Game Theory for Wireless Communications

Mérouane Debbah (*) and Samson Lasaulce (**) 
(*) Alcatel-Lucent Chair on Flexible Radio 
(**) LSS (CNRS–Supelec–Paris 11) 
merouane.debbah@supelec.fr, lasaulce@lss.supelec.fr

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PART I
1. Introduction
4G Mobile Internet

Uplink: Up to 7 Mbps/Terminal
Downlink: Up to 15 Mbps/Terminal
The spectral efficiency barrier: energy and bandwidth

\[ C = W \log(1 + \text{SNR}) \]

\[ \text{SNR} = C \frac{E_b}{N_0} \]

\[ \lim_{C \to 0} \frac{E_b}{N_0} = \frac{2^C - 1}{C} = \ln(2) \]

The two degrees of freedom (Energy and Bandwidth) are the barrier limit of communication.

There is minimum energy to transmit a natural unit of information over a noisy channel.
Is there a limit to spectral efficiency?

"As the number of users in the network increases, interference becomes the bottleneck"...the resources must be shared...
Mobile Flexible Networks do not consider wireless resources as a "cake to be shared" among users but take benefit of the high number of interacting devices to increase the spectral efficiency frontier.

More devices represent more opportunities to schedule information which enhances the global throughput.

Interference is considered as an opportunity rather than a drawback by exploiting intelligently the degrees of freedom of wireless communications:

- Space (MIMO Network)
- Frequency/time (Cognitive Network)
- User (Opportunistic Network).
"We build to many walls and not enough bridges", Isaac Newton

Combat interference through frequency/space reuse or power control.

Exploit interference through coordination and cooperation.
MIMO Network

In theory, the only limiting factor for increasing the spectral efficiency is the number of base stations.

The technology, adequate for dense networks, relies on sophisticated tools of dirty paper coding and cooperative Beam-forming.

Network MIMO can help improve operating SINR thus closing the gap between interference limited and noise limited regime.

Network MIMO can greatly improve the efficiency of multiple antennas in cellular networks.
Cognitive Networks

Cognitive networks coordinate the transmission over various bands of frequencies/technologies by exploiting the vacant bands/technologies in idle periods.

It requires **base stations** able to work in a large range of bandwidth for sensing the different signals in the network and reconfigure smartly.

**Example:** They will sense the different technologies, the energy consumption of the terminals and reconfigure (changing from a **GSM** to **UMTS** base station if **UMTS terminals** are present) to adapt to the standards or services to be delivered at a given time.
Opportunistic Networks

Fluctuations in Rate over Time, ND

Rate, mean 1

Time
As the number of users increase, the spectral efficiency of opportunistic network increases, as the probability of having a user with a good channel increases with the number of users.
Flexible Networks: a Shannon historic perspective

THREE LANDMARK PAPERS FROM Bell-Labs


XXII. Programming a Computer for Playing Chess
By CLAUDE E. SHANNON

Bell Telephone Laboratories, Inc., Murray Hill, N.J.
[Received November 8, 1949]

1. INTRODUCTION

This paper is concerned with the problem of constructing a computing routine or "program" for a modern general purpose computer which will enable it to play chess. Although perhaps of no practical importance, the question is of theoretical interest, and it is hoped that a satisfactory solution of this problem will act as a wedge in attacking other problems of a similar nature and of greater significance. Some possibilities in this direction are:
(1) Machines for designing filters, equalizers, etc.
(2) Machines for designing relay and switching circuits.
(3) Machines which will handle routing of telephone calls based on the individual circumstances rather than by fixed patterns.
(4) Machines for performing symbolic (non-numerical) mathematical operations.
(5) Machines capable of translating from one language to another.
(6) Machines for making strategic decisions in simplified military operations.
(7) Machines capable of orchestrating a melody.
(8) Machines capable of logical deduction.
Computers and Automata*

CLAUDE E. SHANNON†, FELLOW, IRE

C. E. Shannon first became known for a paper in which he applied Boolean Algebra to relay switching circuits; this laid the foundation for the present extensive application of Boolean Algebra to computer design. Dr. Shannon, who is engaged in mathematical research at Bell Telephone Laboratories, is an authority on information theory. More recently he received wide notice for his ingenious maze-solving mechanical mouse, and he is well-known as one of the leading explorers into the exciting, but uncharted world of new ideas in the computer field.

