Continuous-Discrete Interval Observers for Systems with Discrete Measurements

Frédéric Mazenc and Thach Ngoc Dinh

Abstract— We consider linear continuous-time systems with input, output and additive disturbances when measurements are only available at discrete instants. For these systems, we solve a state estimation problem by constructing a family of continuous-discrete time-invariant interval observers. These interval observers are composed of four copies of the studied system accompanied with appropriate outputs which give upper and lower bounds for the solutions.

Index Terms—Interval observer, continuous-discrete, asymptotic stability.

I. INTRODUCTION

It is a well-known fact that, in practice, the access to the variables of a system is often difficult. In particular, frequently, the measures are available at discrete instants only. For this reason, the problem of constructing observers for continuous time systems with discrete time measurements has been addressed by many researchers for more than forty years [15], [7], [23], [14]. In [7] the state variables are estimated through an observer which (i) is a simple copy of the system when no new measurement is available (ii) makes an impulsive correction of the estimate when a new measurement is available. A continuous-discrete observer design methodology of this type is also proposed in [3], where is considered a family of nonlinear globally lipschitz systems.

The state estimation approach based on the notions of framer and interval observer is more recent (but noticed that guaranteed state estimation can be traced back to [25] and that many contributions have followed [16], [1], [4], [5].). Due to the advantages it offers, it becomes more and more popular. The key benefits it presents are (i) it makes it possible to cope with large disturbances (ii) it provides an information about the value of the state at any time instant because an interval observer is a state estimator composed of a dynamic extension with two outputs giving an upper and a lower bound for the solutions of the considered system at every instant. Designs of interval observers have been proposed in many papers, for both linear and nonlinear systems, see for instance [12], [22], [8], [6], [2]. Most of them are concerned with continuous-time systems with continuous measurements, but recent contributions are devoted to discrete-time systems [19], [8] and the papers [21], [10],

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[11] address the problem of constructing interval observers for systems with discrete-time measurements.

In the present paper we revisit the problem of constructing interval observers in the case of discrete measurements. We consider linear continuous-time systems with input, output and an additive disturbance under the assumption that they are stabilizable and detectable and that measurements are available at discrete instants only. For each system of this family, we propose a continuous-discrete interval observer which possesses two classical observers as subsystems. As a consequence, when no disturbances are present, the system can be exponentially stabilized through an output feedback which uses information given by the interval observer. Then both the closed-loop system and its interval observer are exponentially stable. It is worth noticing that the idea of stabilizing systems through the information provided by an interval observer is not new; it has been used for instance in [9], [24].

This family of estimators is significantly different from the one proposed in [21]: the interval observers in [21] are continuous-time time-varying systems, even when the system studied is time-invariant and those we construct in the present work are continuous-discrete time-invariant systems. They also significantly differ from the continuous-discrete framers presented for nonlinear systems in [10], [11] because their asymptotic stability is not guaranteed. Moreover, it is worth pointing out that it is not derived directly from the interval observers constructed for the continuous-time systems in [17] and for discrete time systems in [19], [8], [20], although some of the key ideas of these works are used to construct them. Although, in all these works, the dimension of the proposed interval observers is twice the dimension of the studied system, those constructed in the present work include four copies of the system to be observed. Each copy, or its associated error equation, does not possess the property of being nonnegative systems (see, for instance, [13] for the definition of nonnegative system). This fact may sound surprising since all the designs of interval observers available in the literature rely on this property, except the one in [20]. In fact, as in [20], we will use the notion of nonnegative system as well, but only indirectly to select for the interval observer appropriate initial conditions and upper and lower bounds for the solutions of the studied system. This feature of our design is crucial: it is the reason why the interval observers we propose are given by simple equations.

The paper is organized as follows. The main result is stated and proved in Section II. Conclusions are drawn in Section III.

