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Brief paper

Construction of interval observers for continuous-time systems with discrete measurements☆

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ABSTRACT

We consider continuous-time systems with input, output and additive disturbances in the particular case where the measurements are only available at discrete instants and have disturbances. To solve a state estimation problem, we construct continuous–discrete interval observers that are asymptotically stable in the absence of disturbances. These interval observers are composed of two copies of the studied system and of a framer, accompanied with appropriate outputs which give, componentwise, upper and lower bounds for the solutions of the studied system.

1. Introduction

Guaranteed state estimation can be traced back to the seminal paper Schweppe (1968) and many contributions have followed Alamo, Bravo, and Camacho (2005), Chernousko (1994), Chisci, Garulli, and Zappa (1996) and Kieffer and Walter (2011). The state estimation approach based on the notions of framer and interval observer is more recent. Due to its usefulness, it becomes more and more popular. The key benefits it offers can be briefly described as follows. First, framers and interval observers, (which are framers possessing a stability property), make it possible to cope with large disturbances. Second, they provide information about the value of each component of the state variable at every instant.

Designs of framers and interval observers have been proposed in many contributions, for both linear and nonlinear systems, see for instance Alcaraz-Gonzalez and Gonzalez-Alvarez (2007), Combastel and Raka (2011), Efimov, Perruquetti, Raissi, and Zolghadri (2013a); Efimov, Raissi, Chebotarev, and Zolghadri (2013c), Gouzé, Rapaport, and Hadj-Sadok (2000), Mazenc, Niculescu, and Bernard (2012c), Raissi, Efimov, and Zolghadri (2012); Raissi, Videau, and Zolghadri (2010) and the references therein. Most of them are concerned with continuous-time systems with continuous measurements or discrete-time systems. But it is a well-known fact that the access to the state variables of a system is often difficult. It turns out that in practice, for continuous-time models, the measures are often delivered at discrete instants only. For this reason, for many decades, many researchers have addressed the problem of determining observers for continuous-time systems with discrete-time measurements, see e.g. Deza, Busvelle, Gauthier, and Rakotopara (1992), Dinh, Andrieu, Nadri, and Serres (accepted for publication), Hammouri, Nadri, and Mota (2006), Jazwinski (1970), Nadri, Hammouri, and Astorga (2004), and a few recent contributions are devoted to the problem of constructing interval observers in the same context: Goffaux, Vande Wouwer, and Bernard (2009), Mazenc and Dinh (2013) and Mazenc, Kieffer, and Walter (2012b). In the present paper, we revisit this problem. We consider continuous-time linear systems with input, output and additive disturbances under the assumptions of stabilizability and detectability. For any system in this family, we design a continuous–discrete interval observer. It consists of *two standard observers* and of a framer as subsystems. We shall prove that a consequence of this is that the proposed interval observer possesses a strong stability property: the difference between its two bounds satisfies an inequality of ISS type (see Sontag & Wang, 1995 for the notion of ISS) and converges exponentially to zero in the absence of disturbances. Moreover, the system can be globally asymptotically stabilized by using the information provided by the interval observer and this stability depends entirely on the stability properties achieved by the state feedback. It is worth mentioning that the idea

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of stabilizing systems through the information provided by interval observers is not new. It has been used in particular in the contributions Efimov, Raissi, and Zolghadri (2011), Mazenc, Dinh, and Niculescu (2013b) and Polyakov, Efimov, Perruquetti, and Richard (2012).

The estimators we propose are significantly different from those presented in Mazenc et al. (2012b), which have, as dynamic part, continuous-time systems which are time-varying even when the studied system is time-invariant, while those we construct are continuous–discrete systems that are time-invariant when the studied system is. They are also distinct from the continuous–discrete framers presented in Goffaux et al. (2009) because their asymptotic stability is not guaranteed. The design we propose owes a great deal to the papers Andrieu and Nadri (2010) and Deza et al. (1992), where the state variables are estimated through an observer which (i) is a simple copy of the system when no new measurement is available (ii) makes an impulsive correction of the estimate when a new measurement is available.

It is worth pointing out that the estimators we propose are not derived directly from the interval observers constructed for continuous-time systems in Mazenc and Bernard (2011) and for discrete-time systems in Efimov et al. (2013a), Mazenc, Dinh, and Niculescu (2012a, in press), although some of the key ideas of these works are used along our construction. Although, in these papers, the dimension of the proposed interval observers is twice the dimension of the system to be observed, those constructed in the present work are four times this dimension. The two subsystems which give estimates of the solutions at each instant where a new measurement is available as well as their associated error equations, are not nonnegative systems (see, for instance, Haddad, Chellaboina, & Hui, 2010 for the definition of nonnegative system). This feature may sound astonishing since most of the designs of framers available in the literature rely on this property. In fact, as in Mazenc et al. (in press), we will use the notion of nonnegative system only *indirectly*, to select for the interval observer appropriate upper and lower bounds for the solutions of the studied system at the instants of the impulsive corrections. This feature of our design is fundamental: it is the reason the interval observers we propose are given by rather simple equations. The present paper complements the preliminary conference paper Mazenc and Dinh (2013). In particular, by contrast with Mazenc and Dinh (2013), we consider the case where the output is affected by a disturbance and, to slightly simplify the design, we choose a new framer to estimate the state variable between the instants of discontinuity. Moreover, the assumptions we introduce make it possible to asymptotically stabilize the system by bounded feedback when the studied system possesses a property of robust stabilizability by bounded state feedback.

