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Interval Observers For Discrete-time Systems

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Abstract— **First, time-invariant interval observers are proposed for a family of nonlinear systems. Second, it is shown that, for any time-invariant exponentially stable discrete-time linear system with additive disturbances, time-varying exponentially stable discrete-time interval observers can be constructed. The result relies on the design of time-varying changes of coordinates which transform a linear system into a nonnegative linear system.**

Index Terms—**Interval observer, discrete-time system.**

I. INTRODUCTION

Twelve years ago, a technique of state estimation has been introduced. It is based on the notion of interval observers, which are tools allowing one to cope with uncertainties of various types that affect some classes of systems. The technique, which originates in [4], has been developed in several contexts: in particular some works are devoted to families of linear systems [3], [8], [9], [11], [10] and others are devoted to nonlinear systems [15], [16], [13]. A common feature of all the results available in the literature is that they apply only to continuous-time systems. On the other hand, the family of the discrete-time systems is very important and many constructions of observers or dynamic output feedbacks have been proposed for it (see [7], [2], [17], [6, Chapt. 6]). The interest of the discrete-time systems partially stems from the fact that discretization techniques transform continuous-time systems into discrete-time systems. Moreover, systems with sampled data often lead to discrete-time systems, as explained for instance in [1]. These systems are frequently affected by disturbances, which motivates the development of robust state estimation techniques, like the one based on interval observers.

This motivates the present work. First, we shall consider a nonlinear system of the form

$$x_{k+1} = \mathcal{F}(x_k) + w_k \ , \ k \in \mathbb{N} \tag{1}$$

with $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^n$ and construct time-invariant interval observers, under a condition on the function $\mathcal{F}$. Next, we shall focus our attention on the family of the linear time-invariant discrete-time systems

$$x_{k+1} = \mathcal{A}x_k + w_k \ , \ k \in \mathbb{N} \tag{2}$$

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with $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^n$ when the spectral radius of $\mathcal{A}$ is smaller than 1. We will show that in some cases, the technique of construction of time-invariant interval observers developed for (1) does not lead to interval observers for (2). To overcome this limitation, we will show how time-varying interval observers can be constructed for a family of linear systems with outputs which encompasses the family of systems (2). The construction we will propose relies on time-varying changes of coordinates that transform linear discrete-time systems into nonnegative discrete-time systems. We will obtain the change of coordinates by using the fact that any real matrix can be transformed into a matrix of the Jordan canonical form (see [14, Section 1.8]) and next by finding suitable changes of coordinates for elementary Jordan blocks. Surprisingly, although the changes of coordinates we shall apply are time-varying, the transformed systems are autonomous. However, the interval observers we will construct are time-varying because they involve a time-varying change of coordinates, and thus they give lower and upper bounds for the state of the system studied that depend on the time.

The part of the present paper that is devoted to linear systems owes a great deal to [9], which presents constructions of interval observers for continuous-time linear systems by using extensively time-varying changes of coordinates. However, there are fundamental differences between the main results of [9] and the second part of the present paper because it turns out that a continuous-time system $\dot{x} = \mathcal{A}x$ is positive if and only if the matrix $\mathcal{A}$ is cooperative whereas a discrete-time system $x_{k+1} = \mathcal{A}x_k$ is positive if and only if no entry of $\mathcal{A}$ is negative. Consequently, the time-varying changes of coordinates we shall use to obtain nonnegative linear systems cannot be deduced from the time-varying changes of coordinates used in [9] to transform a Hurwitz matrix into a Hurwitz and cooperative matrix.

It is worth noticing that, to the best of the authors' knowledge, no other contribution is devoted to the design of interval observers for discrete-time systems. The paper is organized as follows. Basic definitions and results are presented in Section II. In Section III, we state and prove results of construction of interval observers for nonlinear systems. In Section IV, we state and prove that any time-invariant exponentially stable linear discrete-time system can be transformed into a block diagonal system with nonnegative and exponentially stable subsystems. A construction of time-varying interval observers for linear systems with output is established in Section V. Concluding remarks are drawn in Section VI.