A. Notation, definitions

The notation will be simplified whenever no confusion can arise from the context. $|\cdot|$ denotes the Euclidean norm of vectors of any dimension and the induced norm of matrices of any dimensions. Any $k \times n$ matrix, whose entries are all 0 is simply denoted 0. I_n denotes the identity matrix in $\mathbb{R}^{n \times n}$. All the inequalities must be understood componentwise (partial order of $\mathbb{R}^r$) i.e. $v_a = (v_{a1}, ..., v_{ar})^\top \in \mathbb{R}^r$ and $v_b = (v_{b1}, ..., v_{br})^\top \in \mathbb{R}^r$ are such that $v_a \leq v_b$ if and only if, for all $i \in \{1, ..., r\}$, $v_{ai} \leq v_{bi}$. $\max(A, B)$ for two matrices $A = (a_{ij}) \in \mathbb{R}^{r \times s}$ and $B = (b_{ij}) \in \mathbb{R}^{r \times s}$ of same dimension is the matrix where each entry is $m_{ij} = \max(a_{ij}, b_{ij})$. For any matrix $A \in \mathbb{R}^{r \times s}$, we let $A^+ = \max(A, 0)$, $A^- = A^+ - A$. A matrix $A \in \mathbb{R}^{r \times r}$ is said to be *cooperative* or *Metzler* if every off-diagonal entry of A is nonnegative. A matrix $A \in \mathbb{R}^{r \times s}$ is said to be *nonnegative* if every entry of A is nonnegative. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be *positive definite* if for all non-zero vectors v with real entries ($v \in \mathbb{R}^n$), $v^\top A v > 0$ and we denote $A \succ 0$. Let $\nu > 0$ be a constant and the sequence t_i be defined by

$$t_0 = 0 , \ t_{i+1} - t_i = \nu , \ \forall i \in \mathbb{N} \qquad (1)$$

Let $r : [0, +\infty) \to \mathbb{R}^r$ be a function continuous and bounded over each interval $[t_i, t_{i+1})$. Then, for all integer $k \geq 1$, we let $r_k^{\mathfrak{s}} = \lim_{t \to t_k, t < t_k} r(t)$ and $r_k = r(t_k)$.

II. MAIN RESULT

In this section, we state and prove the main result of the paper. To this end, we introduce a sequence t_i defined by

$$t_0 = 0 , \ t_{i+1} - t_i = \nu , \ \forall i \in \mathbb{N}, \qquad (2)$$

where $\nu > 0$ is a constant and the linear system with output defined, over every interval $[t_i, t_{i+1})$, by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \delta(t) , \\ y(t) &= Cx(t_i) , \ \forall t \in [t_i, t_{i+1}), \end{aligned} \qquad (3)$$

with $x \in \mathbb{R}^n$ are the state variables, $u \in \mathbb{R}^p$ is the input, $y \in \mathbb{R}^q$ is the output and δ is a disturbance, which is supposed to be piecewise continuous function and where the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$ and $C \in \mathbb{R}^{q \times n}$ are constant.

We introduce two assumptions.

Assumption A1. *There is a matrix $L \in \mathbb{R}^{n \times q}$ such that the spectral radius of the matrix*

$$G = Je^{\nu A}, \qquad (4)$$

with

$$J = I_n - LC, \qquad (5)$$

is smaller than 1, i.e. G is Schur stable. Moreover there exists an invertible matrix $P_1 \in \mathbb{R}^{n \times n}$ such that the matrix $\mathfrak{G} = P_1 G P_1^{-1}$ is nonnegative.

Assumption A2. *There is a matrix $K \in \mathbb{R}^{p \times n}$ such that the matrix*

$$H = A + BK \qquad (6)$$

is Hurwitz. Moreover there exists an invertible matrix $P_2 \in \mathbb{R}^{n \times n}$ such that the matrix $\mathfrak{A} = P_2 A P_2^{-1}$ is Metzler.

Let us introduce some notation:

$$N_1 = P_1^{-1} , \ N_2 = P_2^{-1}$$

and, for all $j \in \mathbb{N}$, $j \geq 1$,

$$\begin{aligned} R_{a,j}(\delta^+, \delta^-) &= \int_{t_{j-1}}^{t_j} \left[\Im_{j,\ell}^+ \delta^+(\ell) - \Im_{j,\ell}^- \delta^-(\ell) \right] d\ell , \\ R_{b,j}(\delta^+, \delta^-) &= \int_{t_{j-1}}^{t_j} \left[\Im_{j,\ell}^+ \delta^-(\ell) - \Im_{j,\ell}^- \delta^+(\ell) \right] d\ell , \end{aligned} \qquad (7)$$

with $\Im_{j,\ell} = P_1 J e^{(t_j - \ell)A}$ where J defined in (5).