The paper is organized as follows. The main result is proposed in Section 2. It is proved in Section 3. Section 4 is devoted to an illustrative example. Conclusions are drawn in Section 5.

Notation, definitions. $|\cdot|$ denotes the Euclidean norm of vectors of any dimension and the induced norm of matrices of any dimensions. Any $k \times n$ matrix, whose entries are all 0 is denoted by 0. I_n denotes the identity matrix in $\mathbb{R}^{n\times n}$. The inequalities must be understood componentwise (partial order of $\mathbb{R}^r$) *i.e.* $v_a = (v_{a1}, \dots, v_{ar})^\top \in \mathbb{R}^r$ and $v_b = (v_{b1}, \dots, v_{br})^\top \in \mathbb{R}^r$ are such that $v_a \le v_b$ if and only if, for all $i \in \{1, \dots, r\}$, $v_{ai} \le v_{bi}$. $\max(A, B)$ for two matrices $A = (a_{ij}) \in \mathbb{R}^{r\times s}$ and $B = (b_{ij}) \in \mathbb{R}^{r\times s}$ of same dimension is the matrix where each entry is $m_{ij} = \max(a_{ij}, b_{ij})$. For any matrix $M \in \mathbb{R}^{r\times s}$, we let $M^+ = \max(M, 0)$, $M^- = M^+ - M$. A matrix $M \in \mathbb{R}^{r\times s}$ is said to be nonnegative if all its entries are nonnegative. A matrix $M \in \mathbb{R}^{r\times r}$ is said to be Metzler or cooperative if each off-diagonal entry of this matrix is nonnegative. A square matrix $M \in \mathbb{R}^{n\times n}$ is said to be positive definite if for all non-zero vectors $v \in \mathbb{R}^n$, the inequality $v^\top M v > 0$ is satisfied and we denote $M \succ 0$. Let $\nu > 0$ be a constant and the sequence t_i be defined by

$$t_0 = 0, \qquad t_i = \nu i, \quad \forall i \in \mathbb{N}. \tag{1}$$

We shall use the simplifying notation: $\Im_k = [t_k, t_{k+1})$, let $m : [0, +\infty) \to \mathbb{R}^l$ be a function that is continuous over each interval $\Im_i$ and such that $\lim_{t\to t_i t<t_i} m(t)$ exists. Then, for all integers $k \in \mathbb{N}$, we let $m_k^s = \lim_{t\to t_k t<t_k} m(t)$ and $m_k = m(t_k)$.

2. Main result

In this section, we state and discuss the main result of the paper. To this end, we consider the sequence of real numbers t_i defined in (1) and the linear system with output defined, over every interval $\Im_i$, by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \delta_1(t), \\ y(t) &= Cx(t_i) + \delta_{2,i}, \quad \forall t \in \Im_i, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ are the state variables, $u : [0, +\infty) \to \mathbb{R}^p$ is the input, $y \in \mathbb{R}^q$ is the output, $\delta_1 : [0, +\infty) \to \mathbb{R}^n$ is a disturbance, which is supposed to be piecewise continuous function, $\delta_{2,i} \in \mathbb{R}^q$ is a disturbance and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{q\times n}$ are constant matrices.

We introduce three assumptions.

Assumption 1. There is a matrix $L \in \mathbb{R}^{n\times q}$ such that the spectral radius of the matrix

$$G = Je^{\nu A}, \tag{3}$$

with

$$J = I_n - LC, \tag{4}$$

is smaller than 1, i.e. G is Schur stable. Moreover there exists an invertible matrix $P \in \mathbb{R}^{n\times n}$ such that the matrix $\mathfrak{G} = PGP^{-1}$ is nonnegative.

Assumption 2. There is a Lipschitz continuous feedback u_s satisfying $u_s(0) = 0$ such that, for every function $d : [0, +\infty) \to \mathbb{R}^n$ for which there are two constants $s_1 > 0$, $s_2 > 0$ such that

$$|d(t)| \le s_1 e^{-s_2 t}, \quad \forall t \ge 0, \tag{5}$$

all the solutions of the system

$$\dot{z}(t) = Az(t) + Bu_s(z(t) + d(t)) \tag{6}$$

converge asymptotically to the origin.

