Interval Observers For Discrete-time Systems

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Introduction

Introduction

- ▷ Interval observers : it originates in the work by Gouzé, Rapaport & Haji-Sadok (2000).
- ▷ C. Combastel, S.A. Raka, F. Mazenc, O. Bernard, M.Kieffer, E. Water, S.-I. Niculescu, M. Moisan, T. Raissi, D. Efimov, A. Zolghadri, N. Ramdani, Y. Candau.
- ▷ Main advantage of the technique : it allows to cope with uncertainties that are known to characterize some classes of systems.

Introduction

- Interest of the discrete-time systems :
  - Discretization techniques transform continuous-time systems into discrete-time systems.
  - Systems with sampled data often lead to discrete-time systems.
- Many researchers have constructed observers or dynamic output feedbacks for discrete-time systems : M. Boutayeb, M. Darouach, I. Karafyllis, C. Kravaris, M. Xiao, N. Kazantzis, C. Kravaris, A.J. Krener, Z.P. Jiang...

Introduction

We discovered that our worked has been created simultaneously with :

D. Efimov, W. Perruquetti, T. Raissi, A. Zolghadri, *On Interval Observer Design for Discrete Systems.* Submitted, 2012.

Introduction

- First we construct time-invariant interval observers for a nonlinear discrete-time system under a specific condition.
- Second we show how time-varying interval observers can be constructed for the linear time-invariant discrete-time systems.

  → Key idea : time-varying changes of coordinates that transform linear discrete-time systems into nonnegative systems.

Remark : fundamental differences between our work and F. Mazenc, O. Bernard (2011) because a continuous-time system $\dot{x} = \mathcal{A}x$ is positive iff the matrix $\mathcal{A}$ is cooperative whereas a discrete-time system $x_{k+1} = \mathcal{A}x_k$ is positive iff no entry of $\mathcal{A}$ is negative.

F. Mazenc, O. Bernard, *Interval observers for linear time-invariant systems with disturbances.* Automatica, Vol. 47, No. 1, pp. 140-147, Jan. 2011.

Time-invariant interval observers

Time-invariant interval observers for nonlinear systems

Consider the system

$$x_{k+1} = \mathcal{F}(x_k) + w_k \ , k \in \mathbb{N}, \tag{1}$$

Assumption 1. There exists $\mathcal{F}_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ s. t.

$$\mathcal{F}(x) = \mathcal{F}_c(x, x) \ , \ \forall x \in \mathbb{R}^n, \tag{2}$$

$\mathcal{F}_c$ is nondecreasing with respect to each of its n first variables and nonincreasing with respect to each of its n last variables and

$$\left\{ \begin{array}{rcl} a_{k+1} & = & \mathcal{F}_c(a_k, b_k), \\ b_{k+1} & = & \mathcal{F}_c(b_k, a_k), \end{array} \right. \tag{3}$$

admits the origin as a GAS equilibrium point.

Time-invariant interval observers for nonlinear systems

Theorem

Let the system (1) satisfy Assumption 1. Let (w_k) *be bounded by two known sequences* (w_k^+)*,* (w_k^-) *:* $w_k^- \leq w_k \leq w_k^+, \forall k \geq 0$*. Then the system*

$$\left\{ \begin{array}{rcl} z_{k+1}^+ & = & \mathcal{F}_c(z_k^+, z_k^-) + w_k^+, \\ z_{k+1}^- & = & \mathcal{F}_c(z_k^-, z_k^+) + w_k^-, \\ z_{k_0}^+ & = & x_{k_0}^+ \ , \ z_{k_0}^- = x_{k_0}^-, \\ x_k^+ & = & z_k^+ \ , \ x_k^- = z_k^- \ , \end{array} \right. \tag{4}$$

is an interval observer for system (1).

Time-invariant interval observers for nonlinear systems

Discussion on this result :

- ▷ We proved that if $\mathcal{F}$ is of class C^1, then there exists an infinite family of functions $\mathcal{F}_c$ s.t. $\mathcal{F}(x) = \mathcal{F}_c(x, x)$.
- ▷ The restrictive part of Assumption 1 is the stability property of the system (3).
- ▷ Finding the function $\mathcal{F}_c$ which gives the tighter enclosures of the state vectors is an open problem.
- ▷ If a system can be transformed through a change of coordinates into a system that satisfies Assumption 1, then an interval observer can be constructed.