We are ready to state and prove the following result:

Theorem 1: Assume that the system (3) satisfies Assumptions A1 and A2. Let $(x_0^-, x(0), x_0^+) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ be vectors such that

$$x_0^- \leq x(0) \leq x_0^+ . \qquad (8)$$

Assume that the unknown disturbance δ is piecewise continuous and such that

$$\delta^-(t) \leq \delta(t) \leq \delta^+(t) , \quad \forall t \geq 0 , \qquad (9)$$

where $\delta^+ : [0, +\infty) \to +\infty$ and $\delta^- : [0, +\infty) \to +\infty$ are know continuous functions. Then the system defined, for all $k \in \mathbb{N}$, by

$$\begin{cases} \dot{\xi}_a &= A\xi_a + Bu, \ \forall t \in [t_k, t_{k+1}) \\ \xi_{a,k} &= \xi_{a,k}^{\mathfrak{s}} + L(y_k - C\xi_{a,k}^{\mathfrak{s}}) \\ &\quad + N_1 R_{a,k}(\delta^+, \delta^-), \text{ when } k \geq 1 \\ \dot{\xi}_b &= A\xi_b + Bu, \ \forall t \in [t_k, t_{k+1}) \\ \xi_{b,k} &= \xi_{b,k}^{\mathfrak{s}} + L(y_k - C\xi_{b,k}^{\mathfrak{s}}) \\ &\quad + N_1 R_{b,k}(\delta^+, \delta^-), \text{ when } k \geq 1 \\ \dot{\omega}_a &= A\omega_a + Bu \\ &\quad + N_2[P_2^+ \delta^+(t) - P_2^- \delta^-(t)], \ \forall t \in [t_k, t_{k+1}) \\ \omega_{a,k} &= N_2[P_2^+ \rho_{a,k} - P_2^- \rho_{b,k}], \\ \dot{\omega}_b &= A\omega_b + Bu \\ &\quad + N_2[P_2^+ \delta^-(t) - P_2^- \delta^+(t)], \ \forall t \in [t_k, t_{k+1}) \\ \omega_{b,k} &= N_2[P_2^+ \rho_{b,k} - P_2^- \rho_{a,k}], \end{cases} \qquad (10)$$

with

$$\begin{aligned} \rho_{a,k} &= N_1^+ P_1 \xi_{a,k} - N_1^- P_1 \xi_{b,k} , \\ \rho_{b,k} &= N_1^+ P_1 \xi_{b,k} - N_1^- P_1 \xi_{a,k} , \end{aligned} \qquad (11)$$

and the initial conditions

$$\begin{aligned} \xi_{a,0} &= N_1[P_1^+ x_0^+ - P_1^- x_0^-], \\ \xi_{b,0} &= N_1[P_1^+ x_0^- - P_1^- x_0^+], \end{aligned} \qquad (12)$$

with the bounds:

$$\begin{aligned} x^+(t) &= N_2^+ P_2 \omega_a(t) - N_2^- P_2 \omega_b(t), \\ x^-(t) &= N_2^+ P_2 \omega_b(t) - N_2^- P_2 \omega_a(t), \end{aligned} \qquad (13)$$

is a framer for the system (3), i.e., for all $t \geq 0$,

$$x^-(t) \leq x(t) \leq x^+(t) \qquad (14)$$

when $u(t)$ is a continuous function defined over $[0, +\infty)$.

Moreover the feedback

$$u(\xi_a) = K\xi_a, \qquad (15)$$

with K given by Assumption A2, globally exponentially stabilizes the system (10)-(3) when $\delta^+ = \delta^- = 0$, i.e. (10), associated to the initial conditions (12) and the bounds (13), is an interval observer for (3) in closed loop with (15). $\square$

Discussion of Theorem 1.

• If the pair $(e^{\nu A}, C)$ is detectable, there is a matrix $L_1 \in \mathbb{R}^{n \times q}$ such that the matrix $e^{\nu A} + L_1 C$ is Schur stable and therefore the first part of Assumption A1 is satisfied. Then the matrix $e^{-\nu A}[e^{\nu A} + L_1 C]e^{\nu A}$ is Schur stable. It follows that the matrix $[I_n - LC]e^{\nu A}$ with $L = -e^{-\nu A}L_1$ is Schur stable. Detectability of the pair $(e^{\nu A}, C)$ is a mild condition and, for some pairs (A, C), as for instance $\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, [1 \quad 0] \right)$, it is satisfied for all $\nu > 0$. We deduce that our assumptions imply less stringent requirements on the size of ν than the assumptions in [21]. Moreover, it is worth noticing that if there exists a matrix $L_2 \in \mathbb{R}^{n \times q}$ such that $A + L_2 C$ is Hurwitz, then there exists $\nu_* > 0$ such that for all $\nu \in (0, \nu_*]$, the pair $(e^{\nu A}, C)$ is detectable.

• For the sake of simplicity, we have considered the case where there is no disturbance on the measurements. But the case where disturbances affect the measurements is of interest and may be considered in further studies.

• We conjecture that the interval observers (10) in closed-loop with the feedback (15) is Input-to-State-Stable (ISS) with respect to δ^+, δ^- in the sense similar as this defined in [18].