Assumption 3. The unknown disturbance δ_1 is piecewise continuous and such that, for all $t \ge 0$,

$$\underline{\delta}_1(t) \le \delta_1(t) \le \overline{\delta}_1(t), \tag{7}$$

where $\overline{\delta}_1 : [0, +\infty) \to \mathbb{R}^n$ and $\underline{\delta}_1 : [0, +\infty) \to \mathbb{R}^n$ are known continuous functions. The unknown disturbance δ_2 is such that, for all integers $i \in \mathbb{N}$,

$$\underline{\delta}_{2,i} \le \delta_{2,i} \le \overline{\delta}_{2,i}, \tag{8}$$

where $\overline{\delta}_{2,i}$, $\underline{\delta}_{2,i}$ are known constants.

Let us introduce some notation. Let $D_A \in \mathbb{R}^{n\times n}$ denote the diagonal matrix such that all the diagonal entries of $A - D_A$ are equal to zero. Let $M_A = D_A + (A - D_A)^+$, $P_A = (A - D_A)^-$,

$$T_A = \begin{bmatrix} M_A & P_A \\ P_A & M_A \end{bmatrix}, \qquad Q_A = \begin{bmatrix} M_A & -P_A \\ -P_A & M_A \end{bmatrix}, \tag{9}$$

$$N = P^{-1}, \qquad W = PL, \qquad Z_N = \begin{bmatrix} N^+ & -N^- \\ -N^- & N^+ \end{bmatrix} \tag{10}$$

and, for all $j \in \mathbb{N}, j \geq 1$,

$$R_{a,j}(\Delta_*) = \int_{t_{j-1}}^{t_j} \left[\mathfrak{I}_j^+(\ell)\overline{\delta}_1(\ell) - \mathfrak{I}_j^-(\ell)\underline{\delta}_1(\ell) \right] d\ell,$$
$$R_{b,j}(\Delta_*) = \int_{t_{j-1}}^{t_j} \left[\mathfrak{I}_j^+(\ell)\underline{\delta}_1(\ell) - \mathfrak{I}_j^-(\ell)\overline{\delta}_1(\ell) \right] d\ell, \tag{11}$$

with $\Delta_* = (\overline{\delta}_1, \underline{\delta}_1)$, $\mathfrak{I}_j(\ell) = PJe^{(t_j-\ell)A}$ where J is the matrix defined in (4). Let $\Delta_\diamond = (\overline{\delta}_1, \underline{\delta}_1, \overline{\delta}_2, \underline{\delta}_2)$ and, for all $k \in \mathbb{N}, k \geq 1$,

$$S_{1,k}(\Delta_\diamond) = N[R_{a,k}(\Delta_*) - W^+\underline{\delta}_{2,k} + W^-\overline{\delta}_{2,k}],$$
$$S_{2,k}(\Delta_\diamond) = N[R_{b,k}(\Delta_*) - W^+\overline{\delta}_{2,k} + W^-\underline{\delta}_{2,k}]. \tag{12}$$

Notice for later use that there is a constant $s_3 > 0$ such that, for $i = 1, 2$, and all $k \in \mathbb{N}, k \geq 1$,

$$|S_{i,k}(\Delta_\diamond)| \leq s_3 \left[\sup_{\ell \in [t_{k-1}, t_k]} |\Delta_*(\ell)| + |\overline{\delta}_{2,k}| + |\underline{\delta}_{2,k}| \right]. \tag{13}$$

We are ready to state and prove the following result:

Theorem 1. *Assume that the system (2) satisfies Assumptions 1–3. Then the system defined, for all $k \in \mathbb{N}$, by*

$$\begin{cases} \begin{pmatrix} \dot{x}_a(t) \\ \dot{x}_b(t) \end{pmatrix} = \begin{pmatrix} Ax_a(t) + Bu(t) \\ Ax_b(t) + Bu(t) \end{pmatrix}, \quad \forall t \in \mathfrak{I}_k \\ \begin{pmatrix} x_{a,k} \\ x_{b,k} \end{pmatrix} = \begin{pmatrix} x^{\mathfrak{s}}_{a,k} \\ x^{\mathfrak{s}}_{b,k} \end{pmatrix} + \begin{pmatrix} L(y_k - Cx^{\mathfrak{s}}_{a,k}) \\ L(y_k - Cx^{\mathfrak{s}}_{b,k}) \end{pmatrix} \\ \qquad + \begin{pmatrix} S_{1,k}(\Delta_\diamond) \\ S_{2,k}(\Delta_\diamond) \end{pmatrix} \quad \text{when } k \geq 1 \\ \begin{pmatrix} \dot{\overline{x}}(t) \\ \dot{\underline{x}}(t) \end{pmatrix} = Q_A \begin{pmatrix} \overline{x}(t) \\ \underline{x}(t) \end{pmatrix} + \begin{pmatrix} Bu(t) + \overline{\delta}_1(t) \\ Bu(t) + \underline{\delta}_1(t) \end{pmatrix}, \quad \forall t \in \mathfrak{I}_k, \\ \begin{pmatrix} \overline{x}_k \\ \underline{x}_k \end{pmatrix} = Z_N \begin{pmatrix} Px_{a,k} \\ Px_{b,k} \end{pmatrix}, \quad \text{when } k \geq 1, \end{cases} \tag{14}$$

