Robustness of Nonlinear Systems with Respect to Delay and Sampling of the Controls*

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***Abstract*— We consider continuous time nonlinear time varying systems that are globally asymptotically stabilizable by state feedbacks. We study the stability of these systems in closed loop with controls that are corrupted by both delay and sampling. We establish robustness results through a Lyapunov approach of a new type.**

I. Introduction

Sampling in control laws is a well known problem that has been studied in many contributions [16], [17], [18], [21]. Similarly, time delay problems have been studied for a long time and the last two decades have witnessed important research activity devoted to nonlinear systems with delay [1], [7], [11], [13], [19], [20], [22]. Although sampling and delay occur simultaneously in practice in a wide range of applications, not many papers consider systems that are subject to both delay and sampling in the controls. Notable exceptions are [2], [6], [14], [15]. Even more rare are papers that consider *nonlinear* systems with delay and sampling; [8] seems to be the only general result for this problem, and it relies on a prediction strategy that requires knowledge of the delay and the sampling interval.

Given a nonlinear time varying system with a uniformly globally asymptotically stabilizing time varying undelayed continuous time state controller, it is natural to search for conditions under which the closed loop system remains uniformly globally asymptotically stable (UGAS) when delays and sampling are introduced into the controller. To the best of our knowledge, the problem has never been addressed. However, implementing controls with measurement delays frequently leads to sampling of the control laws with delay. Therefore, we consider a nonlinear system

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))u \tag{1}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ for any dimensions n and p where f and g are locally Lipschitz with respect to x and piecewise continuous with respect to t. We assume that (1) is rendered UGAS by some C^1 controller $u_s(t, x)$. Next, we give conditions under which the UGAS property is preserved when the input has sampling and delays, meaning the control value u entering (1) is $u_s(t_i-\tau, x(t_i-\tau))$ for all $t \in [t_i, t_{i+1})$ and $i = 0, 1, 2, \ldots$, where $\{t_i\}$ is a given sequence of sample times and $\tau > 0$ is the given positive pointwise delay. Our conditions give upper bounds for the delay and for the lengths of the sampling intervals.

The result of the present paper cannot be proven by adapting the proofs of [4] or [12]. In fact, we show through examples that we establish our main result under conditions that do not imply the assumptions in [12], including cases where the undelayed unsampled system is not locally exponentially stabilizable. To overcome this obstacle, our proof relies on a functional of Lyapunov type, which is reminiscent of the one used in [3] to study time invariant linear systems and [10] to study neutral time delay systems. For simplicity, we only consider control affine systems, but extensions to systems that are not control affine can be obtained. We illustrate our work through several examples with input delays and sampled inputs, including a tracking problem for a model from [5].

II. Notation

Let $\mathcal{K}_\infty$ denote the set of all continuous functions $\rho : [0, +\infty) \to [0, +\infty)$ for which (i) $\rho(0) = 0$ and (ii) ρ is strictly increasing and unbounded. For any function $\phi : \mathcal{I} \to \mathbb{R}^p$ defined on any interval $\mathcal{I}$, let $|\phi|_{\mathcal{I}}$ denote its (essential) supremum over $\mathcal{I}$. Let $|\cdot|$ denote the Euclidean norm (or the induced matrix norm, depending on the context). For any continuous function $\varphi : \mathbb{R} \to \mathbb{R}^n$ and all $t \geq 0$, the function φ_t is defined by $\varphi_t(\theta) = \varphi(t+\theta)$ for all $\theta \in [-r, 0]$, where the constant $r > 0$ will depend on the context. We say that a function $\varphi(t, x)$ is uniformly bounded with respect to t provided there exists a function ρ of class $\mathcal{K}_\infty$ such that $|\varphi(t, x)| \leq \rho(1 + |x|)$ for all (t, x) in the domain of φ. Throughout this paper, we assume that all of the time varying functions are uniformly bounded with respect to t. We set $\mathcal{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$. The notation will be simplified, e.g., by omitting arguments of functions, whenever no confusion can arise from the context.

III. Assumptions and Main Result

Consider the nonlinear system (1) and let $\{t_i\}$ be a sequence in $[0, +\infty)$ such that $t_0 = 0$ and such that there are two constants $\nu > 0$ and $\delta > \nu$ such that

$$t_{i+1} - t_i \in [\nu, \delta] \quad \forall i \in \mathcal{Z}_{\geq 0}\,. \tag{2}$$

Our first assumption is:

Assumption 1: There exist a C^1 feedback $u_s(t, x)$, a C^1 positive definite and radially unbounded function V, and a

*This work was supported by... (à remplir)

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continuous positive definite function W such that

$$W_b(t,x) = -\left[\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)\big(f(t,x)+g(t,x)u_s(t,x)\big)\right] \quad (3)$$

satisfies

$$W_b(t,x) \geq W(x) \quad (4)$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$. Also, $u_s(t,0)=0$ for all $t \in \mathbb{R}$. Hence, V is a strict Lyapunov function for $\dot{x} = f(t,x) + g(t,x)u_s(t,x)$, and we can fix class $\mathcal{K}_\infty$ functions α_1 and α_2 such that $\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$. Define the function h by

$$\begin{array}{rcl} h(t,x) & = & \frac{\partial u_s}{\partial t}(t,x) + \frac{\partial u_s}{\partial x}(t,x)f(t,x) \\ & & +\frac{\partial u_s}{\partial x}(t,x)g(t,x)u_s(t,x). \end{array} \quad (5)$$

