Interval Observers For Discrete-time Systems

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Introduction


- **Main advantage** of the technique: it allows to cope with uncertainties that are known to characterize some classes of systems.
Introduction

Interest of the discrete-time systems:

- Discretization techniques transform continuous-time systems into discrete-time systems.
- Systems with sampled data often lead to discrete-time systems.

Many researchers have constructed observers or dynamic output feedbacks for discrete-time systems: M. Boutayeb, M. Darouach, I. Karafyllis, C. Kravaris, M. Xiao, N. Kazantzis, C. Kravaris, A.J. Krener, Z.P. Jiang...
Introduction

We discovered that our work has been created simultaneously with:

First we construct time-invariant interval observers for a nonlinear discrete-time system under a specific condition.

Second we show how time-varying interval observers can be constructed for the linear time-invariant discrete-time systems.

→ **Key idea:** time-varying changes of coordinates that transform linear discrete-time systems into nonnegative systems.

**Remark:** fundamental differences between our work and F. Mazenc, O. Bernard (2011) because a continuous-time system \( \dot{x} = Ax \) is positive iff the matrix \( A \) is cooperative whereas a discrete-time system \( x_{k+1} = Ax_k \) is positive iff no entry of \( A \) is negative.

Time-invariant interval observers
Consider the system

$$x_{k+1} = \mathcal{F}(x_k) + w_k , \ k \in \mathbb{N},$$  \hspace{1cm} (1)

**Assumption 1.** There exists $\mathcal{F}_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ s. t.

$$\mathcal{F}(x) = \mathcal{F}_c(x, x) , \ \forall x \in \mathbb{R}^n,$$

(2)

$\mathcal{F}_c$ is nondecreasing with respect to each of its $n$ first variables and nonincreasing with respect to each of its $n$ last variables and

$$\begin{cases} a_{k+1} = \mathcal{F}_c(a_k, b_k), \\ b_{k+1} = \mathcal{F}_c(b_k, a_k), \end{cases}$$

(3)

admits the origin as a GAS equilibrium point.
Theorem

Let the system (1) satisfy Assumption 1. Let \((w_k)\) be bounded by two known sequences \((w_k^+), (w_k^-)\): \(w_k^- \leq w_k \leq w_k^+, \forall k \geq 0\). Then the system

\[
\begin{align*}
  z_{k+1}^+ &= \mathcal{F}_c(z_k^+, z_k^-) + w_k^+, \\
  z_{k+1}^- &= \mathcal{F}_c(z_k^-, z_k^+) + w_k^-, \\
  z_{k_0}^+ &= x_{k_0}^+, \ z_{k_0}^- = x_{k_0}^-, \\
  x_k^+ &= z_k^+, \ x_k^- = z_k^-,
\end{align*}
\]

(4)
is an interval observer for system (1).
Discussion on this result:

- We proved that if $F$ is of class $C^1$, then there exists an infinite family of functions $F_c$ such that $F(x) = F_c(x, x)$.
- The restrictive part of Assumption 1 is the stability property of the system (3).
- Finding the function $F_c$ which gives the tighter enclosures of the state vectors is an open problem.
- If a system can be transformed through a change of coordinates into a system that satisfies Assumption 1, then an interval observer can be constructed.
Corollary

Consider the system

\[
x_{k+1} = Ax_k + w_k, \quad k \in \mathbb{N},
\]

(5)

Assume that the matrix \( A^* = \begin{bmatrix} A^+ & -A^- \\ -A^+ & A^- \end{bmatrix} \) is Schur stable.

Let \((w_k)\) be bounded by two known sequences \((w^+_k), (w^-_k)\):

\[
w^-_k \leq w_k \leq w^+_k, \quad \forall k \geq 0.
\]

Then

\[
\begin{cases}
  z^+_{k+1} = A^+ z^+_k - A^- z^-_k + w^+_k, \\
  z^-_{k+1} = A^+ z^-_k - A^- z^+_k + w^-_k, \\
  z^+_0 = x^+_0, \quad z^-_0 = x^-_0,

  x^+_k = z^+_k, \quad x^-_k = z^-_k,
\end{cases}
\]

(6)

is an interval observer for system (5).
Question 1: If $A$ is Schur stable, is the corresponding matrix $A^*$ necessarily Schur stable?

If the answer to the question was positive $\rightarrow$ construction of interval observers for any exponentially stable linear discrete-time system.

Unfortunately, the answer is negative, e.g., $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$.

Question 2: If $A$ is Schur stable, does it exist a linear time-invariant change of coordinate that transforms the system (5) into the another one to which the Corollary applies?

We conjecture that the answer is negative.

$\rightarrow$ This motivates our main result!
Transformations of linear systems
Theorem

Consider the system

\[ x_{k+1} = Ax_k, \quad k \in \mathbb{N}, \quad (7) \]

with \( x_k \in \mathbb{R}^n \), where \( A \in \mathbb{R}^{n \times n} \) is a Schur stable matrix. Then there exists a time-varying change of coordinates \( y_k = R_k x_k \), where \( (R_k) \) is a sequence of invertible matrices s. t. there exists a constant \( c > 0 \) s. t., for all \( k \in \mathbb{N} \), \( |R_k| + |R_k^{-1}| \leq c \), which transforms (7) into a positive and exponentially stable linear system.
Time-varying interval observers
**Theorem.** Consider a system

\[ x_{k+1} = \alpha x_k + w_k, \quad y_k = Cx_k, \]  

\[ (8) \]

s.t. there exists \( K \) s.t. \( A = \alpha + KC \) is Schur stable. Let \((w_k)\) be bounded by two known sequences \((w_k^+)\), \((w_k^-)\): \( w_k^- \leq w_k \leq w_k^+ \), \( \forall k \geq 0 \). Then there exists a sequence of invertible matrices \((R_k)\) and \( c > 0 \) s.t. for all \( k \in \mathbb{N}, |R_k| + |R_k^{-1}| \leq c \) and \( R_{k+1}AR_k^{-1} = \mathcal{E} \), where \( \mathcal{E} \) is a nonnegative matrix. Then

\[ \begin{align*}
  z^+_{k+1} &= \mathcal{E}z_k^+ - R_{k+1}Ky_k + R_{k+1}^+ w_k^+ - R_{k+1}^- w_k^- , \\
  z^-_{k+1} &= \mathcal{E}z_k^- - R_{k+1}Ky_k + R_{k+1}^+ w_k^- - R_{k+1}^- w_k^+ , \\
  z^+_0 &= R_k x_k^0 - R_k^- x_k^0 , \\
  z^-_0 &= R_k x_k^- , \\
  x^+_k &= S_k^+ z_k^+ - S_k^- z_k^- , \\
  x^-_k &= S_k^+ z_k^- - S_k^- z_k^+ ,
\end{align*} \]

\[ (9) \]

with \( S_k = R_k^{-1} \) is an interval observer for system (8).
Conclusion
We developed:

- a technique of construction of time-invariant interval observers for a family of nonlinear discrete-time time-invariant systems.
- a technique of construction of time-varying interval observers for linear time-invariant systems.

The key point: linear time-varying change of coordinates.

Many possible extensions: … discrete-time systems with delay… triangular systems…. systems with nonlinear globally lipschitz disturbances…
References