with the initial conditions

$$\begin{pmatrix} x_a(0) \\ x_b(0) \\ x_{a,0} \\ x_{b,0} \\ \overline{x}_0 \\ \underline{x}_0 \end{pmatrix} = \begin{pmatrix} N[P^+\overline{x}_0 - P^-\underline{x}_0] \\ N[P^+\underline{x}_0 - P^-\overline{x}_0] \\ N[P^+\overline{x}_0 - P^-\underline{x}_0] \\ N[P^+\underline{x}_0 - P^-\overline{x}_0] \\ \overline{x}_0 \\ \underline{x}_0 \end{pmatrix} \tag{15}$$

and the bounds $\overline{x}$, $\underline{x}$ is an interval observer for the system (2): i.e. when $u(t)$ is a piecewise continuous function defined over $[0, +\infty)$ and bounded over every compact set and

$$\underline{x}_0 \leq x(0) \leq \overline{x}_0 \tag{16}$$

then, for all $t \geq 0$,

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t) \tag{17}$$

and there are constants $c_i > 0$, $i = 1$ to 4 such that, for all $k \in \mathbb{N}, k \geq 1, t \in \mathfrak{I}_k$,

$$|\underline{x}(t) - \overline{x}(t)| \leq c_2 e^{-c_1 t}|\underline{x}_0 - \overline{x}_0| + c_3 \sup_{\ell \in [t_0, t_k]} |\Delta_*(\ell)| + c_4 \sup_{i \in \{1,\ldots,k\}} \left(|\overline{\delta}_{2,i}| + |\underline{\delta}_{2,i}| \right). \tag{18}$$

Moreover all the solutions of the system (14)–(2) in closed-loop with the feedback

$$u(x_a) = u_s(x_a), \tag{19}$$

where u_s is the feedback given by Assumption 2, converge to the origin when $\overline{\delta}_1 = \underline{\delta}_1 = 0, \overline{\delta}_2 = \underline{\delta}_2 = 0$. □

Discussion of Theorem 1.

- If the discrete-time system $g_{k+1} = e^{\nu A}g_k$ with the output Cg_k is detectable, there is a matrix $L_1 \in \mathbb{R}^{n \times q}$ such that the matrix $e^{\nu A} + L_1C$ is Schur stable. Then necessarily the matrix $e^{-\nu A}[e^{\nu A} + L_1C]e^{\nu A}$ is Schur stable. It follows that the matrix $[I_n - LC]e^{\nu A}$ with $L = -e^{-\nu A}L_1$ is Schur stable, which implies that the first part of Assumption 1 is satisfied. Detectability of the pair $(e^{\nu A}, C)$ is a mild condition and, for some pairs (A, C), as for instance $\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, [1 \ 0] \right)$, it is satisfied for all $\nu > 0$. We deduce that our assumptions imply less stringent requirements on the size of ν than the assumptions in Mazenc et al. (2012b). Moreover, it is worth noticing that if the continuous-time system $\dot{h} = Ah$ with the output Ch is detectable, then there exists $\nu_* > 0$ such that for all $\nu \in (0, \nu_*]$, the discrete-time system $g_{k+1} = e^{\nu A}g_k$ with the output Cg_k is detectable.
- For the sake of simplicity, the matrix P in Assumption 1 is constant. However, we conjecture that this assumption can be relaxed by replacing the matrix P by a sequence of matrices that can be deduced from the results in Mazenc et al. (2012a). Then the corresponding interval observer (14)–(15) would be time-varying. If the pair $(e^{\nu A}, C)$ is observable, distinct eigenvalues for the matrix $[I_n - LC]e^{\nu A}$ can be selected and then one can determine a constant matrix P for which Assumption 1 is satisfied.
- The x_a and x_b-subsystems of (14) are classical continuous–discrete observers for the system (2), which belong to the family of continuous–discrete observers used in Andrieu and Nadri (2010). Therefore the interval observer inherits the performances of the classical continuous–discrete observers.
- The goal of the x_a and x_b-subsystems of (14) is to provide with bounds for the solution x at the discrete instants t_k, while the goal of the $\overline{x}$ and $\underline{x}$-subsystems is to provide with bounds for the solution x over the intervals (t_k, t_{k+1}). An alternative choice of dynamics giving bounds over the intervals (t_k, t_{k+1}) is proposed in Mazenc and Dinh (2013).
- Other choices of stabilizing feedback than (19) can be made. Among them, there is in particular $u(x_b) = u_s(x_b)$. Observe that when each eigenvalue of A has a nonpositive real part, then Assumption 2 is satisfied with bounded feedback of arbitrary size (see Sussmann, Sontag, & Yang, 1994). Observe also that Assumption 2 implies that the pair (A, B) is stabilizable, which implies that there is a matrix $K \in \mathbb{R}^{p \times m}$ such that $A + BK$ is Hurwitz. It follows that Assumption 2 is satisfied with the linear feedback $u_s(z) = Kz$. From this linear property and (18), one can easily deduce that this specific choice of feedback results in a closed-loop system possessing an ISS property with respect to δ_1 and δ_2.