Time-invariant interval observers for linear systems

Corollary

Consider the system

$$x_{k+1} = \mathcal{A}x_k + w_k \ , k \in \mathbb{N}, \tag{5}$$

Assume that the matrix $\mathcal{A}^* = \begin{bmatrix} \mathcal{A}^+ & -\mathcal{A}^- \\ -\mathcal{A}^- & \mathcal{A}^+ \end{bmatrix}$ *is Schur stable.*
Let (w_k) *be bounded by two known sequences* (w_k^+), (w_k^-) *:* $w_k^- \leq w_k \leq w_k^+$, $\forall k \geq 0$*. Then*

$$\begin{cases} z_{k+1}^+ &= \mathcal{A}^+ z_k^+ - \mathcal{A}^- z_k^- + w_k^+, \\ z_{k+1}^- &= \mathcal{A}^+ z_k^- - \mathcal{A}^- z_k^+ + w_k^-, \\ z_{k_0}^+ &= x_{k_0}^+ \ , \ z_{k_0}^- = x_{k_0}^-, \\ x_k^+ &= z_k^+ \ , \ x_k^- = z_k^- , \end{cases} \tag{6}$$

is an interval observer for system (5).

Time-invariant interval observers for linear systems

Question 1 : If $\mathcal{A}$ is Schur stable, is the corresponding matrix $\mathcal{A}^*$ necessarily Schur stale ?

> If the answer to the question was positive $\rightarrow$ construction of interval observers for any exponentially stable linear discrete-time system.

Unfortunately, the answer is negative, e.g., $\mathcal{A} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

Question 2 : If $\mathcal{A}$ is Schur stable, does it exist a linear time-invariant change of coordinate that transforms the system (5) into the another one to which the Corollary applies ?
We conjecture that the answer is negative.

$\rightarrow$ This motivates our main result !

Transformations of linear systems

Transformations of linear systems

Theorem

Consider the system

$$x_{k+1} = \mathcal{A} x_k \ , k \in \mathbb{N}, \tag{7}$$

with $x_k \in \mathbb{R}^n$, where $\mathcal{A} \in \mathbb{R}^{n \times n}$ is a Schur stable matrix. Then there exists a time-varying change of coordinates $y_k = \mathcal{R}_k x_k$, where $(\mathcal{R}_k)$ is a sequence of invertible matrices s. t. there exists a constant $c > 0$ s. t., for all $k \in \mathbb{N}$, $|\mathcal{R}_k| + |\mathcal{R}_k^{-1}| \leq c$, which transforms (7) into a positive and exponentially stable linear system.

Time-varying interval observers

Time-varying interval observers

Theorem. Consider a system

$$x_{k+1} = \alpha x_k + w_k \ , \ y_k = Cx_k, \tag{8}$$

s.t. there exists K s.t. $\mathcal{A} = \alpha + KC$ is Schur stable. Let (w_k) be bounded by two known sequences (w_k^+), (w_k^-) : $w_k^- \leq w_k \leq w_k^+$, $\forall k \geq 0$. Then there exists a sequence of invertible matrices $(\mathcal{R}_k)$ and $c > 0$ s.t. for all $k \in \mathbb{N}$, $|\mathcal{R}_k| + |\mathcal{R}_k^{-1}| \leq c$ and $\mathcal{R}_{k+1}\mathcal{A}\mathcal{R}_k^{-1} = \mathcal{E}$, where $\mathcal{E}$ is a nonnegative matrix. Then

$$\begin{array}{rcl} z_{k+1}^+ & = & \mathcal{E}z_k^+ - \mathcal{R}_{k+1}Ky_k + \mathcal{R}_{k+1}^+ w_k^+ - \mathcal{R}_{k+1}^- w_k^-, \\ z_{k+1}^- & = & \mathcal{E}z_k^- - \mathcal{R}_{k+1}Ky_k, + \mathcal{R}_{k+1}^+ w_k^- - \mathcal{R}_{k+1}^- w_k^+, \\ z_{k_0}^+ & = & \mathcal{R}_k^+ x_{k_0}^+ - \mathcal{R}_k^- x_{k_0}^-, \ z_{k_0}^- = \mathcal{R}_k^+ x_{k_0}^- - \mathcal{R}_k^- x_{k_0}^+, \\ x_k^+ & = & \mathcal{S}_k^+ z_k^+ - \mathcal{S}_k^- z_k^-, \ x_k^- = \mathcal{S}_k^+ z_k^- - \mathcal{S}_k^- z_k^+, \end{array} \tag{9}$$

with $\mathcal{S}_k = \mathcal{R}_k^{-1}$ is an interval observer for system (8).

Conclusion

Conclusion

▷ We developed :

- a technique of construction of time-invariant interval observers for a family of nonlinear discrete-time time-invariant systems.
- a technique of construction of time-varying interval observers for linear time-invariant systems.

▷ The key point : linear time-varying change of coordinates.

▷ Many possible extensions : ... discrete-time systems with delay... triangular systems.... systems with nonlinear globally lipschiz disturbances...

References

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