Our second assumption is:

Assumption 2: There are constants $c_i > 0$ for $i = 1,2,3,4$ such that

$$\left|\frac{\partial u_s}{\partial x}(t,x)g(t,x)\right|^2 \leq c_1, \quad (6)$$

$$\left|\frac{\partial V}{\partial x}(t,x)g(t,x)\right|^2 \leq c_2 W(x), \quad (7)$$

$$|h(t,x)|^2 \leq c_3 W(x), \quad (8)$$

and

$$\left|\frac{\partial V}{\partial x}(t,x)g(t,x)u_s(t,x)\right| \leq c_4[V(t,x)+1] \quad (9)$$

hold for all $t \geq 0$ and $x \in \mathbb{R}^n$.

We can now state and prove the main result:

Theorem 1: Let the system (1) satisfy Assumptions 1 and 2. If δ and τ_* are any two positive constants such that

$$\delta + \tau_* \leq \frac{1}{\sqrt{4c_1 + 8c_2c_3}} \quad (10)$$

and if $\tau \in (0, \tau_*]$, then the system (1) in closed loop with

$$u(t) = u_s(t_i - \tau, x(t_i - \tau)) \text{ when } t \in [t_i, t_{i+1}) \quad (11)$$

with the sequence $\{t_i\}$ from (2) is UGAS.

Remark 1: Assumption 1 implies that the origin of (1) in closed loop with $u_s(t,x)$ without delay and sampling is UGAS. Assumptions 1 and 2 do not imply that the functions f and g are globally Lipschitz with respect to x or that (1) is locally exponentially stabilizable, and it allows cases where W may not be radially unbounded; see the examples below.

Remark 2: The system (1) in closed loop with (11) admits solutions whose first derivatives are not continuous. However, their derivatives are piecewise continuous, and continuous over each interval of the form $[t_i, t_{i+1})$.

Remark 3: Theorem 1 can be extended to the case where the delay is time varying. Moreover, we could establish our result by representing the presence of delay and sampling as a time varying discontinuous feedback as was done in [2]. However, we did not make this choice because it does not help to simplify the forthcoming proof.

Remark 4: The requirement (9) is often satisfied in applications. We will see in the proof of Theorem 1 that it prevents the finite escape time phenomenon from occurring.

Remark 5: Theorem 1 applies to systems that are not necessarily globally Lipschitz or locally exponentially stabilizable by continuous feedback. For example, take $n=1$ and

$$\dot{x} = \frac{x^2}{1+x^2}u\ , \quad (12)$$

with $u_s(x) = -x$ and $V(x) = \frac{1}{2}x^2$. Indeed, using the notation from above with the time dependency omitted, we have

$$\begin{array}{l} f(x) = 0\ ,\ g(x) = \frac{x^2}{1+x^2}\ ,\ \frac{\partial u_s}{\partial x}(x)g(x) = -\frac{x^2}{1+x^2}\ , \\ h(x) = \frac{x^3}{1+x^2}\ ,\ L_gV(x) = \frac{x^3}{1+x^2}\ ,\text{ and } W(x) = \frac{x^4}{1+x^2}\ . \end{array}$$

Then Assumptions 1-2 hold with $c_1 = c_2 = c_3 = 1$, so Theorem 1 ensures that the corresponding input delayed sampled system is UGAS if $\delta + \tau < 1/(2\sqrt{3})$. The assumptions also hold with $n=2$ and $\dot{x}_1 = -x_1 - x_1^9 + x_2$, $\dot{x}_2 = u + x_1$ with $u_s(t,x) = -x_1 - x_2$ and $V(t,x) = 0.1x_1^{10} + 0.5x_1^2 + 2x_2^2$.

IV. Proof of Theorem 1

Throughout the proof, all time derivatives are over all trajectories of $\dot{x}(t) = f(t,x(t)) + g(t,x(t))u_s(t_i-\tau, x(t_i-\tau))$. We first prove that the solutions of the closed loop system are defined over $[-\tau, +\infty)$. Let $x(t)$ be any solution of the closed loop system and let t_c be a positive real number or $+\infty$ such that the maximal interval of definition of $x(t)$ includes $[-\tau, t_c)$. The definition of W_b in (3) gives

$$\begin{array}{rcl} \dot{V} & = & -W_b(t,x(t)) - \frac{\partial V}{\partial x}(t,x)g(t,x)u_s(t,x(t)) \\ & & +\frac{\partial V}{\partial x}(t,x)g(t,x)u_s(t_i-\tau, x(t_i-\tau)) \end{array} \quad (13)$$

for all $t \in [t_i, t_{i+1})$, $i \in \mathcal{Z}_{\geq 0}$, and $t \in [0, t_c)$. From (4), (7) and (9), we deduce that

