matrices S_p and S_n are nonnegative, it follows from the previous inequalities that, for all $t \geq t_0$,

$$\begin{array}{c} S_p R\hat{x}^-(t) \leq S_p R x(t) \leq S_p R\hat{x}^+(t), \\ S_n R\hat{x}^-(t) \leq S_n R x(t) \leq S_n R\hat{x}^+(t), \end{array} \tag{25}$$

which implies that

$$\begin{array}{rcl} S_p R\hat{x}^-(t) - S_n R\hat{x}^+(t) & \leq & (S_p - S_n) R x(t) \\ & \leq & S_p R\hat{x}^+(t) - S_n R\hat{x}^-(t). \end{array} \tag{26}$$

From the definitions of x^+ and x^- in (15) and the fact that $S_p - S_n = S$, it follows that, for all $t \geq t_0$, $x^-(t) \leq x(t) \leq x^+(t)$. This allows us to conclude.

IV. Illustrative Example

In this section, we illustrate Theorem 1 with the model of electromechanical system described in [7]:

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2, \\ \dot{x}_2 & = & b_1 x_3 - a_1 \sin(x_1) - a_2 x_2, \\ \dot{x}_3 & = & b_0 u - a_3 x_2 - a_4 x_3, \end{array}\right. \tag{27}$$

with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, with the output $y = x_1$ and where the b_i's and the a_i's are positive real numbers. Assuming that the variable x_1 is measured is realistic from an applied point of view because x_1 represents the angular motor position of the device.

Let us check that Assumption 1 is satisfied. With the notation of previous sections and choosing $\lambda(y) = [-1 \ 0 \ 0]^\top$, we obtain

$$\alpha(y) + \lambda(y)\mathcal{C} = \left[\begin{array}{ccc} -1 & 1 & 0 \\ 0 & -a_2 & b_1 \\ 0 & -a_3 & -a_4 \end{array}\right]. \tag{28}$$

Assumption 1 is satisfied with the positive definite quadratic function

$$V(\xi) = a_2 a_3 \xi_1^2 + a_3 \xi_2^2 + b_1 \xi_3^2 .$$

Indeed, one can check easily that the inequality

$$\frac{\partial V}{\partial \xi}(\xi)[\alpha(y) + \lambda(y)\mathcal{C}]\xi \leq -\omega(\xi),$$

with $\omega(\xi) = a_2 a_3(\xi_1^2 + \xi_2^2) + 2 b_1 a_4 \xi_3^2$ is satisfied and ω is a positive definite quadratic function.

Let us check that Assumption 2 is satisfied. The matrix $\alpha(y) + \lambda(y)\mathcal{C}$ is not Metzler because $a_3 > 0$. But we show that, with the numerical values given below, this matrix can be transformed into a Metzler matrix through a time-invariant transformation. With the notation of [7], we have $b_1 = \frac{1}{M}$, $a_2 = \frac{B}{M}$, $a_3 = \frac{K_B}{L}$, $a_4 = \frac{R}{L}$, with $B = \frac{B_0}{K_\tau}$,

$M = \frac{J}{K_\tau} + \frac{mL_0^2}{3K_\tau} + \frac{M_0 L_0^2}{K_\tau} + \frac{2M_0 R_0^2}{5K_\tau}$. The numerical values given in [7] are: $M_0 = 0.434$, $J = \frac{1.625}{1000}$, $m = 0.506$, $R_0 = 0.023$, $B_0 = 0.01625$, $L_0 = 0.305$, $L = 0.025$, $R = 5$, $K_\tau = K_B = 0.9$. Therefore $0.018 \leq B \leq 0.0181$ and $0.06419 \leq M \leq 0.0642$.

We obtain $15.576 \leq b_1 \leq 15.579, 0.280 \leq a_2 \leq 0.282, a_3 = 36, a_4 = 200$.