3. Proof of Theorem 1

First step: existence and uniqueness of the solutions.

We consider a solution of (2)–(14) with the initial conditions selected in (15) under the assumption that $u(t)$ is piecewise continuous, defined over $[0, +\infty)$ and bounded over every compact interval. One can show that such initial conditions generate one and only one solution of (14) as follows. From (15), it follows that the x_a and x_b-subsystems admit one and only one solution over $[t_0, t_1)$. Moreover, $x^{\mathfrak{s}}_{a,1}$ and $x^{\mathfrak{s}}_{b,1}$ exist because u is piecewise continuous. On the other hand, from (15) we can also deduce that $\overline{x}$ and $\underline{x}$-subsystems admit one and only one solution over $[t_0, t_1)$. Next, arguing similarly over each interval $\mathfrak{I}_k$, one establishes the existence of one and only one solution over $[0, +\infty)$.

Second step: framer.

In this part of the proof, we show that (14) is a framer for (2).

Since $P^+ \geq 0$ and $P^- \geq 0$, (16) implies that the inequalities

$$\begin{aligned} &P^+\underline{x}_0 \leq P^+x(0) \leq P^+\overline{x}_0, \\ &P^-\underline{x}_0 \leq P^-x(0) \leq P^-\overline{x}_0, \end{aligned} \tag{20}$$

are satisfied. Since $P = P^+ - P^-$, it follows that

$$P^+\underline{x}_0 - P^-\overline{x}_0 \leq Px(0) \leq P^+\overline{x}_0 - P^-\underline{x}_0. \tag{21}$$

Using (15), we obtain

$$Px_{b,0} \leq Px(0) \leq Px_{a,0}. \tag{22}$$

Now, to ease the analysis, we define two variables

$$e_a = x_a - x, \qquad e_b = x - x_b. \tag{23}$$

They satisfy, for all $k \in \mathbb{N}$,

$$\begin{cases} \dot{e}_a = Ae_a - \delta_1(t), \quad \forall t \in \mathfrak{I}_k \\ e_{a,k} = e^{\mathfrak{s}}_{a,k} - LCe^{\mathfrak{s}}_{a,k} + L\delta_{2,k} + S_{1,k}(\Delta_\diamond), \quad k \geq 1 \\ \dot{e}_b = Ae_b + \delta_1(t), \quad \forall t \in \mathfrak{I}_k \\ e_{b,k} = e^{\mathfrak{s}}_{b,k} - LCe^{\mathfrak{s}}_{b,k} - L\delta_{2,k} - S_{2,k}(\Delta_\diamond), \quad k \geq 1. \end{cases} \tag{24}$$

By integrating, for any $k \in \mathbb{N}$ any $t \in \mathfrak{I}_k$ over $[t_k, t)$, we deduce that, for all $k \in \mathbb{N}$, for all $t \in \mathfrak{I}_k$,

$$e_a(t) = e^{A(t-t_k)}e_{a,k} - \int_{t_k}^{t} e^{(t-\ell)A}\delta_1(\ell)d\ell, \tag{25}$$

which implies that

$$e^{\mathfrak{s}}_{a,k+1} = e^{\nu A}e_{a,k} - \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell)A}\delta_1(\ell)d\ell.$$

Similarly, one can prove that

$$e^{\mathfrak{s}}_{b,k+1} = e^{\nu A}e_{b,k} + \int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell)A}\delta_1(\ell)d\ell.$$

These equalities and (24) give, for all $k \in \mathbb{N}$,

$$\begin{cases} e_{a,k+1} = Ge_{a,k} - J\displaystyle\int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell)A}\delta_1(\ell)d\ell \\ \qquad + S_{1,k+1}(\Delta_\diamond) + L\delta_{2,k+1}, \\ e_{b,k+1} = Ge_{b,k} + J\displaystyle\int_{t_k}^{t_{k+1}} e^{(t_{k+1}-\ell)A}\delta_1(\ell)d\ell \\ \qquad - S_{2,k+1}(\Delta_\diamond) - L\delta_{2,k+1}, \end{cases} \tag{26}$$

where G is the matrix defined in (3).

From the definitions of $\mathfrak{I}_j$, N, W, S_1, S_2, and $\mathfrak{G}$ in Section 2, we deduce that, for all $k \in \mathbb{N}$,

$$\begin{cases} Pe_{a,k+1} = \mathfrak{G}Pe_{a,k} + R_{a,k+1}(\Delta_*) - \displaystyle\int_{t_k}^{t_{k+1}} \Im_{k+1}(\ell)\delta_1(\ell)d\ell \\ \qquad + W\delta_{2,k+1} - W^+\underline{\delta}_{2,k+1} + W^-\overline{\delta}_{2,k+1}, \\ Pe_{b,k+1} = \mathfrak{G}Pe_{b,k} - R_{b,k+1}(\Delta_*) + \displaystyle\int_{t_k}^{t_{k+1}} \Im_{k+1}(\ell)\delta_1(\ell)d\ell \\ \qquad - W\delta_{2,k+1} + W^+\overline{\delta}_{2,k+1} - W^-\underline{\delta}_{2,k+1}. \end{cases} \tag{27}$$