$$\begin{array}{rcl} \dot{V} & \leq & -W(x(t)) + c_4[V(t,x(t))+1] \\ & & +\sqrt{c_2W(x(t))}|u_s(t_i-\tau, x(t_i-\tau))| \\ & \leq & c_4V(t,x(t)) + c_2|u_s(t_i-\tau, x(t_i-\tau))|^2 + c_4 \end{array} \quad (14)$$

for all $t \in [t_i, t_{i+1})$, $i \in \mathcal{Z}_{\geq 0}$, and $t \in [0, t_c)$, where the last inequality is a consequence of the triangle inequality. Therefore,

$$\begin{array}{rcl} V(t,x(t)) & \leq & e^{c_4(t-t_i)}V(t_i,x(t_i)) + \frac{e^{c_4(t-t_i)}-1}{c_4}\big[c_4 + \\ & & c_2|u_s(t_i-\tau, x(t_i-\tau))|^2\big] \\ & \leq & e^{c_4\delta}V(t_i,x(t_i)) + \frac{e^{c_4\delta}-1}{c_4}\big[c_4 + \\ & & c_2|u_s(t_i-\tau, x(t_i-\tau))|^2\big] \end{array} \quad (15)$$

for all $t \in [t_i, t_{i+1})$, $i \in \mathcal{Z}_{\geq 0}$, and $t \in [0, t_c)$. This readily implies that the solutions are defined over $[-\tau, +\infty)$.

We now prove our UGAS result. In what follows, all equalities and inequalities should be understood to hold for all $t \in [t_i, t_{i+1})$ and all $i \in \mathcal{Z}_{\geq 0}$. We also set $\Delta u_s(t) = u_s(t_i-\tau, x(t_i-\tau)) - u_s(t, x(t))$. We introduce an operator Γ such that

$$\Gamma(t, x_t) = \frac{\epsilon}{\delta+\tau_*}\int_{t-\delta-\tau_*}^{t}\int_{\ell}^{t}|\psi(m,x_m)|^2\,\mathrm{d}m\mathrm{d}\ell \quad (16)$$

along the trajectories of (1), where $\epsilon > 0$ is a constant to be selected later and

$$\psi(t,x_t) = \frac{\partial u_s}{\partial t}(t,x(t)) + \frac{\partial u_s}{\partial x}(t,x(t))\dot{x}(t). \quad (17)$$

Since $\dot{x}$ is a piecewise continuous function of t, the function

Γ is piecewise differentiable and satisfies

$$\dot{\Gamma}(t) = \epsilon \left|\psi(t,x_t)\right|^2 - \frac{\epsilon}{\delta+\tau_*} \int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m. \tag{18}$$

Next, we define

$$U(t,x_t) = V(t,x(t)) + \Gamma(t,x_t) \tag{19}$$

along all trajectories of (1).

From (13), (18), the expression for $\dot{x}$ and the definition of h in (5), it follows that

$$\begin{aligned} \dot{U} \;=\; & -W_b(t,x(t)) - \tfrac{\epsilon}{\delta+\tau_*} \int_{t-\delta-\tau_*}^{t} |\psi(m,x_m)|^2 \mathrm{d}m \\ & + \epsilon \left| h(t,x(t)) + \tfrac{\partial u_s}{\partial x}(t,x(t)) g(t,x(t)) \Delta u_s(t) \right|^2 \\ & + \tfrac{\partial V}{\partial x}(t,x) g(t,x) \Delta u_s(t) \end{aligned} \tag{20}$$

along all trajectories of the delayed sampled dynamics. Using the inequality $(a+b)^2 \le 2a^2 + 2b^2$ which is valid for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$, (4), and Assumption 2, we get

$$\begin{aligned} \dot{U} \;\le\; & -W(x(t)) - \tfrac{\epsilon}{\delta+\tau_*} \int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m \\ & + 2\epsilon |h(t,x(t))|^2 + \left|\tfrac{\partial V}{\partial x}(t,x) g(t,x)\right| |\Delta u_s(t)| \\ & + 2\epsilon \left|\tfrac{\partial u_s}{\partial x}(t,x(t)) g(t,x(t))\right|^2 |\Delta u_s(t)|^2 \\ \le\; & -W(x(t)) - \tfrac{\epsilon}{\delta+\tau_*} \int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m \\ & + 2\epsilon c_3 W(x(t)) + 2\epsilon c_1 |\Delta u_s(t)|^2 \\ & + \sqrt{c_2 W(x(t))} |\Delta u_s(t)|. \end{aligned} \tag{21}$$

From the triangle inequality, we deduce that

$$\sqrt{c_2 W(x(t))} |\Delta u_s(t)| \;\le\; \tfrac{1}{4} W(x(t)) + c_2 |\Delta u_s(t)|^2. \tag{22}$$