To simplify, let us choose values close to those given in [7]: $b_1 = 15, a_2 = \frac{1}{4}, a_3 = 36, a_4 = 200$. We replace their own values in (28), then the equality

$$R[\alpha(y) + \lambda(y)\mathcal{C}]R^{-1} = \begin{bmatrix} -1 & \frac{13}{2} & 37 \\ 0 & -160 & \frac{225}{4} \\ 0 & 104 & -\frac{161}{4} \end{bmatrix}$$

with

$$R = \begin{bmatrix} 1 & 6 & 1 \\ 0 & 1 & 6 \\ 0 & -1 & -4 \end{bmatrix}$$

is satisfied. Therefore $R[\alpha(y) + \lambda(y)\mathcal{C}]R^{-1}$ is Metzler and Assumption 2 is satisfied.

Let us check that Assumption 3 is satisfied. The system (27) in closed-loop with the feedback

$$u_s(y, x) = \frac{1}{b_0 b_1}\left[-b_1 a_2 a_3 y + a_1 a_4 \sin(y) + a_1 \cos(y) x_2\right] \tag{29}$$

is

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = b_1 x_3 - a_1 \sin(x_1) - a_2 x_2, \\ \dot{x}_3 = -a_2 a_3 x_1 + \frac{a_1 a_4}{b_1}\sin(x_1) \\ \qquad +\frac{a_1}{b_1}\cos(x_1) x_2 - a_3 x_2 - a_4 x_3. \end{cases} \tag{30}$$

Let $h = b_1 x_3 - a_1 \sin(x_1)$, $g = a_2 x_1 + x_2$. Then

$$\begin{cases} \dot{x}_2 = -a_2 x_2 + h, \\ \dot{g} = h, \\ \dot{h} = -b_1 a_3 g - a_4 h. \end{cases} \tag{31}$$

This system is exponentially stable. Next, through simple, lengthy and useless calculations, one can establish that Assumption 3 is satisfied with a positive definite quadratic function for γ and a function κ identically equal to a positive constant.

Since Assumptions 1 to 3 are satisfied, Theorem 1 applies and provides us with the following interval observer:

$$\begin{cases} \dot{\hat{x}}_1^+ = \hat{x}_2^+ + y - \hat{x}_1^+, \\ \dot{\hat{x}}_2^+ = b_1 \hat{x}_3^+ - a_1 \sin(y) - a_2 \hat{x}_2^+, \\ \dot{\hat{x}}_3^+ = b_0 u_s(y, \hat{x}^+) - a_3 \hat{x}_2^+ - a_4 \hat{x}_3^+, \\ \dot{\hat{x}}_1^- = \hat{x}_2^- + y - \hat{x}_1^-, \\ \dot{\hat{x}}_2^- = b_1 \hat{x}_3^- - a_1 \sin(y) - a_2 \hat{x}_2^-, \\ \dot{\hat{x}}_3^- = b_0 u_s(y, \hat{x}^+) - a_3 \hat{x}_2^- - a_4 \hat{x}_3^-, \end{cases} \tag{32}$$

where $a_1 = \frac{N}{M}$ with $N = \frac{mL_0 G}{2K_\tau} + \frac{M_0 L_0 G}{K_\tau}$ and $b_0 = \frac{1}{L} = \frac{1}{0.025} = 40$, with the initial conditions

$$\hat{x}_0^+ = SR_p x_0^+ - SR_n x_0^-, \quad \hat{x}_0^- = SR_p x_0^- - SR_n x_0^+ \tag{33}$$

where $SR_p = \begin{bmatrix} 1 & \frac{35}{2} & 70 \\ 0 & -2 & -12 \\ 0 & \frac{1}{2} & 3 \end{bmatrix}$, $SR_n = \begin{bmatrix} 0 & \frac{35}{2} & 70 \\ 0 & -3 & -12 \\ 0 & \frac{1}{2} & 2 \end{bmatrix}$ and the bounds

$$x^+ = S_p R \hat{x}^+ - S_n R \hat{x}^-, \quad x^- = S_p R \hat{x}^- - S_n R \hat{x}^+ \tag{34}$$

where $S_p R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $S_n R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Figure 1 below illustrates the result in the case where there is no disturbances. A trajectory with $t_0 = 0$, $x_0 = (20, 10, 1)^\top$, $x_0^+ = \left(\frac{41}{2}, \frac{11}{2}, \frac{3}{2}\right)^\top$, $x_0^- = \left(\frac{39}{2}, \frac{9}{2}, \frac{1}{2}\right)^\top$, $\hat{x}_0^+ = \left(108, -\frac{9}{2}, 4\right)^\top$, $\hat{x}_0^- = \left(-68, \frac{49}{2}, -2\right)^\top$ as initial condition is simulated. We see clearly that the bounds and the global asymptotically stability are ensured.