• The ξ_a and ξ_b subsystems of (10) are classical continuous-discrete observers for the system (3), which belong to the family of continuous-discrete observers used in [3].

• For the sake of simplicity, the matrices P_1 and P_2 in Assumptions A1 and A2 are constant. However, this assumption can be relaxed by replacing the matrices P_1 and P_2 to be respectively a sequence and a function [17], [19]. Then the corresponding interval observer (10)-(12)-(13) would be time-varying.

• The goal of the ξ_a and ξ_b subsystems of (10) is to provide with bounds for the solution x at the discrete instants t_k, while the goal of the ω_a and ω_b subsystems is to provide with bounds for the solution x over the intervals (t_k, t_{k+1}).

• Other choices of stabilizing feedback than (15) can be made. Among them, there are in particular

$$u(\omega_a) = K\omega_a \ , \ u(\omega_b) = K\omega_b. \tag{16}$$

Proof of Theorem 1.

First step: existence and uniqueness of the solutions.

Throughout the proof, we consider a solution of (3)-(10) with initial conditions such that

$$x_0^- \le x(0) \le x_0^+ \tag{17}$$

under the assumption that $u(t)$ is continuous and defined over $[0, +\infty)$, Such initial conditions generate one and only one solution of (10). Indeed, from (12), it follows that

$$\xi_{a,0} = N_1[P_1^+ x_0^+ - P_1^- x_0^-] \ , \ \xi_{b,0} = N_1[P_1^+ x_0^- - P_1^- x_0^+] \ . \tag{18}$$

Therefore the ξ_a and ξ_b subsystems admit one and only one solution over $[t_0, t_1)$. Moreover, from the expressions of $\omega_{a,k}$, $\omega_{b,k}$, we deduce that

$$\omega_a(0) = N_2[P_2^+ \rho_{a,0} - P_2^- \rho_{b,0}],$$
$$\omega_b(0) = N_2[P_2^+ \rho_{b,0} - P_2^- \rho_{a,0}],$$

with

$$\rho_{a,0} = N_1^+ P_1 \xi_{a,0} - N_1^- P_1 \xi_{b,0},$$
$$\rho_{b,0} = N_1^+ P_1 \xi_{b,0} - N_1^- P_1 \xi_{a,0}.$$

Therefore the ω_a and ω_b-subsystems admit one and only one solution over $[t_0, t_1)$. Next, arguing similarly over each interval $[t_k, t_{k+1})$, one obtains one and only one solution over $[0, +\infty)$.

Second step: framer.

In this part of the proof, we show that (10) is a framer for (3), when $u(t)$ is a continuous function defined over $[0, +\infty)$. Since $P_1^+ \ge 0$ and $P_1^- \ge 0$, the inequalities

$$P_1^+ x_0^- \le P_1^+ x(0) \le P_1^+ x_0^+,$$
$$P_1^- x_0^- \le P_1^- x(0) \le P_1^- x_0^+, \tag{19}$$

are satisfied. Since $P_1 = P_1^+ - P_1^-$, it follows that

$$P_1^+ x_0^- - P_1^- x_0^+ \le P_1 x(0) \le P_1^+ x_0^+ - P_1^- x_0^- \ . \tag{20}$$

Using (18), we obtain

$$P_1 \xi_{b,0} \le P_1 x(0) \le P_1 \xi_{a,0} \ . \tag{21}$$

These inequalities rewrite

$$0 \le P_1[x(0) - \xi_{b,0}] \ , \ 0 \le P_1[\xi_{a,0} - x(0)] \ . \tag{22}$$

Now, to ease the analysis, we define two variables

$$e_a = \xi_a - x \ , \ e_b = x - \xi_b \ . \tag{23}$$

According to (22), the inequalities

$$0 \le P_1 e_{a,0} \ , \ 0 \le P_1 e_{b,0} \tag{24}$$

are satisfied. Moreover, one can check readily that e_a and e_b satisfy, for all $k \in \mathbb{N}$,

$$\begin{cases} \dot{e}_a &= Ae_a - \delta(t), \ \forall t \in [t_k, t_{k+1}) \\ e_{a,k} &= e_{a,k}^{\mathfrak{s}} - LCe_{a,k}^{\mathfrak{s}} + P_1^{-1} R_{a,k}(\delta^+, \delta^-), \\ \dot{e}_b &= Ae_b + \delta(t), \ \forall t \in [t_k, t_{k+1}) \\ e_{b,k} &= e_{b,k}^{\mathfrak{s}} - LCe_{b,k}^{\mathfrak{s}} - P_1^{-1} R_{b,k}(\delta^+, \delta^-). \end{cases} \tag{25}$$