Combining (21) and (22), we deduce that

$$\begin{aligned} \dot{U} \;\le\; & \left(-\tfrac{3}{4} + 2\epsilon c_3\right) W(x(t)) + (2\epsilon c_1 + c_2) |\Delta u_s(t)|^2 \\ & - \tfrac{\epsilon}{\delta+\tau_*} \int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m \\ =\; & \left(-\tfrac{3}{4} + 2\epsilon c_3\right) W(x(t)) \\ & - \tfrac{\epsilon}{\delta+\tau_*} \int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m \\ & + (2\epsilon c_1 + c_2) \left| \int_{t_i-\tau}^{t} \psi(m,x_m) \mathrm{d}m \right|^2. \end{aligned} \tag{23}$$

From Jensen's inequality, it follows that

$$\begin{aligned} \dot{U} \;\le\; & \left(-\tfrac{3}{4} + 2\epsilon c_3\right) W(x(t)) \\ & - \tfrac{\epsilon}{\delta+\tau_*} \int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m \\ & + (2\epsilon c_1 + c_2)(\delta+\tau_*) \int_{t_i-\tau}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m. \end{aligned} \tag{24}$$

By grouping terms and using the fact that $t_i - \tau \ge t - \tau_* - \delta$ when $t \in [t_i, t_{i+1})$ to upper bound the second integral in (24) by the first integral, and then taking $\epsilon = \frac{1}{4c_3}$, we get

$$\begin{aligned} \dot{U} \le & -\tfrac{1}{4} W(x(t)) + \tfrac{1}{\delta+\tau_*} \left[-\tfrac{1}{4c_3} + \left(\tfrac{c_1}{2c_3} + c_2\right)(\delta+\tau_*)^2 \right] \\ & \times \int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m. \end{aligned}$$

From the bound (10) on $\delta + \tau_*$, we deduce that

$$\begin{aligned} \dot{U} \;\le\; & -\tfrac{1}{4} W(x(t)) \\ & - \tfrac{1}{8c_3(\delta+\tau_*)} \int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m\,. \end{aligned} \tag{25}$$

Let κ be a C^1 function of class $\mathcal{K}_\infty$ such that κ' is nondecreasing and $\kappa'(0) = 8c_3(\delta+\tau_*)$. Then $U_\kappa = \kappa(U)$ satisfies

$$\begin{aligned} \dot{U}_\kappa \;\le\; & -\tfrac{1}{4} \kappa'(U(t,x_t)) W(x(t)) \\ & - \tfrac{1}{8c_3(\delta+\tau_*)} \kappa'(U(t,x_t)) \int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m. \end{aligned}$$

One can choose κ such that there exists a function $\rho \in \mathcal{K}_\infty$ satisfying $\rho(V(t,x(t))) \le \frac{1}{4}\kappa'(V(t,x(t))) W(x(t))$ for all t; see [9, Lemma A.7, p.354]. We may assume that $\rho(s) \le s$ for all $s \ge 0$. (Otherwise, replace it by $\min\{s, \rho(s)\}$ which is also of class $\mathcal{K}_\infty$.) Hence,

$$\begin{aligned} \dot{U}_\kappa & \le -\rho(V(t,x(t))) - \int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m \\ & \le -\rho(V(t,x(t))) - \rho\left(\int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m \right) \\ & \le -\rho\left(\tfrac{1}{2} \left[V(t,x(t)) + \int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m \right] \right) \\ & \le -\rho\left(\tfrac{1}{2(1+\varepsilon)} \left[V(t,x(t)) + \epsilon \int_{t-\delta-\tau_*}^{t} \left|\psi(m,x_m)\right|^2 \mathrm{d}m \right] \right) \\ & \le -\rho\left(\tfrac{V(t,x(t)) + \Gamma(t,x_t)}{2(1+\epsilon)} \right) \;=\; -\rho\left(\tfrac{\kappa^{-1}(U_\kappa(t,x_t))}{2(1+\epsilon)} \right)\,. \end{aligned}$$

Since $s \to \rho(\kappa^{-1}(s)/[2(1+\epsilon)])$ is of class $\mathcal{K}_\infty$, and since $U(t,x_t) \ge V(t,x(t)) \ge \alpha_1(|x(t)|)$ for all t, and U admits a function $\bar{U} \in \mathcal{K}_\infty$ such that $U(t,x_0) \le \bar{U}(|x_0|_{[-\tau_*-\delta,0]})$ for all x_0 and t (which exists by (8) and our assumption that $u_s(t,0) = 0$ for all $t \in \mathbb{R}$), this gives the UGAS estimate.