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II. Classical definitions and results

A. Notation, definitions, basic result

The notation will be simplified whenever no confusion can arise from the context. Any $k \times n$ matrix, whose entries are all 0 is denoted 0. We denote by $\mathrm{spec}(A)$ the spectrum of a matrix $A \in \mathbb{R}^{r\times n}$. Then the spectral radius of A is the real number $\rho(A) = \max\{|\lambda| : \lambda \in \mathrm{spec}(A)\}$. All the inequalities must be understood componentwise *i.e.* $v_a = (v_{a1}, ..., v_{ar})^\top \in \mathbb{R}^r$ and $v_b = (v_{b1}, ..., v_{br})^\top \in \mathbb{R}^r$ are such that $v_a \leq v_b$ if and only if, for all $i \in \{1, ..., r\}$, $v_{ai} \leq v_{bi}$. $\max(A, B)$ for two matrices $A = (a_{ij}) \in \mathbb{R}^{r\times s}$ and $B = (b_{ij}) \in \mathbb{R}^{r\times s}$ of same dimension is the matrix where each entry is $m_{ij} = \max(a_{ij}, b_{ij})$. For a matrix $A \in \mathbb{R}^{r\times s}$, $A^+ = \max(A, 0)$, $A^- = \max(-A, 0)$. Thus, $A = A^+ - A^-$. A matrix $A \in \mathbb{R}^{r\times s}$ is said to be *nonnegative* if every entry of A is nonnegative. A sequence (u_i) is nonnegative if for all integer k, u_k is nonnegative. The discrete-time dynamical system (1) is nonnegative if for every nonnegative initial condition x_0 and nonnegative sequence (w_k), the corresponding solution x_k is nonnegative for all k. $\mathrm{diag}\{B_1, ..., B_j\}$ denotes the block diagonal matrix $\begin{bmatrix} B_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_j \end{bmatrix}$.

B. Definition of interval observer

For the sake of generality, we first introduce a general definition of interval observer for discrete-time time-varying nonlinear systems.

Definition 1: Consider a time-varying system

$$x_{k+1} = f_1(k, x_k, w_k), \; k \in \mathbb{N}, \tag{3}$$

with an output $y_k = m(x_k, w_k)$, with $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^\ell$, and where f_1 and m are two functions. The uncertainties w_k are such that there exists a sequence $\overline{w}_k = (w_k^+, w_k^-) \in \mathbb{R}^{2\ell}$ such that, for all integer $k \geq 0$,

$$w_k^- \leq w_k \leq w_k^+. \tag{4}$$

Moreover, the initial condition $x_0 \in \mathbb{R}^n$ is assumed to be bounded by two known bounds:

$$x_0^- \leq x_0 \leq x_0^+. \tag{5}$$

Then, the dynamical system

$$z_{k+1} = f_2(k, z_k, y_k, \overline{w}_k), \; k \in \mathbb{N}, \tag{6}$$

associated with the initial condition $z_0 = g(k_0, x_0^+, x_0^-) \in \mathbb{R}^{n_z}$ and bounds for the solution x_k:

$$x_k^+ = h^+(k, z_k), \; x_k^- = h^-(k, z_k) \tag{7}$$

where f_2, g, h^+ and h^- are functions, is called an interval observer for (3) if

(i) any solution (x_k, z_k) of (3)-(6) with $\overline{w}_k = 0$ for all $k \in \mathbb{N}$ is such that $\lim_{k\to+\infty} |h^+(k, z_k) - h^-(k, z_k)| = 0$.

(ii) for any vectors x_0, x_0^- and x_0^+ in $\mathbb{R}^n$ satisfying (5), the solutions of (3), (6) with respectively x_0, $z_0 = g(k_0, x_0^+, x_0^-)$ as initial condition at $k = k_0$, denoted respectively x_k and z_k satisfy, for all $k \geq k_0$, the inequalities

$$x_k^- = h^-(k, z_k) \leq x_k \leq h^+(k, z_k) = x_k^+. \tag{8}$$

C. Basic result

The following result, which is a direct consequence of [5, Chapt. 5, Proposition 5.6], is instrumental in establishing one of our main results.