Now we consider (27) in the presence of additive disturbances. For our simulations, we choose $\delta(t) = \left(\frac{1}{9}\sin(t), \frac{1}{9}\sin(t), \frac{1}{9}\sin(t)\right)^\top$ which are bounded by $\delta^+(t) = \left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right)^\top$ and $\delta^-(t) = -\delta^+(t)$. Then, from (13), we deduce that the corresponding interval observer is:

$$\begin{cases} \dot{\hat{x}}_1^+ = \hat{x}_2^+ + y - \hat{x}_1^+ + \frac{176}{9}, \\ \dot{\hat{x}}_2^+ = b_1 \hat{x}_3^+ - a_1 \sin(y) - a_2 \hat{x}_2^+ - \frac{29}{9}, \\ \dot{\hat{x}}_3^+ = b_0 u_s(y, \hat{x}^+) - a_3 \hat{x}_2^+ - a_4 \hat{x}_3^+ + \frac{2}{3}, \\ \dot{\hat{x}}_1^- = \hat{x}_2^- + y - \hat{x}_1^- - \frac{176}{9}, \\ \dot{\hat{x}}_2^- = b_1 \hat{x}_3^- - a_1 \sin(y) - a_2 \hat{x}_2^- + \frac{29}{9}, \\ \dot{\hat{x}}_3^- = b_0 u_s(y, \hat{x}^+) - a_3 \hat{x}_2^- - a_4 \hat{x}_3^- - \frac{2}{3}, \end{cases} \tag{35}$$

Figure 2 below shows that in the case where the system is affected by additive disturbances, the interval observer still provided the solutions with bounds, which do not converge to the solution of the studied system. The initial conditions we selected are the same as those chosen in the undisturbed case of Figure 1.

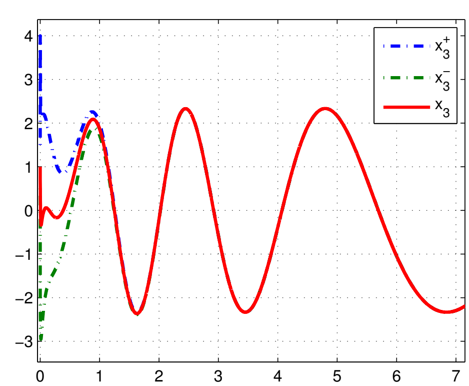

Fig. 1. Evolution of x_3, x_3^+ and x_3^- without uncertainties $\delta(t)$

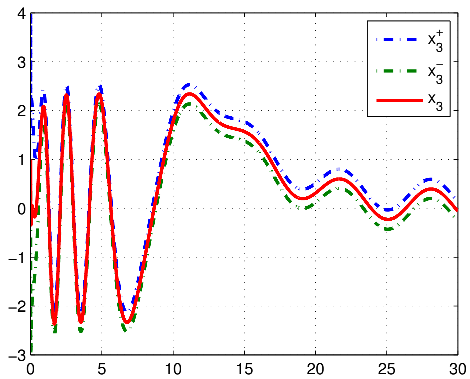

Fig. 2. Evolution of x_3, x_3^+ and x_3^- with uncertainties $\delta(t)$

V. Conclusion

For a family of systems affine in the unmeasured variables, we have presented a new technique of construction of interval observers, which makes it possible to cope with the presence of additive disturbances. The dynamic part of interval observers is simply composed of two copies of a classical observer and therefore they can be used when it comes to stabilize the systems through a dynamic output feedback. Extensions to systems with special structures, in the spirit of what is done in [21] and in [8] and extensions to time-varying systems, systems with delay and systems with discrete measurements are expected.

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