V. Examples

A. Saturating Controller

Our work [12] used Lyapunov-Krasovskii functionals to prove robustness of closed loop control affine systems with respect to small enough input delays. We next give an example that satisfies our Assumptions 1-2 and so is covered by Theorem 1, but does not satisfy the assumptions imposed to establish the main result in [12]. It will be key to the higher dimensional tracking dynamics in the next subsection. Take $\dot{x} = u$, where the state x and input u are one dimensional. This is rendered UGAS and locally exponentially stable by

$$u_s(x) = -\frac{\xi x}{\sqrt{1+x^2}}, \tag{26}$$

where ξ is any positive constant. Then, with the notation of Section III, we have $f(x) = 0$ and $g(x) = 1$. We choose the positive definite radially unbounded function $V(x) = \sqrt{1+x^2} - 1$. Then Assumption 1 is satisfied with $W_b(x) = W(x) = \xi x^2/(1+x^2)$. Omitting the time dependency, we have

$$\begin{aligned} & \left|\tfrac{\partial u_s}{\partial x}(x) g(x)\right|^2 \le \xi^2, \;\; |L_gV(x)|^2 \;=\; \tfrac{1}{\xi} W(x), \\ & |h(x)|^2 = \tfrac{\xi^4 x^2}{(1+x^2)^4} \le \xi^3 W(x), \text{ and} \\ & |L_gV(x) u_s(x)| \le \tfrac{\xi x^2}{1+x^2} \le \xi[V(x)+1] \end{aligned} \tag{27}$$

so Assumption 2 holds. Hence, Theorem 1 applies to the system $\dot{x} = u$ with the feedback (26). The upper bound for $\delta + \tau_*$ is then $1/(2\sqrt{3}\xi)$, which can be made arbitrarily large if ξ is sufficiently small. However, this example violates [12, Assumption H]. This follows from:

Lemma 1: If a system $\dot{x} = f(x) + u$ with f bounded is rendered GAS on $\mathbb{R}^n$ by a bounded feedback $u_s(x)$, then for each Lyapunov function $V(t,x)$ of the closed loop system, the requirements [12, Assumption H] on the delayed system $\dot{x}(t) = f(x(t)) + u_s(x(t-\tau))$ fail to hold.

Proof. Suppose the contrary. Then Assumption H provides a function $\sigma \in \mathcal{K}_\infty$ such that $V_t(t,x) + V_x(t,x)[f(x) + u_s(x)] \le -\sigma^2(\sqrt{n}|x|)$ along all trajectories of the undelayed system. We claim that for each $x \in \mathbb{R}^n$, we can find a value $t_x \ge 0$ such that $|V_t(t_x,x)| \le 0.5\sigma^2(\sqrt{n}|x|)$. To prove this claim, we can assume that there is no $t_x \ge 0$ such that $V_t(t_x,x) = 0$ and therefore that $V_t(t,x) < 0$ for all $t \ge 0$ for our given x, or that $V_t(t,x) > 0$ for all $t \ge 0$ for our chosen x. In the former case, we have $0 < -\int_0^t V_t(s,x)\mathrm{d}s = V(0,x) - V(t,x) \le V(0,x)$ for all $t > 0$, so letting $t \to +\infty$ gives $V_t(s,x) \to 0$ as $s \to +\infty$ by the divergence test. The case where $V_t(t,x) > 0$ for all t is handled similarly, since there is a function $\bar{\alpha} \in \mathcal{K}_\infty$ such that $V(t,x) \le \bar{\alpha}(|x|)$ for all $t \ge 0$ and $x \in \mathbb{R}^n$. Therefore, $V_x(t_x,x)[f(x) + u_s(x)] \le -0.5\sigma^2(\sqrt{n}|x|)$ for all $x \in \mathbb{R}^n$. Moreover, Assumption H gives a constant $K_1 > 0$ such that $|V_x(t_x,x)| \le K_1\sigma(|x|)$ for all $x \in \mathbb{R}^n$. Since f and u_s are bounded, this provides a constant $\bar{K} > 0$ such that $\sigma^2(\sqrt{n}|x|) \le \bar{K}\sigma(|x|)$ and so also $\sigma(\sqrt{n}|x|) \le \bar{K}\sigma(|x|)/\sigma(\sqrt{n}|x|) \le \bar{K}$ for all nonzero $x \in \mathbb{R}^n$, contradicting the unboundedness of σ. □

B. Tracking Example

We consider the system

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& \omega x_2 \\ \dot{x}_2 &=& -\omega x_1 + \lambda \\ \dot{x}_3 &=& \omega, \end{array} \right. \tag{28}$$

where λ and ω are the controls, which is obtained from the kinematics of a wheeled mobile robot after a change of coordinates; see [5]. Let $\zeta > 0$ be any constant. We aim to track the periodic trajectory $(0,0,-\cos(\zeta t))^\top$, using sampled delayed feedback. Hence, the change of coordinates $z = x_3 + \cos(\zeta t)$ gives the tracking dynamics

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& \omega x_2 \\ \dot{x}_2 &=& -\omega x_1 + \lambda \\ \dot{z} &=& -\zeta\sin(\zeta t) + \omega. \end{array} \right. \tag{29}$$

Case 1: The change of feedback $\omega = \zeta\sin(\zeta t) + \mu$ leads to

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& (\zeta\sin(\zeta t) + \mu)x_2 \\ \dot{x}_2 &=& -(\zeta\sin(\zeta t) + \mu)x_1 + \lambda \\ \dot{z} &=& \mu. \end{array} \right. \tag{30}$$

(See Case 2 below for the case where the full feedback ω has sampling, i.e., $\omega(t_i - \tau) = \zeta\sin(\zeta(t_i - \tau)) + \mu(t_i - \tau)$.) The z subsystem of (30) can be stabilized easily using

$$\mu(z(t_i - \tau)) = -\zeta\frac{\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}, \tag{31}$$

where $\mho$ is a positive constant. In fact, our analysis from Section V-A shows that

$$\dot{z}(t) = -\frac{\zeta\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}} \tag{32}$$

is UGAS if $\delta + \tau_* < 1/(2\sqrt{3}\zeta\mho)$. Assume that $\mho \le \frac{1}{30}$.