By integrating, for any $k \in \mathbb{N}$ any $t \in [t_k, t_{k+1})$ over $[t_k, t)$, we deduce that, for all $k \in \mathbb{N}$, for all $t \in [t_k, t_{k+1})$,

$$e_a(t) = e^{A(t-t_k)} e_{a,k} - \int_{t_k}^{t} e^{(t-\ell)A} \delta(\ell) d\ell \tag{26}$$

and

$$e_b(t) = e^{A(t-t_k)} e_{b,k} + \int_{t_k}^{t} e^{(t-\ell)A} \delta(\ell) d\ell, \tag{27}$$

which implies that

$$e_{a,k+1}^{\mathfrak{s}} = e^{\nu A} e_{a,k} - \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell)A} \delta(\ell) d\ell$$

and

$$e_{b,k+1}^{\mathfrak{s}} = e^{\nu A}e_{b,k} + \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell)A}\delta(\ell)d\ell .$$

We deduce that, for all $k \in \mathbb{N}$,

$$\begin{cases} e_{a,k+1} = Ge_{a,k} + P_1^{-1}R_{a,k+1}(\delta^+, \delta^-) \\ \qquad\qquad -J\int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell)A}\delta(\ell)d\ell , \\ e_{b,k+1} = Ge_{b,k} - P_1^{-1}R_{b,k+1}(\delta^+, \delta^-) \\ \qquad\qquad +J\int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell)A}\delta(\ell)d\ell , \end{cases} \quad (28)$$

where G is the matrix defined in (4).

From the definition of $\mathfrak{G}$ in Assumption A1, we deduce that, for all $k \in \mathbb{N}$,

$$\begin{cases} P_1 e_{a,k+1} = \mathfrak{G}P_1 e_{a,k} + R_{a,k+1}(\delta^+, \delta^-) \\ \qquad\qquad -\int_{t_k}^{t_{k+1}} P_1 J e^{(t_{k+1}-\ell)A}\delta(\ell)d\ell , \\ P_1 e_{b,k+1} = \mathfrak{G}P_1 e_{b,k} - R_{b,k+1}(\delta^+, \delta^-) \\ \qquad\qquad +\int_{t_k}^{t_{k+1}} P_1 J e^{(t_{k+1}-\ell)A}\delta(\ell)d\ell , \end{cases} \quad (29)$$

We recall that $\mathfrak{I}_{j,\ell} = P_1 J e^{(t_j-\ell)A}$ where J defined in (5), it gives

$$\begin{cases} P_1 e_{a,k+1} = \mathfrak{G}P_1 e_{a,k} + R_{a,k+1}(\delta^+, \delta^-) \\ \qquad\qquad -\int_{t_k}^{t_{k+1}} \mathfrak{I}_{j,\ell}\delta(\ell)d\ell , \\ P_1 e_{b,k+1} = \mathfrak{G}P_1 e_{b,k} - R_{b,k+1}(\delta^+, \delta^-) \\ \qquad\qquad +\int_{t_k}^{t_{k+1}} \mathfrak{I}_{j,\ell}\delta(\ell)d\ell . \end{cases} \quad (30)$$

From the expressions of $R_{a,j}$ and $R_{b,j}$ in (7) we deduce that, for all $k \in \mathbb{N}$,

$$R_{a,k+1}(\delta^+, \delta^-) - \int_{t_k}^{t_{k+1}} \mathfrak{I}_{j,\ell}\delta(\ell)d\ell \geq 0$$

and

$$\int_{t_k}^{t_{k+1}} \mathfrak{I}_{j,\ell}\delta(\ell)d\ell - R_{b,k+1}(\delta^+, \delta^-) \geq 0 .$$

These inequalities and the inequalities $\mathfrak{G} \geq 0$ and (24) imply that, for all $k \in \mathbb{N}$,

$$0 \leq P_1 e_{b,k} , \ 0 \leq P_1 e_{a,k} , \quad (31)$$

which implies that, for all $k \in \mathbb{N}$,

$$0 \leq P_1[x_k - \xi_{b,k}] , \ 0 \leq P_1[\xi_{a,k} - x_k] . \quad (32)$$

Therefore, for all $k \in \mathbb{N}$,

$$P_1\xi_{b,k} \leq P_1 x_k \leq P_1\xi_{a,k} . \quad (33)$$

Since $N_1^+ \geq 0$ and $N_1^- \geq 0$, we have, for all $k \in \mathbb{N}$,

$$\begin{aligned} N_1^+ P_1\xi_{b,k} \leq N_1^+ P_1 x_k \leq N_1^+ P_1\xi_{a,k} , \\ N_1^- P_1\xi_{b,k} \leq N_1^- P_1 x_k \leq N_1^- P_1\xi_{a,k} . \end{aligned} \quad (34)$$