Lemma 1: The system (2) is nonnegative if and only if the matrix $\mathcal{A}$ is nonnegative. □

III. Time-invariant interval observers

The result of this section consists in a construction of interval observers for nonlinear systems of the form

$$x_{k+1} = \mathcal{F}(x_k) + w_k \;, k \in \mathbb{N}, \tag{9}$$

with $x_k \in \mathbb{R}^n$. We introduce an assumption:

Assumption 1. There exists a function $\mathcal{F}_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mathcal{F}(x) = \mathcal{F}_c(x, x) \;, \; \forall x \in \mathbb{R}^n, \tag{10}$$

$\mathcal{F}_c$ is nondecreasing with respect to each of its n first variables and nonincreasing with respect to each of its n last variables and the system

$$\left\{ \begin{array}{rcl} a_{k+1} & = & \mathcal{F}_c(a_k, b_k), \\ b_{k+1} & = & \mathcal{F}_c(b_k, a_k), \end{array} \right. \tag{11}$$

admits the origin as a globally asymptotically stable equilibrium point.

We state and prove the following result.

Theorem 1: Assume that the system (9) satisfies Assumption 1. Let the sequence (w_k) be bounded by two known sequences (w_k^+), (w_k^-): for all integer $k \geq 0$,

$$w_k^- \leq w_k \leq w_k^+. \tag{12}$$

Then the system

$$\left\{ \begin{array}{rcl} z_{k+1}^+ & = & \mathcal{F}_c(z_k^+, z_k^-) + w_k^+, \\ z_{k+1}^- & = & \mathcal{F}_c(z_k^-, z_k^+) + w_k^-, \end{array} \right. \tag{13}$$

associated with the initial conditions

$$z_{k_0}^+ = x_{k_0}^+ \;, \; z_{k_0}^- = x_{k_0}^-, \tag{14}$$

and the bounds for the solutions x_k

$$x_k^+ = z_k^+ \;, \; x_k^- = z_k^-, \tag{15}$$

is an interval observer for system (9). □

Remark 1. From Lemma 6 in Appendix, one deduces easily that if $\mathcal{F}$ is of class C^1, then there exists an infinite family of functions $\mathcal{F}_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathcal{F}(x) = \mathcal{F}_c(x, x)$ for all $x \in \mathbb{R}^n$, $\mathcal{F}_c$ is nondecreasing with respect to each of its n first variables and nonincreasing with respect to each of its n last variables. So the restrictive part of Assumption 1 is the stability property of the system (11). Finding the function $\mathcal{F}_c$ which gives the tighter enclosures of the state vectors is an open problem. In [12], the result of Lemma 6 is proved for the family of the global Lipschitz functions.

Remark 2. Extensions of Theorem 1 to the case where the system (9) is endowed with an output and to time-varying systems can be easily obtained. For the sake of brevity, we omit them.

Proof. Let us consider vectors x_{k_0}, $x^+_{k_0}$, $x^-_{k_0}$, $z^+_{k_0}$ $z^-_{k_0}$ in $\mathbb{R}^n$ such that

$$z^-_{k_0} = x^-_{k_0} \le x_{k_0} \le x^+_{k_0} = z^+_{k_0}. \tag{16}$$

Next, let us consider the solutions (x_k), (z^+_k), (z^-_k) of the systems (9), (13) with initial conditions x_{k_0}, $z^+_{k_0}$, $z^-_{k_0}$. Using the equality (10), we obtain, for all integer $k \ge k_0$,

$$\begin{array}{l} z^+_{k+1} - x_{k+1} = \mathcal{F}_c(z^+_k, z^-_k) - \mathcal{F}_c(x_k, x_k) + w^+_k - w_k, \\ x_{k+1} - z^-_{k+1} = \mathcal{F}_c(x_k, x_k) - \mathcal{F}_c(z^-_k, z^+_k) + w_k - w^-_k. \end{array} \tag{17}$$

Now, we prove by induction that for all $k \ge k_0$, $z^+_k - x_k \ge 0$, $x_k - z^-_k \ge 0$. According to (16), the property is satisfied at the instant k_0. Assume that it is satisfied at the step $k \ge k_0$. Then, the monotonicity properties of $\mathcal{F}_c$ imply that $\mathcal{F}_c(z^+_k, z^-_k) - \mathcal{F}_c(x_k, x_k) \ge 0$, $\mathcal{F}_c(x_k, x_k) - \mathcal{F}_c(z^-_k, z^+_k) \ge 0$. Since, for all integer k, $w^+_k - w_k \ge 0$ and $w_k - w^-_k \ge 0$, it follows that $z^+_{k+1} - x_{k+1} \ge 0$ and $x_{k+1} - z^-_{k+1} \ge 0$. Consequently, the induction assumption is satisfied at the step $k+1$. Finally, we conclude by observing that the global asymptotic stability of the system (11) implies that the solutions of the system (13), when the sequences (w^+_k) and (w^-_k) are identically equal to zero are such that $\lim_{k \to +\infty} |z^+_k - z^-_k| = 0$.