Next, we choose the control $\lambda(x(t)) = -\zeta x_2(t)$ to get

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& \zeta\left(\sin(\zeta t) - \frac{\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right)x_2 \\ \dot{x}_2 &=& -\zeta\left(\sin(\zeta t) - \frac{\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right)x_1 - \zeta x_2 \\ \dot{z} &=& -\frac{\zeta\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}. \end{array} \right. \tag{33}$$

By Lemma A.1 in the appendix, we deduce that the time derivative of the time varying positive definite and proper quadratic function $Q_\zeta(t,x)$ given in (A.1) along all trajectories of (33) satisfies

$$\dot{Q}_\zeta \le -\tfrac{\zeta}{4}[x_1^2 + x_2^2]. \tag{34}$$

We replace the control λ by the delayed sampled controller $\lambda(x(t_i - \tau)) = -\zeta x_2(t_i - \tau)$. This gives

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& \zeta\left(\sin(\zeta t) - \frac{\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right)x_2 \\ \dot{x}_2 &=& -\zeta\left(\sin(\zeta t) - \frac{\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right)x_1 \\ && -\zeta x_2(t_i - \tau) \\ \dot{z} &=& -\frac{\zeta\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}. \end{array} \right. \tag{35}$$

We study (35) using the following strategy. We fix a particular solution of the z subsystem and focus on the system

$$\left\{ \begin{array}{rcl} \dot{x}_1 &=& \zeta(\sin(\zeta t) + \gamma(t))x_2 \\ \dot{x}_2 &=& -\zeta(\sin(\zeta t) + \gamma(t))x_1 + \lambda, \end{array} \right. \tag{36}$$

where

$$\gamma(t) = -\frac{\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}. \tag{37}$$

Then, we apply Theorem 1 to (36), with λ playing the role of u in (1) and $\lambda(x_2) = -\zeta x_2$ playing the role of u_s.

With the notation of (1) we choose

$$f(t,x) = \begin{pmatrix} \zeta(\sin(\zeta t) + \gamma(t))x_2 \\ -\zeta(\sin(\zeta t) + \gamma(t))x_1 \end{pmatrix},\ g(t,x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$V(t,x) = Q_\zeta(t,x)$ and $u_s(t,x) = -\zeta x_2$, so $h(t,x) = \zeta^2[(\sin(\zeta t) + \gamma(t))x_1 + x_2]$. We check that Theorem 1 applies to (36). Since (34) holds along all trajectories of (36), Assumption 1 holds with $W(x) = \zeta|x|^2/4$. Condition (6) holds with $c_1 = \zeta^2$. We have $V_x(t,x)g(t,x) = (9/2)x_2 + 2\sin(\zeta t)x_1$, so (7) is satisfied, with $c_2 = 162/\zeta$. Also, (8) holds with $c_3 = 32\zeta^3$, by the formula for h. Finally, it is clear that (9) is satisfied. Therefore Theorem 1 provides an upper bound on $\delta + \tau_*$ that is independent of the choice of the solution z. Combining this with our analysis of the z subsystem (32) gives the upper bound

$$\delta + \tau_* < \frac{1}{2\zeta}\min\left\{\frac{1}{\sqrt{3}\mho}, \frac{1}{\sqrt{10369}}\right\}, \tag{38}$$

which ensures that (35) is UGAS.

Case 2: Next consider the case where there is also sampling in the sin part of the change of feedback, i.e., $\omega = \zeta\sin(\zeta(t_i - \tau)) + \mu$, which we substitute into (28) to get

$$\left\{\begin{array}{lcl}\dot{x}_1 & = & [\zeta\sin(\zeta(t_i-\tau))+\mu]x_2\\ \dot{x}_2 & = & -[\zeta\sin(\zeta(t_i-\tau))+\mu]x_1+\lambda\\ \dot{x}_3 & = & \zeta\sin(\zeta(t_i-\tau))+\mu.\end{array}\right. \tag{39}$$

We assume that $t_i=\delta i$ where $\delta=\pi/(\zeta L)$ for some positive integer L, and we set $\varphi(t)=t_i-\tau$ for all $t\in[t_i,t_{i+1})$ and $i\in\mathcal{Z}_{\geq 0}$. Let

$$z=x_3-\int_0^t\zeta\sin(\zeta\varphi(m))\mathrm{d}m \tag{40}$$

This gives the dynamics

$$\left\{\begin{array}{lcl}\dot{x}_1 & = & [\zeta\sin(\zeta(t_i-\tau))+\mu]x_2\\ \dot{x}_2 & = & -[\zeta\sin(\zeta(t_i-\tau))+\mu]x_1+\lambda\\ \dot{z} & = & \mu\ .\end{array}\right. \tag{41}$$