Since $N_1^+ P_1 - N_1^- P_1 = I_n$, it follows that, for all $k \in \mathbb{N}$,

$$\rho_{b,k} \leq x_k \leq \rho_{a,k}, \quad (35)$$

with $\rho_{a,k}$ and $\rho_{b,k}$ defined in (11). As an immediate consequence, since $P_2^+ \geq 0$ and $P_2^- \geq 0$, the inequalities

$$P_2^+ \rho_{b,k} \leq P_2^+ x_k \leq P_2^+ \rho_{a,k} , \ P_2^- \rho_{b,k} \leq P_2^- x_k \leq P_2^- \rho_{a,k} \quad (36)$$

are satisfied, which implies that

$$P_2^+ \rho_{b,k} - P_2^- \rho_{a,k} \leq P_2 x_k \leq P_2^+ \rho_{a,k} - P_2^- \rho_{b,k} . \quad (37)$$

Therefore, for all $k \in \mathbb{N}$, the inequalities

$$P_2\omega_{b,k} \leq P_2 x_k \leq P_2\omega_{a,k} \quad (38)$$

hold. Besides, for all $t \in [t_k, t_{k+1})$,

$$\begin{cases} P_2\dot\omega_a = P_2 A P_2^{-1} P_2\omega_a + P_2 Bu \\ \qquad\qquad +P_2^+\delta^+(t) - P_2^-\delta^-(t) , \\ P_2\dot x = P_2 A P_2^{-1} P_2 x + P_2 Bu + P_2\delta(t) , \\ P_2\dot\omega_b = P_2 A P_2^{-1} P_2\omega_b + P_2 Bu \\ \qquad\qquad +P_2^+\delta^-(t) - P_2^-\delta^+(t) . \end{cases} \quad (39)$$

Therefore, according to the definition of $\mathfrak{A}$ in Assumption A2, for all $t \in [t_k, t_{k+1})$,

$$\begin{cases} P_2\dot\omega_a = \mathfrak{A}P_2\omega_a + P_2 Bu + P_2^+\delta^+(t) - P_2^-\delta^-(t) , \\ P_2\dot x = \mathfrak{A}P_2 x + P_2 Bu + P_2\delta(t) , \\ P_2\dot\omega_b = \mathfrak{A}P_2\omega_b + P_2 Bu + P_2^+\delta^-(t) - P_2^-\delta^+(t) . \end{cases} \quad (40)$$

Since the matrix $\mathfrak{A}$ is Metzler and the inequalities (38) are satisfied, we deduce that, for all $t \geq 0$,

$$P_2\omega_b(t) \leq P_2 x(t) \leq P_2\omega_a(t) . \quad (41)$$

Then, using the fact that $N_2^+ \geq 0$, $N_2^- \geq 0$, we obtain the inequalities

$$\begin{aligned} N_2^+ P_2\omega_b(t) \leq N_2^+ P_2 x(t) \leq N_2^+ P_2\omega_a(t) , \\ N_2^- P_2\omega_b(t) \leq N_2^- P_2 x(t) \leq N_2^- P_2\omega_a(t) . \end{aligned} \quad (42)$$

Since $N_2^+ P_2 - N_2^- P_2 = I_n$, we deduce that, for all $t \geq 0$,

$$x^-(t) \leq x(t) \leq x^+(t) , \quad (43)$$

with x^+, x^- defined in (13).

Third step: stability analysis of the closed-loop system.
In this part, we show that all the solutions of the system (3)-(10) in closed loop with (15) go exponentially to the origin when $\delta^+ = \delta^- = 0$.
From (28) and Assumption A1, we deduce that there are real numbers $r_1 > 0$, $r_2 > 0$ such that, for all $i \in \mathbb{N}$,

$$|e_{a,i}| \leq r_1 e^{-ir_2}|e_{a,0}| , \ |e_{b,i}| \leq r_1 e^{-ir_2}|e_{b,0}| . \quad (44)$$

From this property and (26)-(27), we deduce that there are real numbers $r_3 > 0$, $r_4 > 0$ such that, for all $t \geq 0$,

$$|e_a(t)| \leq r_3 e^{-tr_4}|e_a(0)| , \ |e_b(t)| \leq r_3 e^{-tr_4}|e_b(0)| . \quad (45)$$