We show now that if a system can be transformed through a change of coordinates into a system that satisfies Assumption 1, then again an interval observer can be constructed. For the sake of simplicity, we consider the case where no disturbance is present, but extensions to cases where there are disturbances can be established under additional assumptions. □

Theorem 2: Consider the system

$$s_{k+1} = \mathcal{G}(s_k) \ , k \in \mathbb{N}, \tag{18}$$

with $s_k \in \mathbb{R}^n$. Assume that there is a diffeomorphism $\theta : \mathbb{R}^n \to \mathbb{R}^n$ such that $\theta(0) = 0$ and the change of coordinates $x_k = \theta(s_k)$ transforms (18) into a system

$$x_{k+1} = \mathcal{F}(x_k) \ , k \in \mathbb{N}, \tag{19}$$

that satisfies Assumption 1. Let $\theta_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\rho_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be two functions such that, for all $x \in \mathbb{R}^n$,

$$\theta(x) = \theta_c(x, x) \ , \ \theta^{-1}(x) = \rho_c(x, x) \tag{20}$$

and both θ_c and ρ_c are nondecreasing with respect to each of their n first variables and nonincreasing with respect to their last n variables.

Then the system

$$\left\{ \begin{array}{lcl} z^+_{k+1} & = & \mathcal{F}_c(z^+_k, z^-_k), \\ z^-_{k+1} & = & \mathcal{F}_c(z^-_k, z^+_k), \end{array} \right. \tag{21}$$

where $\mathcal{F}_c$ is the function provided by Assumption 1, associated with the initial conditions

$$z^+_{k_0} = \theta_c(s^+_{k_0}, s^-_{k_0}) \ , \ z^-_{k_0} = \theta_c(s^-_{k_0}, s^+_{k_0}) \tag{22}$$

and the bounds for the solutions s_k

$$s^+_k = \rho_c(z^+_k, z^-_k) \ , \ s^-_k = \rho_c(z^-_k, z^+_k), \tag{23}$$

is an interval observer for the system (18). □

Proof. To begin with, we observe that the existence of the functions θ_c, ρ_c satisfying the conditions of Theorem 2 is a consequence of Lemma 6 in Appendix. Now, let us consider the solutions (s_k), (z^+_k), (z^-_k) of the systems (18), (21) with initial conditions $(s_{k_0}, s^+_{k_0}, s^-_{k_0}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, $(z^+_{k_0}, z^-_{k_0}) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$s^-_{k_0} \le s_{k_0} \le s^+_{k_0} \tag{24}$$

and the equalities (22) are satisfied.

From (24) and the monotonicity properties of θ_c, it follows that the inequalities

$$\theta_c(s^-_{k_0}, s^+_{k_0}) \le \theta_c(s_{k_0}, s_{k_0}) \le \theta_c(s^+_{k_0}, s^-_{k_0}) \tag{25}$$

are satisfied. From (22) and (20), it follows that

$$z^-_{k_0} \le \theta(s_{k_0}) \le z^+_{k_0}. \tag{26}$$

On the other hand, we know that, for all $k \in \mathbb{N}$,

$$\begin{array}{rcl} z^+_{k+1} & = & \mathcal{F}_c(z^+_k, z^-_k), \\ \theta(s_{k+1}) & = & \mathcal{F}(\theta(s_k)) = \mathcal{F}_c(\theta(s_k), \theta(s_k)), \\ z^-_{k+1} & = & \mathcal{F}_c(z^-_k, z^+_k). \end{array} \tag{27}$$