The Editors asked Dr. Shannon to write a paper describing current experiments, and speculations concerning future developments in computer logic. Here is a real challenge for those in search of a field where creative ability, imagination, and curiosity will undoubtedly lead to major advances in human knowledge.—The Editor

Summary—This paper reviews briefly some of the recent developments in the field of automata and nonnumerical computation. A number of typical machines are described, including logic machines, game-playing machines and learning machines. Some theoretical questions and developments are discussed, such as a comparison of computers and the brain, Turing's formulation of computing machines and von Neumann's models of self-reproducing machines.

* Decimal classification: 621.385.2. Original manuscript received by the Institute, July 17, 1953.
† Bell Telephone Laboratories, Murray Hill, N. J.

INTRODUCTION

SAMUEL BUTLER, in 1871, completed the manuscript of a most engaging social satire, Erewhon. Three chapters of Erewhon, originally appearing under the title "Darwin Among the Machines," are a witty parody of The Origin of Species. In the topsy-turvy logic of satirical writing, Butler sees machines as gradually evolving into higher forms. He considers the classification of machines into genera, species and vari-
Machines of this general type are an extension over the ordinary use of numerical computers in several ways. First, the entities dealt with are not primarily numbers, but rather chess positions, circuits, mathematical expressions, words, etc. Second, the proper procedure involves general principles, something of the nature of judgement, and considerable trial and error, rather than a strict, unalterable computing process. Finally, the solutions of these problems are not merely right or wrong but have a continuous range of "quality" from the best down to the worst. We might be satisfied with a machine that designed good filters even though they were not always the best possible.
REFERENCES.


Flexible Networks: a Shannon historic perspective

Claude Shannon and his mouse Theseus, in one of the first experience on artificial intelligence in 1950. Shannon’s mouse appears to have been the first learning device of this level. Like the learning mouse, flexible radio intends to build learning networks, just on a much larger, and more complex scale.
The big dilemma

One of the most challenging problems in the development of this technology is to manage complexity. The key is to develop:

- The right abstractions to reason about the spatial and temporal dynamics of complex systems
- Understand how information can be processed, stored, and transferred in the system with bounded delay.

We refer to this as the "theoretical foundations of Mobile Flexible Networks".
The research is highly interdisciplinary and is a blend of

- **Statistical inference methods** to build devices which would carry plausible reasoning (Maximum Entropy methods,..).
- **Random matrix theory techniques** to reduce the dimensionality of the problem i.e find the parameters of interest in a network rather than optimizing through simulations with 1 billion parameters.
- **Game theoretic techniques** (based on rational players) to promote decentralized/adaptive resource allocation schemes.
The research is highly interdisciplinary and is a blend of

- **Control theory** with the use of feedback mechanisms in the realm of cybernetics.
- **Physics** to study how information can be processed, stored, and transferred in the network.
- **Network Information theory** to understand the fundamental limits achievable with intelligent devices.
New Mathematical Tools

- Statistical Physics
- Convex Optimization
- Random Matrix theory
From a game perspective, where and how to split the intelligence...which is the purpose of the course.
2. Historical Development of Game Theory
Game Theory
Game Theory

Theme of the Academy Award-winning film "A Beautiful Mind" inspired by the Nobel Prize (Economics) winning mathematician John Nash.
What is Game Theory?

Basically, Game Theory is the mathematics of strategy.

Game Theory aims to help us to understand situations in which decision makers interact.

The primary theory is the Minimax Theorem which basically says that if all the players of a game play the best, most rational strategy, the resulting outcome of the game is predictable.
Is it useful?

Players are considered rational and should determine what is the best

Game Theory aims to help us to understand situation in which decision makers interact.

The primary theory is the Minimax Theorem which basically says that if all the players of a game play the best, most rational strategy, the resulting outcome of the game is predictable.
Applications of Game Theory

- Game theory and Economics (Auman, Schelling (Nobel Prize in 2005)).
- Game Theory and War (Cold war, war on terrorism).
- Game theory and Philosophy (morality from self-interest).
- Game theory and Social Science (Explanation for the democratic peace).
- Game theory and computer science (modelling interactive computations).
Previous applications of game theoretical tools to all areas of networking.

- Security issues modeled as zero-sum games (Kodialam et al., 2003).
- Non-zero sum games in the context of access to a shared wireless channel (Altman et al., 2004, MacKenzie et al., 2003).
- Evolutionary games (Bonneau et al., 2005, Menasche et al., 2005).
- Nash Bargaining (Fang et al., 2005, Han et al., 2005).
- Potential games (Sandholm et al., 2001, Shakkottai et al., 2006).
- Match making games applied to output queueing in switches (Chuang et al., 1999).
- S-modular games in the context of power control in wireless networks (Altman et al., 2003).
- Non-cooperative repeated games and dynamic games (Cao et al., 2002, Felegyhazi et al., 2005).