Using the inequalities

$$\begin{aligned} &(\Im_{k+1}(\ell))^+\,\underline{\delta}_1(\ell) \leq (\Im_{k+1}(\ell))^+\,\delta_1(\ell), \\ &(\Im_{k+1}(\ell))^+\,\delta_1(\ell) \leq (\Im_{k+1}(\ell))^+\,\overline{\delta}_1(\ell), \\ &-(\Im_{k+1}(\ell))^-\,\overline{\delta}_1(\ell) \leq -(\Im_{k+1}(\ell))^-\,\delta_1(\ell) \end{aligned}$$

and

$$-(\Im_{k+1}(\ell))^-\,\delta_1(\ell) \leq -(\Im_{k+1}(\ell))^-\,\underline{\delta}_1(\ell),$$

we deduce that

$$R_{a,k+1}(\Delta_*) - \int_{t_k}^{t_{k+1}} \Im_{k+1}(\ell)\delta_1(\ell)d\ell \geq 0 \tag{28}$$

and

$$\int_{t_k}^{t_{k+1}} \Im_{k+1}(\ell)\delta_1(\ell)d\ell - R_{b,k+1}(\Delta_*) \geq 0. \tag{29}$$

Moreover, using the inequalities

$$\begin{aligned} &W^+\underline{\delta}_{2,k+1} \leq W^+\delta_{2,k+1} \leq W^+\overline{\delta}_{2,k+1}, \\ &-W^-\overline{\delta}_{2,k+1} \leq -W^-\delta_{2,k+1} \leq -W^-\underline{\delta}_{2,k+1}, \end{aligned}$$

we deduce that

$$W\delta_{2,k+1} - W^+\underline{\delta}_{2,k+1} + W^-\overline{\delta}_{2,k+1} \geq 0 \tag{30}$$

and

$$-W\delta_{2,k+1} + W^+\overline{\delta}_{2,k+1} - W^-\underline{\delta}_{2,k+1} \geq 0. \tag{31}$$

From (27), the inequalities (28)–(31), the inequality $\mathfrak{G} \geq 0$ and the fact that the inequalities in (22) are equivalent to

$$0 \leq Pe_{a,0}, \qquad 0 \leq Pe_{b,0}, \tag{32}$$

it follows that, for all $k \in \mathbb{N}$,

$$0 \leq Pe_{b,k}, \qquad 0 \leq Pe_{a,k}, \tag{33}$$

we deduce that, for all $k \in \mathbb{N}$,

$$Px_{b,k} \leq Px_k \leq Px_{a,k}. \tag{34}$$

Since $N^+ \geq 0$ and $N^- \geq 0$, we have, for all $k \in \mathbb{N}$,

$$\begin{aligned} &N^+Px_{b,k} \leq N^+Px_k \leq N^+Px_{a,k}, \\ &N^-Px_{b,k} \leq N^-Px_k \leq N^-Px_{a,k}. \end{aligned} \tag{35}$$

We deduce that, for all $k \in \mathbb{N}$,

$$\underline{x}_k \leq x_k \leq \overline{x}_k. \tag{36}$$

Next, let us analyze x, $\overline{x}$, and $\underline{x}$ over an interval $\mathfrak{I}_k$.

Since $A = M_A - P_A$, we have, for all $t \in \mathfrak{I}_k$,

$$\begin{cases} \dot{\overline{x}} - \dot{x} = M_A(\overline{x} - x) + P_A(x - \underline{x}) + \overline{\delta}_1(t) - \delta_1(t), \\ \dot{x} - \dot{\underline{x}} = P_A(\overline{x} - x) + M_A(x - \underline{x}) + \delta_1(t) - \underline{\delta}_1(t). \end{cases} \tag{37}$$

Since, according to (36), for all $k \in \mathbb{N}$, $\overline{x}(t_k) - x(t_k) \geq 0$, $x(t_k) - \underline{x}(t_k) \geq 0$ and for all $t \geq 0$, $\overline{\delta}_1(t) - \delta_1(t) \geq 0$, $\delta_1(t) - \underline{\delta}_1(t) \geq 0$, we deduce that

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t) \tag{38}$$

for all $t \in \mathfrak{I}_k$ because the matrix T_A defined in (9) is cooperative. This allows us to conclude.

Third step: stability analysis of the framer.