Also, Lemma A.2 in the appendix ensures that $z(t)-2\pi\leq x_3(t)\leq z(t)+2\pi$ for all t, so if $z(t)$ is bounded, then $x_3(t)$ is bounded as well. Next, we choose

$$\mu(z(t_i-\tau))=-\tfrac{\zeta\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}, \tag{42}$$

where $\mho$ is such that $\mho\leq\frac{1}{60}$. Then we have

$$\left\{\begin{array}{lcl}\dot{x}_1 & = & \zeta\left[\sin(\zeta(t_i-\tau))-\tfrac{\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right]x_2\\ \dot{x}_2 & = & -\zeta\left[\sin(\zeta(t_i-\tau))-\tfrac{\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right]x_1+\lambda\\ \dot{z} & = & -\tfrac{\zeta\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}.\end{array}\right. \tag{43}$$

We rewrite the (x_1,x_2) subsystem as

$$\left\{\begin{array}{lcl}\dot{x}_1 & = & \zeta\left[\sin(\zeta t)+\omega(t)\right]x_2\\ \dot{x}_2 & = & -\zeta\left[\sin(\zeta t)+\omega(t)\right]x_1+\lambda\end{array}\right. \tag{44}$$

with

$$\omega(t)=\sin(\zeta(t_i-\tau))-\sin(\zeta t)-\tfrac{\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}. \tag{45}$$

We have $|\omega(t)|\leq\zeta(\delta+\tau_*)+\mho$. Therefore, if $\delta+\tau_*\leq 1/(60\zeta)$, then $|\omega(t)|\leq 1/30$. Hence, our analysis of (36) applies to (44) with the sampled feedback $\lambda=-\zeta x_2(t_i-\tau)$ to give a bound on the admissible values of $\tau_*+\delta$. It follows that

$$\left\{\begin{array}{lcl}\dot{x}_1 & = & \zeta\left[\sin(\zeta(t_i-\tau))-\tfrac{\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right]x_2\\ \dot{x}_2 & = & -\zeta\left[\sin(\zeta(t_i-\tau))-\tfrac{\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right]x_1\\ & & -\zeta x_2(t_i-\tau)\\ \dot{z} & = & -\tfrac{\zeta\mho z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\end{array}\right. \tag{46}$$

is also UGAS when (38) is satisfied.

VI. Conclusions

Taking delays and sampling in the inputs into account is a challenging, central problem that has been studied by several authors using a variety of methods. We considered nonlinear control affine systems with feedbacks corrupted by both delay and sampling. We gave conditions on the size of the delay and the maximal sampling interval that ensure uniform global asymptotic stability. We used a new Lyapunov approach, and we covered systems that were beyond the scope of standard results. We applied our result to a tracking problem, where the bound on the sampling interval and delay can be arbitrarily large. Extensions to nonaffine systems are possible. We conjecture that our main result can also be adapted to systems that can be locally but not globally asymptotically stabilized.

Appendix: Two Technical Lemmas

We used the following in our analysis of (30):

Lemma A.1: Let $\zeta>0$ be any constant. Then: (a) The time derivative of

$$Q_\zeta(t,x)=\frac{9}{4}|x|^2+2\sin(\zeta t)x_1x_2-\sin(\zeta t)\cos(\zeta t)x_1^2 \tag{A.1}$$

along all trajectories of the two dimensional system

$$\left\{\begin{array}{lcl}\dot{x}_1(t) & = & \zeta\sin(\zeta t)x_2(t)\\ \dot{x}_2(t) & = & -\zeta\sin(\zeta t)x_1(t)-\zeta x_2(t)\end{array}\right. \tag{A.2}$$

satisfies $\dot{Q}_\zeta(t,x)\leq-\zeta|x|^2/2$. (b) For any piecewise continuous function $\aleph$ satisfying $|\aleph(t)|\leq 1/30$ for all $t\geq 0$, the time derivative of (A.1) along all trajectories of

$$\left\{\begin{array}{lcl}\dot{x}_1(t) & = & \zeta\sin(\zeta t)x_2(t)+\zeta\aleph(t)x_2(t)\\ \dot{x}_2(t) & = & -\zeta\sin(\zeta t)x_1(t)-\zeta x_2(t)-\zeta\aleph(t)x_1(t)\end{array}\right. \tag{A.3}$$

satisfies $\dot{Q}_\zeta(t,x)\leq-\frac{\zeta}{4}|x|^2$. Also, $5|x|^2\geq Q_\zeta(t,x)\geq\frac{1}{4}|x|^2$ for all $t\in\mathbb{R}$ and $x\in\mathbb{R}^2$.