First formulation of the communication process as a game:

N. M. Blachman, "Communication as a Game", in Proc. IRE WESCON conf., pages 61-66, August 1961.
Landmark papers

Antoine Augustin Cournot, ”Researches into the Mathematical Principles of the Theory of Wealth”, 1838

J. Von Neumann, ”Zur Theorie der Gesellschaftsspiele”, Mathematische Annalen 100, 295-320, 1928


J. von Neumann made important contributions in quantum mechanics, functional analysis, informatics, economics as well as many other fields of mathematics and physics.
The Birth of Game Theory

Oskar Morgenstern, 1902-1977

Oskar Morgenstern worked on the application of mathematics to economy and laid with J. Von Neumann the foundations of game theory.
Formal aspects of Game Theory

Number of players

- One player: Optimization, control theory.
- Two players: Classical case and the best understood.
- Many players: Rich and complicated due to coalitions.

Temporal structure

- Simultaneous: prisoner’s dilemma
- Sequential: Chess.
Random factors

- Deterministic: chess
- Probabilistic: Stock market

Access to information

- Complete: chess
- Incomplete: poker
3. Static Games
Static games in Strategic Games

Strategic Games: A strategic game consists of:

- A set of players.
- for each player, a set of actions.
- for each player, preferences over the set of action profiles.

Example 1: the players may be firms, the actions prices and the preferences the firm’s profit.

Example 2: the players may be candidates for political office, the actions campaign expenditures and the preferences a reflection of the candidates’s probabilities of winning.

We consider for the moment only static games (the players have only one move as a strategy).
Strategic Games: some notations

Players: Let $P$ be the set of players. The subscript $-i$ designates all the players belonging to $P$ except $i$ himself.

Remark: These players are often designated as being opponents of $i$. In a two player games, player $i$ has one opponent referred as $j$. 
Strategic Games: some notations

**Strategy:** Let $S$ be the joint set of the strategy space of all players $S = S_1 \ast \ldots \ast S_N$. $S_i$ corresponds to the pure strategy space of player $i$ whereas the pure strategy of the opponents of player $i$ is denoted by $S_{-i} = S/S_i$.

The set of chosen strategies constitutes a strategy profile $s = \{s_1, s_2, \ldots \}$.

**Remark:** For each player, the strategy assigns zero probability to all moves except one (it clearly determines one move). The player could also use mixed strategies in the sense that they choose different strategies with different probabilities.
Utility (or payoff function): Let $u_i(s)$ be the benefit of player $i$ given the strategy profile $s$. In the two players case, we have $U = \{u_1(s), u_2(s)\}$.

**Remark:** The payoff function represents a decision maker’s preferences in the sense that if he prefers $a \in S$ to $b \in S$ then $u(a) > u(b)$.

**Remark:** The decision’s maker’s preferences convey only ordinal information. They tell us that he prefers the action $a$ to the action $b$ but not how much he prefers $a$ to $b$. 
**Definition:** A game with complete information is a game in which each player has full knowledge of all aspects of the game.

In particular, the players know:

- who the other players are.
- what their possible strategies are
- What payoff will result for each player for any combination of moves.

**Remark:** Complete information is different from perfect information. A game is of perfect information if all players know the moves previously made by all other players (application for sequential games mainly). Complete information requires that every player know the strategies and payoffs of the other players but not necessarily the actions!
Let us start: the prisoner’s dilemma

Context: Two suspects in a major crime are held in a separate cells.

- If they both stay quiet, each will be convicted of the minor offense and spend one year in prison.
- If one and only one of them finks, he will be freed and used as a witness against the other who will spend four years in prison.
- If they both fink, each will spend three years in prison.
Modeling the game

Players: The two suspects

Actions: Each player’s set of actions is \{Quiet, Fink\}

Preferences: We need a function \( u_1 \) such as:

\[
u_1(\text{Fink}, \text{Quiet}) > u_1(\text{Quiet}, \text{Quiet}) > u_1(\text{Fink}, \text{Fink}) > u_1(\text{Quite}, \text{Fink})
\]

For example,

- \( u_1(\text{Fink}, \text{Quiet}) = 3 \).
- \( u_1(\text{Quiet}, \text{Quiet}) = 2 \).
- \( u_1(\text{Fink}, \text{Fink}) = 1 \).
- \( u_1(\text{Quite}, \text{Fink}) = 0 \).
Games and matrices

<table>
<thead>
<tr>
<th></th>
<th>Suspect 2</th>
<th>Suspect 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Suspect 1</td>
<td>Quiet</td>
<td>Fink</td>
</tr>
<tr>
<td>Quiet</td>
<td>2,2</td>
<td>0,3</td>
</tr>
<tr>
<td>Fink</td>
<td>3,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

- There are gains from cooperation (each player prefers that both players choose Quite than they both choose Fink).
- However, each player has an incentive to "free ride" (choose fink?) whatever the other player does.
- In any case, for those who don’t understand game theory, YOU will always go to jail!