Arguing as we did to prove Theorem 1, we deduce that, for all integer $k \ge k_0$, the inequalities

$$z^-_k \le \theta(s_k) \le z^+_k \tag{28}$$

are satisfied. From the monotonous properties of ρ_c and the inequalities (28), we deduce that

$$\rho_c(z^-_k, z^+_k) \le \rho_c(\theta(s_k), \theta(s_k)) \le \rho_c(z^+_k, z^-_k). \tag{29}$$

From (20), we deduce that, for all integer $k \ge k_0$,

$$\rho_c(z^-_k, z^+_k) \le \theta^{-1}(\theta(s_k)) \le \rho_c(z^+_k, z^-_k). \tag{30}$$

Thus the inequalities

$$s^-_k \le s_k \le s^+_k \tag{31}$$

are satisfied for all integer $k \ge k_0$. Using $\rho_c(0,0) = \theta^{-1}(0) = 0$, we can conclude the proof. □

A. Interval observers for linear systems

In this section we analyze the consequences of the results of Section III when particularized to the family of the linear time-invariant systems.

As an immediate consequence of Theorem 1, we have the following result:

Corollary 1: Consider the system

$$x_{k+1} = \mathcal{A}x_k + w_k \ , k \in \mathbb{N}, \tag{32}$$

with $x_k \in \mathbb{R}^n$, where $\mathcal{A} \in \mathbb{R}^{n\times n}$ is a constant matrix. Assume that the spectral radius of the matrix

$$\mathcal{A}^* = \begin{bmatrix} \mathcal{A}^+ & -\mathcal{A}^- \\ -\mathcal{A}^- & \mathcal{A}^+ \end{bmatrix} \tag{33}$$

is smaller than 1. Let (w_k) be a sequence bounded by two known sequences (w_k^+), (w_k^-): for all integer $k \geq 0$,

$$w_k^- \leq w_k \leq w_k^+. \tag{34}$$

Then the system

$$\begin{cases} z_{k+1}^+ &= \mathcal{A}^+ z_k^+ - \mathcal{A}^- z_k^- + w_k^+, \\ z_{k+1}^- &= \mathcal{A}^+ z_k^- - \mathcal{A}^- z_k^+ + w_k^-, \end{cases} \tag{35}$$

associated with the initial conditions

$$z_{k_0}^+ = x_{k_0}^+ \;,\; z_{k_0}^- = x_{k_0}^- \tag{36}$$

and the bounds for the solutions x_k

$$x_k^+ = z_k^+ \;,\; x_k^- = z_k^-, \tag{37}$$

is an interval observer for system (32). □

From Corollary 1, a question arises. If the spectral radius of a matrix $\mathcal{A}$ is smaller than 1, is the spectral radius of the corresponding matrix $\mathcal{A}^*$ necessarily smaller than 1 ? If the answer to the question was positive, then by Corollary 1 it would be possible to construct interval observers for any exponentially stable linear discrete-time system. Unfortunately, the answer is negative. For instance, the matrix

$$\mathcal{A} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \tag{38}$$

has a spectral radius smaller than 1, but the spectral radius of the corresponding matrix $\mathcal{A}^*$ is larger than 1.

Then, from Theorem 2, another question arises. If a linear time-invariant discrete-time system (32) has a spectral radius smaller than 1, is it always possible to apply Theorem 2 with a linear change of coordinates θ ? If the answer was positive, then one might always transform an exponentially stable linear discrete-time system into a system for which an interval observer could be designed. But we conjecture that the answer is negative. The following lemma is the reason why we conjecture this.

Lemma 2: Let

$$\mathcal{A} = \begin{bmatrix} -\omega & \omega \\ -\omega & -\omega \end{bmatrix} \in \mathbb{R}^{2\times 2} \tag{39}$$

with $\omega \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$. The spectral radius of $\mathcal{A}$ is $\sqrt{2}\omega < 1$. For any invertible matrix

$$L = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{40}$$

the matrix

$$\mathcal{A}_L = L\mathcal{A}L^{-1} \tag{41}$$

is such that the spectral radius of

$$\mathcal{A}_L^* = \begin{bmatrix} \mathcal{A}_L^+ & -\mathcal{A}_L^- \\ -\mathcal{A}_L^- & \mathcal{A}_L^+ \end{bmatrix} \in \mathbb{R}^{4\times 4} \tag{42}$$

is larger than 1. □

Proof. Due to space limitation, the proof is omitted. □

IV. Transformations of Linear Systems into Nonnegative Systems

The previous section motivates the main results of the present and the next section. In this section, we establish that any discrete-time exponentially stable time-invariant linear system can be transformed into a nonnegative and exponentially stable time-invariant system through a linear time-varying change of coordinates. In the next section, we will use this result to construct interval observers for linear systems.