Let $\tilde{e} = x_a - x_b$. Then, for all $k \in \mathbb{N}$,

$$\begin{cases} \dot{\tilde{e}}(t) = A\tilde{e}(t), \quad \forall t \in \mathfrak{I}_k, \\ \tilde{e}_k = J\tilde{e}^{\mathfrak{s}}_k + S_{3,k}(\Delta_\diamond), \quad \text{when } k \geq 1, \end{cases} \tag{39}$$

with

$$S_{3,k}(\Delta_\diamond) = S_{1,k}(\Delta_\diamond) - S_{2,k}(\Delta_\diamond). \tag{40}$$

By adding the two equations in (26), for all $k \geq 0$,

$$\tilde{e}_{k+1} = G\tilde{e}_k + S_{3,k+1}(\Delta_\diamond). \tag{41}$$

From Assumption 1, we deduce that there exist real numbers $r_i > 0$, $i = 1$ to 3 such that, for all $k \in \mathbb{N}$, $k \geq 1$,

$$|\tilde{e}_k| \leq r_1 e^{-kr_2}|\tilde{e}_0| + r_3 \sup_{j\in\{1,\ldots,k\}} |S_{3,j}(\Delta_\diamond)|. \tag{42}$$

By integrating the differential equation in (39), we deduce that for all $t \in \mathfrak{I}_k$,

$$|\tilde{e}(t)| \leq e^{\nu|A|}|\tilde{e}(t_k)| \leq r_1 e^{\nu|A|}e^{-kr_2}|\tilde{e}_0| + e^{\nu|A|}r_3 \sup_{j\in\{1,\ldots,k\}} |S_{3,j}(\Delta_\diamond)|. \quad (43)$$

Now, observe that (14) implies that, when $k \geq 1$,

$$\begin{pmatrix} \overline{x}_k \\ \underline{x}_k \end{pmatrix} = Z_N \begin{pmatrix} Px_{b,k} \\ Px_{b,k} \end{pmatrix} + Z_N \begin{pmatrix} P\tilde{e}_k \\ 0 \end{pmatrix}. \quad (44)$$

It follows that, for all $k \geq 1$,

$$\overline{x}_k - \underline{x}_k = (N^+ + N^-)P\tilde{e}_k. \quad (45)$$

We deduce that there is a constant $r_4 > 0$ such that, for all $t \in \mathfrak{I}_k$,

$$|\overline{x}(t) - \underline{x}(t)| \leq r_4|\tilde{e}_k|. \quad (46)$$

Bearing in mind (42), we deduce that there are constants $r_5 > 0$, $r_6 > 0$ such that

$$|\overline{x}(t) - \underline{x}(t)| \leq r_5 e^{-kr_2}|\tilde{e}_0| + r_6 \sup_{j\in\{1,\ldots,k\}} |S_{3,j}(\Delta_\diamond)|. \quad (47)$$

Bearing in mind (13), we deduce that (18) is satisfied.

Fourth step: stability analysis of the closed-loop system.

Let us establish that all the solutions of the system (14)–(2) in closed-loop with the feedback in (19) converge to the origin when $\overline{\delta}_1 = \underline{\delta}_1 = 0$, $\overline{\delta}_2 = \underline{\delta}_2 = 0$.

Observe that, for all $t \geq 0$,

$$\dot{x}(t) = Ax(t) + u_s(x(t) + e_a(t)). \quad (48)$$

We deduce from (26) that there are constants $r_7 > 0$ and $r_8 > 0$ such that, for all $k \in \mathbb{N}$,

$$|e_a(t_k)| \leq r_7 e^{-r_8 k}|\tilde{e}_0|. \quad (49)$$

We deduce that there is a constant $r_9 > 0$ such that for all $t \geq 0$,

$$|e_a(t)| \leq r_9 e^{-r_8 t}|\tilde{e}_0|. \quad (50)$$

This inequality, (48) and Assumption 2 allow us to conclude. □

4. Example

Throughout this section, we use the notation of Section 2 and denote by v_i the ith component of the vector v. To illustrate Theorem 1, we consider the two dimensional system of the family (2) with the matrices:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1\ 0], \quad (51)$$

with disturbances admitting the bounds $\overline{\delta}_1 = c[1\ 1]^\top$, $\underline{\delta}_1 = -\overline{\delta}_1$, $\overline{\delta}_{2,k} = c$, $\underline{\delta}_{2,k} = -\overline{\delta}_{2,k}$, where c is a constant.

Next, we construct a continuous–discrete interval observer. Let us choose

$$L = [1\ 1]^\top. \quad (52)$$

Then $J = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$, $e^{\nu A} = \begin{bmatrix} \cos(\nu) & \sin(\nu) \\ -\sin(\nu) & \cos(\nu) \end{bmatrix}$, which implies that the choice $\nu = \frac{\pi}{4}$ gives the matrix

$$G = -\begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix}, \quad (53)$$

which is Schur stable, but is not nonnegative. One can check readily that Assumption 1 is satisfied with $P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ since $\mathfrak{G} = PGP^{-1} = -G$. Since Assumption 2 is satisfied with, for instance,