**Question:** How to solve it? In other words, how to predict the strategy of each player, considering the information the game offers and assuming that the players are rational?
Methods to solve static games

- Iterated dominance.
- Nash Equilibrium.
- Mixed Strategies.

In all these techniques, we assume non-cooperative games. Cooperative games require agreements between the decision makers and might be more difficult to realize (additional signalization).
Illustration of the methods in basic examples

- Single hop Packet forwarding.
- Joint packet forwarding.
- Multiple access games.
- Jamming games.
Single hop packet forwarding

**Context:** We consider two devices $p_1$ and $p_2$ wanting to send packets respectively to $r_1$ and $r_2$ in each time slot using the player as a forwarder.

- If player $p_1$ forwards the packet of $p_2$, it cost player $p_1$ a fixed cost $c$ ($0 < c \leq 1$, $c$ represents the energy spent).
- If he does so, he enables communication between $p_2$ and $r_2$, which gives $p_2$ a reward of 1.
- What would be the pay-off?

Of course, each player is tempted to drop the packet to save resources. but the other player would act the same way....

**Question:** Represent the Single hop packet dilemma forwarding through a matrix.
## Single hop packet forwarding

<table>
<thead>
<tr>
<th></th>
<th>$p_2$</th>
<th>$p_2$</th>
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<tbody>
<tr>
<td>$p_1$</td>
<td>$p_1$</td>
<td>$p_1$</td>
</tr>
<tr>
<td>F</td>
<td>F (1-c,1-c)</td>
<td>D (-c,1)</td>
</tr>
<tr>
<td>D</td>
<td>(1,-c)</td>
<td>(0,0)</td>
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</tbody>
</table>
Joint packet forwarding

**Context:** A sender wants to send a packet to his receiver in each time slot. He needs both devices $p_1$ and $p_2$ to forward for him.

- There is a forwarding cost $c$ ($0 < c \leq 1$) if a player forwards the packet to the sender.
- If both players forward, then they each receive a reward of 1 (from the receiver or the sender).
- What would be the pay-off?

**Question:** Represent the joint packet dilemma forwarding through a matrix.
Joint packet forwarding

<table>
<thead>
<tr>
<th></th>
<th>$p_2$</th>
<th>$p_2$</th>
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<tbody>
<tr>
<td>$p_1$</td>
<td>$p_1$</td>
<td>$p_1$</td>
</tr>
<tr>
<td>F</td>
<td>F (1-c, 1-c)</td>
<td>D (-c, 0)</td>
</tr>
<tr>
<td>D</td>
<td>(0, 0)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>
Multiple Access Game

Context: Two players $p_1$ and $p_2$ want to send some packets to their receivers $r_1$ and $r_2$ using a shared medium.

- Players have a packet to send at each time slot and they can decide to transmit it or not.
- Their transmission mutually interfere.
- When $p_1$ transmits, it incurs a cost of $c$.
- The transmission is successful if $p_2$ does not transmit. In this case, $p_1$ gets a reward of 1 from the successful packet transmission.

Question: Represent the multiple access game through a matrix.
## Multiple Access Game

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Q$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>(0,0)</td>
<td>(0,1-c)</td>
</tr>
<tr>
<td>$Q$</td>
<td>(1-c,0)</td>
<td>(-c,-c)</td>
</tr>
</tbody>
</table>
Jamming Game

**Context:** The objective of the malicious player $p_2$ is to prevent player $p_1$ from a successful transmission by transmitting on the same channel in a given time slot. The objective of $p_1$ is to succeed in spite of the presence of $p_2$. We suppose that the wireless medium is split into two channel $C_1$ and $C_2$.

- The player receives a payoff of 1 if the attacker cannot jam.
- The player receives a payoff of -1 if the attacker jams his packet.
- We neglect the transmission cost as it applies to each payoff.