Theorem 3: Consider the system

$$x_{k+1} = \mathcal{A}x_k \;, k \in \mathbb{N}, \tag{43}$$

with $x_k \in \mathbb{R}^n$, where $\mathcal{A} \in \mathbb{R}^{n\times n}$ is a constant matrix with a spectral radius smaller than 1. Then there exists a time-varying change of coordinates $y_k = \mathcal{R}_k x_k$, where $(\mathcal{R}_k)$ is a sequence of invertible matrices such that there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$, $|\mathcal{R}_k| + |\mathcal{R}_k^{-1}| \leq c$, which transforms (43) into a positive and exponentially stable linear system. □

Proof. The proof splits up into two steps. First we recall that any real matrix admits a real Jordan canonical form. In the second step, we transform systems in Jordan canonical form into positive systems.

Step 1: Jordan canonical forms.

From [14, Section 1.8], we deduce that for some integers $r \in \{0, 1, ..., n\}$, $s \in \{0, 1, ..., n-1\}$ there exists a linear time-invariant change of coordinates

$$y_k = \mathcal{P}x_k, \tag{44}$$

that transforms (43) into

$$y_{k+1} = \overline{\mathcal{J}}y_k, \tag{45}$$

with

$$\overline{\mathcal{J}} = \mathrm{diag}\{\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_s\} \in \mathbb{R}^{n\times n}, \tag{46}$$

where the matrices $\mathcal{J}_i$ are partitioned into two groups: the first r matrices are associated with the r real eigenvalues of multiplicity n_i of $\mathcal{A}$ and the others are associated with the imaginary eigenvalues of multiplicity m_i of $\mathcal{A}$. Therefore $n = \sum_{i=1}^{r} n_i + \sum_{r+1}^{s} 2m_i$ and, for $i = 1$ to r,

$$\mathcal{J}_i = \begin{bmatrix} -\mu_i & 1 & \dots & 0 \\ 0 & -\mu_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -\mu_i \end{bmatrix} \in \mathbb{R}^{n_i\times n_i}, \tag{47}$$

where the μ_i's are real numbers and, for $i = r+1$ to s,

$$\mathcal{J}_i = \begin{bmatrix} \mathcal{M}_i & I_2 & 0 & \dots & 0 \\ 0 & \mathcal{M}_i & I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I_2 \\ 0 & \dots & \dots & 0 & \mathcal{M}_i \end{bmatrix} \in \mathbb{R}^{2m_i\times 2m_i}, \tag{48}$$

with

$$\mathcal{M}_i = \begin{bmatrix} -\kappa_i & \omega_i \\ -\omega_i & -\kappa_i \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad (49)$$

and

$$I_2 = \text{diag}\{1, 1\} \in \mathbb{R}^{2\times 2}, \quad (50)$$

where the ω_i's are non-zero real numbers and the κ_i are real numbers. Notice also that if $\mathcal{A}$ has no real eigenvalues, then $r = 0$ and if all of the eigenvalues of $\mathcal{A}$ are real, then $n = \sum_{i=1}^{r} n_i$.

Step 2: time-varying change of coordinates. We consider the system (45). From Lemmas 4 and 5, we deduce that, for any system

$$a_{k+1} = \mathcal{J}_i a_k, \quad (51)$$

there exist a matrix $\mathcal{H}_i$ nonnegative and with a spectral radius smaller than 1 and a sequence $(\mathcal{Q}_{k,i})$ of invertible matrices bounded in norm by 1 and with inverses bounded in norm by 1 such that the change of coordinates

$$b_k = \mathcal{Q}_{k,i} a_k \quad (52)$$

gives

$$b_{k+1} = \mathcal{H}_i b_k. \quad (53)$$

Next, we consider the change of coordinates

$$r_k = \overline{\mathcal{Q}}_k y_k, \quad (54)$$

with

$$\overline{\mathcal{Q}}_k = \text{diag}\{\mathcal{Q}_{k,1}, \mathcal{Q}_{k,2}, ..., \mathcal{Q}_{k,s}\} \in \mathbb{R}^{n\times n}. \quad (55)$$