It follows that

$$\begin{aligned} |\xi_a(t) - x(t)| &\leq r_3 e^{-tr_4}|e_a(0)| , \\ |\xi_b(t) - x(t)| &\leq r_3 e^{-tr_4}|e_b(0)| . \end{aligned} \quad (46)$$

We deduce that

$$|\xi_a(t) - \xi_b(t)| \leq r_3 e^{-tr_4}(|e_a(0)| + |e_b(0)|) . \quad (47)$$

From the definitions of $\rho_{a,k}$ and $\rho_{b,k}$ in (11), we deduce that, for all $k \in \mathbb{N}$, $\rho_{a,k} - \xi_{a,k} = N_1^- P_1(\xi_{a,k} - \xi_{b,k})$ and $\rho_{b,k} - \xi_{b,k} = N_1^- P_1(\xi_{b,k} - \xi_{a,k})$. We deduce that

$$
\begin{array}{rcl}
|\rho_{a,k} - \xi_{a,k}| & \leq & |N_1^- P_1| r_3 e^{-k\nu r_4}(|e_a(0)| + |e_b(0)|), \\
|\rho_{b,k} - \xi_{b,k}| & \leq & |N_1^- P_1| r_3 e^{-k\nu r_4}(|e_a(0)| + |e_b(0)|), \\
|\rho_{b,k} - \rho_{a,k}| & \leq & r_3 e^{-k\nu r_4}(|e_a(0)| + |e_b(0)|) .
\end{array}
$$
(48)

Now for all $t \in [t_k, t_{k+1})$, observe that (10) with the feedback (15) rewrites

$$
\left\{
\begin{array}{rcl}
\dot{\xi}_a & = & H\xi_a, \\
\xi_{a,k} & = & \xi_{a,k}^{\mathfrak{s}} + LC(x_k - \xi_{a,k}^{\mathfrak{s}}), \\
\dot{\xi}_b & = & A\xi_b + BK\xi_a, \\
\xi_{b,k} & = & \xi_{b,k}^{\mathfrak{s}} + LC(x_k - C\xi_{b,k}^{\mathfrak{s}}), \\
\dot{\omega}_a & = & A\omega_a + BK\xi_a, \\
\omega_{a,k} & = & \xi_{a,k} + (\rho_{a,k} - \xi_{a,k}) \\
& & + P_2^{-1} P_2^-(\rho_{a,k} - \rho_{b,k}), \\
\dot{\omega}_b & = & A\omega_b + BK\xi_a, \\
\omega_{b,k} & = & \xi_{b,k} + (\rho_{b,k} - \xi_{b,k}) \\
& & + P_2^{-1} P_2^-[\rho_{b,k} - \rho_{a,k}] .
\end{array}
\right.
$$
(49)

From Lemma 1 in appendix and Assumption A2, which ensures that the matrix H is Hurwitz, we deduce that all the solutions of the ξ_a-subsystem of (49) converge exponentially to the origin. From (47), we deduce that all the solutions of the ξ_b-subsystem of (49) converge exponentially to the origin. From the equality satisfied by $\omega_{a,k}$ in (49) and (48), we deduce that $\omega_{a,k}$ converges exponentially to the origin. Similarly, we deduce from (48) that $\omega_{b,k}$ converges exponentially to the origin. Now, observe that, for all $t \in [t_k, t_{k+1})$,

$$
\omega_a(t) = e^{A(t-t_k)}\omega_a(t_k) + \int_{t_k}^t e^{A(t-m)} BK\xi_a(m)dm .
$$

It follows that, for all $t \in [t_k, t_{k+1})$,

$$
|\omega_a(t)| \leq e^{\nu|A|}|\omega_a(t_k)| + e^{\nu|A|}|BK| \sup_{m \in [0,\nu]} |\xi_a(t_k + m)| .
$$

We deduce easily that $\omega_a(t)$ converges exponentially to the origin. Similarly, one can prove that $\omega_b(t)$ converges exponentially to the origin. These equalities, the definitions of x^+ and x^- in (13) and the inequalities (43) imply that $x(t)$ converges exponentially to the origin. This concludes the proof. $\qquad \square$

III. CONCLUSION

We have developed a new technique of construction of continuous-discrete interval observers for continuous-time systems with discrete measurements. It is based on four copies of the studied system endowed with suitably chosen outputs. This approach relies on a mild detectability assumption, which is less restrictive than the one imposed in [21].