$$u_s(x) = -\frac{5}{2}x_2, \quad (54)$$

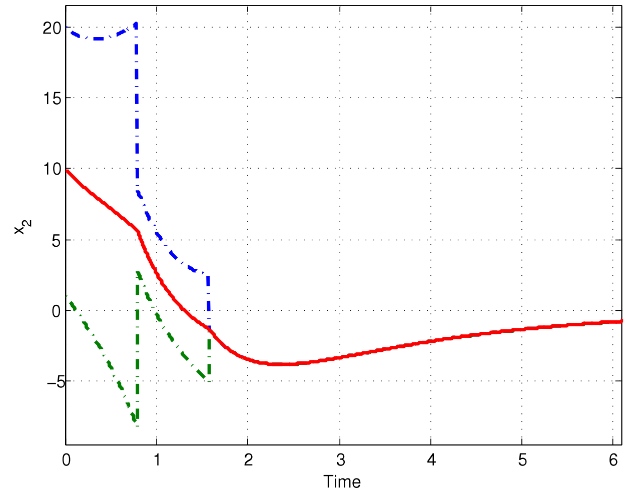

Fig. 1. Evolution with number of iterations of the state component x_2 and its bounds $\overline{x}_2$, $\underline{x}_2$ without uncertainty.

both Assumptions 1 and 2 are satisfied. Therefore, Theorem 1 applies. Then, through simple calculations, we obtain that the system defined, for all $k \in \mathbb{N}$, by

$$\begin{cases} \dot{x}_a(t) = Ax_a(t) + Bu(t), \quad \forall t \in \mathfrak{I}_k \\ \dot{x}_b(t) = Ax_b(t) + Bu(t), \quad \forall t \in \mathfrak{I}_k \\ x_{a,k} = \begin{pmatrix} y_k + c \\ y_k + x^s_{2,a,k} - x^s_{1,a,k} - (1+\sqrt{2})c \end{pmatrix}, \\ \quad \text{when } k \geq 1 \\ x_{b,k} = \begin{pmatrix} y_k - c \\ y_k + x^s_{2,b,k} - x^s_{1,b,k} + (1+\sqrt{2})c \end{pmatrix}, \\ \quad \text{when } k \geq 1 \\ \dot{\overline{x}}(t) = \begin{pmatrix} \overline{x}_2(t) + c \\ -\underline{x}_1(t) + c + u(t) \end{pmatrix}, \quad \forall t \in \mathfrak{I}_k \\ \dot{\underline{x}}(t) = \begin{pmatrix} \underline{x}_2(t) - c \\ -\overline{x}_1(t) - c + u(t) \end{pmatrix}, \quad \forall t \in \mathfrak{I}_k \\ \overline{x}_k = \begin{pmatrix} x_{1,a,k} \\ x_{2,b,k} \end{pmatrix}, \quad \underline{x}_k = \begin{pmatrix} x_{1,b,k} \\ x_{2,a,k} \end{pmatrix} \end{cases} \quad (55)$$

with $y_k = x_1(t_k)$ and the initial conditions

$$x_a(0) = x_{a,0} = \begin{pmatrix} \overline{x}_{1,0} \\ \underline{x}_{2,0} \end{pmatrix}, \quad (56)$$

$$x_b(0) = x_{b,0} = \begin{pmatrix} \underline{x}_{1,0} \\ \overline{x}_{2,0} \end{pmatrix} \quad (57)$$

such that

$$\underline{x}_0 \leq x(0) \leq \overline{x}_0 \quad (58)$$

and the bounds $\overline{x}$, $\underline{x}$ is an interval observer for the considered system.

We present simulations for the system (51)–(55) with u_s defined in (54) as feedback (see Figs. 1 and 2).

5. Conclusion

We have developed a new technique of construction of continuous–discrete interval observers for continuous-time systems with discrete measurements and disturbances in the measurements and the dynamics.

Many extensions of this result are possible. We plan to investigate the case where the sequence of the differences between two

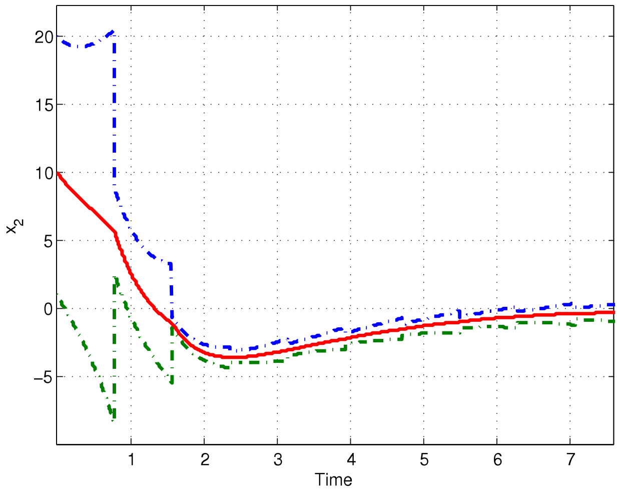

Fig. 2. Evolution with number of iterations of the state component x_2 and its bounds $\overline{x}_2$, $\underline{x}_2$ with the uncertainties.

consecutive instants at which the measurements are available is not constant and to design reduced order interval observers in the spirit of what is done in Efimov, Perruquetti, and Richard (2013b). Moreover, nonlinear systems with globally Lipschitz nonlinearities or triangular structures, time-varying systems and systems with delay (notably in the input) may be considered.

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