Then

$$r_{k+1} = \overline{\mathcal{Q}}_{k+1} \mathcal{J} y_k = \overline{\mathcal{Q}}_{k+1} \mathcal{J} \overline{\mathcal{Q}}_k^{-1} r_k = \overline{\mathcal{H}} r_k \quad (56)$$

with

$$\overline{\mathcal{H}} = \text{diag}\{\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_s\} \in \mathbb{R}^{n\times n}.$$

Finally, we conclude by observing that the change of coordinates

$$s_k = \overline{\mathcal{Q}}_k \mathcal{P} x_k \quad (57)$$

with $\mathcal{R}_k = \overline{\mathcal{Q}}_k \mathcal{P}$ gives the nonnegative exponentially stable time-invariant system

$$s_{k+1} = \overline{\mathcal{H}} s_k$$

and that $\mathcal{R}_k^{-1} = \mathcal{P}^{-1} \overline{\mathcal{Q}}_k^{-1}$, which implies that the sequences $(\mathcal{R}_k)$ and $(\mathcal{R}_k^{-1})$ are bounded in norm. □

V. Time-varying interval observers

The result of this section shows how Theorem 3 can be used when it comes to constructing interval observers for linear systems with output.

Theorem 4: Consider the system

$$\begin{array}{rcl} x_{k+1} & = & \alpha x_k + w_k \ , k \in \mathbb{N}, \\ y_k & = & C x_k \end{array} \quad (58)$$

with $x_k \in \mathbb{R}^n$, the output $y_k \in \mathbb{R}^q$, $w_k \in \mathbb{R}^n$, where $\alpha \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{q\times n}$ are matrices such that there exists $K \in \mathbb{R}^{n\times q}$ such that the matrix $\mathcal{A} = \alpha + KC$ has a spectral radius smaller than 1. Let the sequence (w_k) be bounded by two known sequences $(w_k^+), (w_k^-)$: for all integer $k \geq 0$,

$$w_k^- \leq w_k \leq w_k^+. \quad (59)$$

Then there exists a sequence of real matrices $(\mathcal{R}_k)$ invertible for all $k \in \mathbb{N}$ such that there exists $c > 0$ such that for all $k \in \mathbb{N}$, $|\mathcal{R}_k| + |\mathcal{R}_k^{-1}| \leq c$ and

$$\mathcal{R}_{k+1} \mathcal{A} \mathcal{R}_k^{-1} = \mathcal{E}, \quad (60)$$

where $\mathcal{E} \in \mathbb{R}^{n\times n}$ is a nonnegative matrix whose spectral radius is smaller than 1. Let $\mathcal{S}_k = \mathcal{R}_k^{-1}$ for all $k \in \mathbb{N}$. Then the system

$$\begin{array}{l} z_{k+1}^+ = \mathcal{E} z_k^+ - \mathcal{R}_{k+1} K y_k + \mathcal{R}_{k+1}^+ w_k^+ - \mathcal{R}_{k+1}^- w_k^-, \\ z_{k+1}^- = \mathcal{E} z_k^- - \mathcal{R}_{k+1} K y_k + \mathcal{R}_{k+1}^+ w_k^- - \mathcal{R}_{k+1}^- w_k^+, \end{array} \quad (61)$$

associated with the initial conditions

$$z_{k_0}^+ = \mathcal{R}_k^+ x_{k_0}^+ - \mathcal{R}_k^- x_{k_0}^- \ , \ z_{k_0}^- = \mathcal{R}_k^+ x_{k_0}^- - \mathcal{R}_k^- x_{k_0}^+ \quad (62)$$

and the bounds for the solutions x_k

$$x_k^+ = \mathcal{S}_k^+ z_k^+ - \mathcal{S}_k^- z_k^- \ , \ x_k^- = \mathcal{S}_k^+ z_k^- - \mathcal{S}_k^- z_k^+ \quad (63)$$

is an interval observer for system (58). □

Proof. The proof is omitted. □

VI. Conclusion

We have developed a technique of construction of time-invariant interval observers for a family of nonlinear discrete-time time-invariant systems and a technique of construction of time-varying interval observers for linear time-invariant systems. Much remains to be done. A discrete-time version of the contribution [11], which is devoted to linear systems with delays, can be expected. Extensions of the results of the present work to families of discrete-time nonlinear systems with delays can be expected too.