Many extensions of this technique are possible. We plan to investigate the case where the sequence of the differences between two consecutive instants at which the measurements are available is not constant. Moreover, nonlinear systems with triangular structures, impulsive systems, time-varying systems and systems with delay may be considered.

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APPENDIX

Lemma 1: Consider the sequence t_i defined by

$$t_0 = 0 , \; t_{i+1} - t_i = \nu , \; \forall i \in \mathbb{N} ,$$

where $\nu > 0$ is a constant and the system

$$
\begin{aligned}
\dot{\zeta}(t) &= H\zeta(t) , \; \forall t \in [t_k, t_{k+1}) , \\
\zeta(t_k) &= \zeta(t_k)^5 + e^{-c_1 t_k} v , \; \forall k \in \mathbb{N}, k \geq 1 ,
\end{aligned}
\tag{50}
$$

with $\zeta \in \mathbb{R}^m$, where $H \in \mathbb{R}^{m \times m}$ is a Hurwitz matrix, $v \in \mathbb{R}^m$ is a vector and c_1 is a positive constant. Then all the solutions of this system converge exponentially to the origin.

Proof. First, observe that existence and uniqueness of the solutions of the system are guaranteed. By integrating the first equation in (50), we deduce that, for all $k \in \mathbb{N}$,

$$\zeta(t_{k+1}) = E\zeta(t_k) + e^{-c_1 t_{k+1}} v ,\tag{51}$$

with $E = e^{\nu H}$. Since the matrix H is Hurwitz and $\nu > 0$, the matrix E is Schur stable. It follows that there exists a matrix $Q \succ 0$ such that, for all $w \in \mathbb{R}^m$,

$$w^\top (E^\top Q E - Q) w \leq -2|w|^2 .\tag{52}$$

Then the function

$$V(\zeta) = \zeta^\top Q \zeta \tag{53}$$

is a positive definite quadratic function and

$$
\begin{aligned}
&V(\zeta(t_{k+1})) \\
=& \; \zeta(t_{k+1})^\top Q \zeta(t_{k+1}) \\
=& \; [E\zeta(t_k) + v e^{-c_1 t_{k+1}}]^\top Q [E\zeta(t_k) + v e^{-c_1 t_{k+1}}] \\
=& \; \zeta(t_k)^\top E^\top Q E \zeta(t_k) + 2 e^{-c_1 t_{k+1}} v^\top Q E \zeta(t_k) \\
& + e^{-2c_1 t_{k+1}} v^\top Q v .
\end{aligned}
\tag{54}
$$

From (52), it follows that

$$
\begin{aligned}
&V(\zeta(t_{k+1})) - V(\zeta(t_k)) \\
\leq& \; -2|\zeta(t_k)|^2 + 2 e^{-c_1 t_{k+1}} v^\top Q E \zeta(t_k) \\
& + e^{-2c_1 t_{k+1}} v^\top Q v \\
\leq& \; -2|\zeta(t_k)|^2 + 2 e^{-c_1 t_{k+1}} |v| |QE| |\zeta(t_k)| \\
& + |Q| e^{-2c_1 t_{k+1}} |v|^2 \\
\leq& \; -|\zeta(t_k)|^2 + c_3 e^{-2c_1 t_k} ,
\end{aligned}
\tag{55}
$$

with $c_3 = \left[|QE|^2 + |Q| \right] |v|^2 e^{-2c_1 \nu}$.
Now, observe that

$$e^{-2c_1 t_{k+1}} - e^{-2c_1 t_k} = -c_4 e^{-2c_1 t_k} ,\tag{56}$$

with $c_4 = 1 - e^{-2c_1 \nu} > 0$.
It follows that

$$
\begin{aligned}
&V(\zeta(t_{k+1})) + \frac{c_3 + 1}{c_4} e^{-2c_1 t_{k+1}} - V(\zeta(t_k)) - \frac{c_3 + 1}{c_4} e^{-2c_1 t_k} \\
&\leq -|\zeta(t_k)|^2 - e^{-2c_1 t_k} .
\end{aligned}
$$

Since, for all $k \in \mathbb{N}$, the term $V(\zeta(t_k)) + \frac{c_3 + 1}{c_4} e^{-2c_1 t_k}$ is positive, we deduce that the sequence $|\zeta(t_k)|$ converges exponentially to zero. Now, observe that, for all $t \in [t_k, t_{k+1})$, the inequality

$$|\zeta(t)| \leq e^{\nu |H|} |\zeta(t_k)| \tag{57}$$

is satisfied. We deduce that all the solutions of (50) go exponentially to the origin when the time goes to the infinity.