Proof. We only prove the case where $\zeta=1$. The general case will then follow from a scaling argument. Part (a) follows because along all trajectories of (A.2), we have

$$\begin{array}{lcl}\dot{Q}_1 & = & \left(-9/2+2\sin^2(t)\right)x_2^2-x_1^2\\ & & +2\left(\cos(t)-\sin(t)-\sin^2(t)\cos(t)\right)x_1x_2\\ & = & \left(-9/2+2\sin^2(t)\right)x_2^2\\ & & +2\left(\cos^3(t)-\sin(t)\right)x_1x_2-x_1^2\\ & \leq & \Big[-9/2+2\sin^2(t)\\ & & +2\left(\cos^3(t)-\sin(t)\right)^2\Big]x_2^2-\frac{1}{2}x_1^2\\ & \leq & -\frac{1}{2}[x_1^2+x_2^2],\end{array} \tag{A.4}$$

where the last inequality used

$$\max_t\left\{\sin^2(t)+(\cos^3(t)-\sin(t))^2\right\}=2. \tag{A.5}$$

Therefore, along all trajectories of (A.3),

$$\begin{array}{l}\dot{Q}_1\leq-\frac{1}{2}[x_1^2+x_2^2]\\ +\left(9x_1/2-2\sin(t)\cos(t)x_1+2\sin(t)x_2\right)\aleph(t)x_2\\ -\left(9x_2/2+2\sin(t)x_1\right)\aleph(t)x_1.\end{array} \tag{A.6}$$

Since $|\aleph(t)|\leq\frac{1}{30}$ for all $t\geq 0$, we deduce that

$$\begin{array}{lcl}\dot{Q}_1 & \leq & -\frac{1}{2}[x_1^2+x_2^2]+11\frac{1}{30}|x_1x_2|+2\frac{1}{30}x_2^2+2\frac{1}{30}x_1^2\\ & \leq & -\frac{1}{4}[x_1^2+x_2^2],\end{array} \tag{A.7}$$

which proves the decay estimate in part (b). □

We used the following in Case 2 in our tracking example:

Lemma A.2: Let $t_i=i\delta$, where $\delta=\pi/(\zeta L)$, L is any positive integer, and $\zeta>0$ is any constant. Let $\varphi(t)=t_i-\tau$

for all $t \in [t_i, t_{i+1})$ and all $i \in \mathcal{Z}_{\geq 0}$. Then

$$\sin(\zeta\varphi(t+\pi/\zeta)) = -\sin(\zeta\varphi(t)) \tag{A.8}$$

for all $t \in \mathbb{R}$,

$$\int_0^{2\pi/\zeta} \sin(\zeta\varphi(m))\mathrm{d}m = 0 \, , \tag{A.9}$$

and $\sin(\zeta\varphi(t))$ is periodic of period $2\pi/\zeta$.

Proof. Let t and $i \in \mathcal{Z}_{\geq 0}$ be such that $t \in [t_i, t_{i+1})$. Then $t+\pi/\zeta \in [t_i+\pi/\zeta, t_{i+1}+\pi/\zeta)$, so our formulas for δ and t_i give $t+\pi/\zeta \in [t_{i+L}, t_{i+L+1})$. Hence, $\varphi(t+\pi/\zeta) = t_{i+L} - \tau = t_i - \tau + \pi/\zeta = \varphi(t) + \pi/\zeta$. Hence (A.8) holds, which implies that $\sin(\zeta\varphi(t))$ is periodic of period $\frac{2\pi}{\zeta}$. Next notice that

$$\int_0^{\frac{2\pi}{\zeta}} \sin(\zeta\varphi(m))\mathrm{d}m = \sum_{i=0}^{2L-1} \int_{t_i}^{t_{i+1}} \sin(\zeta\varphi(m))\mathrm{d}m. \tag{A.10}$$

Since φ is constant on $[t_i, t_{i+1})$, we get

$$\int_0^{\frac{2\pi}{\zeta}} \sin(\varphi(m))\mathrm{d}m = \delta \sum_{i=0}^{2L-1} \sin\left(\zeta\varphi\left(\frac{i\pi}{\zeta L}\right)\right). \tag{A.11}$$

Then we deduce successively that

$$\begin{aligned}
&\textstyle\int_0^{\frac{2\pi}{\zeta}} \sin(\varphi(m))\mathrm{d}m \\
&= \delta \sum_{i=0}^{L-1} \sin\left(\zeta\varphi\left(\frac{i\pi}{\zeta L}\right)\right) + \delta \sum_{i=L}^{2L-1} \sin\left(\zeta\varphi\left(\frac{i\pi}{\zeta L}\right)\right) \\
&= \delta \sum_{i=0}^{L-1} \sin\left(\zeta\varphi\left(\frac{i\pi}{\zeta L}\right)\right) \\
&\quad + \delta \sum_{j=0}^{L-1} \sin\left(\zeta\varphi\left(\frac{j\pi}{\zeta L} + \frac{\pi}{\zeta}\right)\right) \\
&= \delta \sum_{i=0}^{L-1} \sin\left(\zeta\varphi\left(\frac{i\pi}{\zeta L}\right)\right) - \delta \sum_{j=0}^{L-1} \sin\left(\zeta\varphi\left(\frac{j\pi}{\zeta L}\right)\right),
\end{aligned}$$

where the last equality used (A.8). Hence, all terms cancel and the lemma follows. □

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