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APPENDIX

In this section we give technical results whose proofs are omitted.

Lemma 3: Let

$$\mathcal{M} = \begin{bmatrix} -\kappa & \omega \\ -\omega & -\kappa \end{bmatrix} \tag{64}$$

where κ and ω are two real numbers such that $\kappa^2 + \omega^2 > 0$. Let α be any real number such that

$$\sin(\alpha) = -\frac{\omega}{\sqrt{\kappa^2+\omega^2}} \ , \ \cos(\alpha) = -\frac{\kappa}{\sqrt{\kappa^2+\omega^2}} \tag{65}$$

and let, for all $j \in \mathbb{N}$,

$$\mathcal{L}_j = \begin{bmatrix} \cos(\alpha j) & \sin(\alpha j) \\ -\sin(\alpha j) & \cos(\alpha j) \end{bmatrix}. \tag{66}$$

Then, for all $k \in \mathbb{N}$, the equality

$$\mathcal{L}_{k+1}\mathcal{M}\mathcal{L}_k^{-1} = \sqrt{\kappa^2+\omega^2} I_2 \tag{67}$$

is satisfied. □

Lemma 4: We consider the system

$$a_{k+1} = \mathcal{J} a_k \ , k \in \mathbb{N} \tag{68}$$

with

$$\mathcal{J} = \begin{bmatrix} \mathcal{M} & I_2 & 0 & \dots & 0 \\ 0 & \mathcal{M} & I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I_2 \\ 0 & \dots & \dots & 0 & \mathcal{M} \end{bmatrix} \in \mathbb{R}^{2p\times 2p}, \tag{69}$$

with $\mathcal{M}$ defined in (64). Then there exists a constant α such that the time-varying change of coordinates

$$b_k = \overline{\mathcal{L}}_k a_k \tag{70}$$

with

$$\overline{\mathcal{L}}_k = \mathrm{diag}\{\mathcal{L}_{k-p+1}, \mathcal{L}_{k-p+2}, ..., \mathcal{L}_{k-1}, \mathcal{L}_k\} \in \mathbb{R}^{2p\times 2p} \tag{71}$$

with, for all integer j, $\mathcal{L}_j$ defined in (66), transforms the system (68) into the system

$$b_{k+1} = \sqrt{\kappa^2+\omega^2} b_k + \mathcal{K} b_k \tag{72}$$

with

$$\mathcal{K} = \begin{bmatrix} 0 & I_2 & 0 & \dots & 0 \\ 0 & 0 & I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I_2 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{2p\times 2p}. \tag{73}$$

□

Lemma 5: We consider the system

$$a_{k+1} = \mathcal{J}_a a_k \ , k \in \mathbb{N} \tag{74}$$

with, for all $k \in \mathbb{N}$,

$$\mathcal{J}_a = \begin{bmatrix} -\mu & 1 & \dots & 0 \\ 0 & -\mu & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -\mu \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{75}$$

where μ is a positive real number. Then, the time-varying change of coordinates

$$b_k = \mathcal{G}_k a_k \tag{76}$$

with

$$\mathcal{G}_k = \mathrm{diag}\{(-1)^k, (-1)^{k+1}, ..., (-1)^{k+n-1}\} \in \mathbb{R}^{n\times n} \tag{77}$$

gives

$$b_{k+1} = \mathcal{J}_b b_k \ , k \in \mathbb{N} \tag{78}$$

with

$$\mathcal{J}_b = \begin{bmatrix} \mu & 1 & \dots & 0 \\ 0 & \mu & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \mu \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{79}$$

□

Lemma 6: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^1. Then there exists a function $f_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ nondecreasing with respect to each of its n first variables and nonincreasing with respect to each of its n last variables such that, for all $x \in \mathbb{R}^n$, the equality

$$f_c(x, x) = f(x) \tag{80}$$

is satisfied. □