**Question:** Represent the multiple access game through a matrix.
### Jamming Game

<table>
<thead>
<tr>
<th>sender</th>
<th>jammer 1</th>
<th>jammer 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>C_1</td>
<td>(-1,1)</td>
<td>(1,-1)</td>
</tr>
<tr>
<td>C_2</td>
<td>(1,-1)</td>
<td>(-1,1)</td>
</tr>
</tbody>
</table>
Zero and non-zero sum games

The Single hop packet forwarding, the joint packet forwarding and the multiple access games are non-zero sum games as the players can mutually increase their payoff by cooperation.

Where is the conflict?

- Single hop packet forwarding: they have to provide the packet forwarding service for each other.
- Joint packet forwarding: they have to establish the packet forwarding service.
- Multiple Access Game: They have to share a common resource.

The Jamming game is a zero sum game since the reward of one player represents the loss of the other player.
Iterated Strict Dominance

**Definition** Strategy \( s'_i \) of player \( i \) is said to be strictly dominated by his strategy \( s_i \) if

\[
u_i(s'_i, s_{-i}) < u_i(s_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}
\]

**Exercise:** Apply the technique of iterated strict dominance to the Single hop Packet forwarding and the packet forwarding game.
A weaker condition is often needed.

**Definition** Strategy $s'_i$ of player $i$ is said to be weakly dominated by his strategy $s_i$ if

$$u_i(s'_i, s_{-i}) \leq u_i(s_i, s_{-i}) \forall s_{-i} \in S_{-i}$$

with strict inequality for at least one $s_{-i} \in S_{-i}$.

The solution of iterated strict dominance is unique, which is not the case for the weak dominance case.

These techniques are useful to reduce the size of the strategy space.

**Exercise:** Apply the technique of iterated weakly dominance to the packet forwarding game and to the multiple access game.
Best response framework

In the multiple access game,

- If player $p_1$ transmits, then the best response of player $p_2$ is to be quiet.
- If player $p_2$ is quiet, then $p_1$ is better off transmitting a packet.

**Definition** Denote $b_i(s_{-i})$ the best response of player $i$ to the opponent’s strategy vector $s_{-i}$. The best response $b_i(s_{-i})$ of player $i$ to the profile of strategies $s_{-i}$ is a strategy $s_i$ such that:

$$b_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

One can see that if two strategies are mutual best responses to each other, then no player would have a reason to deviate from the given strategy profile.

This is the start of the Nash equilibrium
The Birth of the Nash Equilibrium


John Forbes Nash, Jr., 1928-
**Nash Equilibrium**

**Definition** The pure strategy profile \( s^* \) constitutes a Nash equilibrium if, for each player \( i \),

\[
u_i(s^*_i, s^*_{-i}) \geq u_i(s_i, s^*_{-i}), \quad \forall s_i \in S_i
\]

Expressed differently, a Nash equilibrium embodies a steady stable "social norm": if everyone else adheres to it, no individual wishes to deviate from it.

**Remark:** It can be shown that a solution derived by iterated strict dominance is a Nash equilibrium.

**Exercise:** What is the Nash equilibrium for the Prisoner’s dilemma?

**Exercise:** Show that the Jamming game:

- Cannot be solved by iterated strict dominance.
- has no pure strategy Nash equilibrium.
- Can we find an equilibrium of another nature in this case?
Mixed Strategies

Players can decide to play each the pure strategies with some probabilities.

**Definition** The mixed strategy $\sigma_i(s_i)$ (also denoted $\sigma_i$) of player $i$ is a probability distribution over his pure strategies $s_i \in S_i$.

Player’s $i$’s utility to profile $\sigma$ is then given by:

$$u_i(\sigma) = \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i})$$
Denote $x$ the probability with which player $p_1$ decides to transmit and $y$ the probability for $p_2$. $p_1$ and $p_2$ stay quiet with probability $1 - x$ and $1 - y$, respectively.

The payoff of player $p_1$ is:

$$u_1 = x(1 - y)(1 - c) - xyc = x(1 - c - y)$$

The payoff of player $p_2$ is:

$$u_2 = y(1 - x)(1 - c) - xyc = y(1 - c - x)$$

Each user wants to maximize his utility.
Example with the multiple access game

Analysis of the best response of \( p_2 \) for each strategy of \( p_1 \).

If \( x < 1 - c \), then \( u_2 \) is maximized by setting \( y \) to the highest possible value i.e \( y = 1 \).

If \( x > 1 - c \), then \( u_2 \) is maximized by setting \( y \) to the lowest possible value i.e \( y = 0 \).

However, if \( x = 1 - c \), any strategy of \( p_2 \) is a best response. The game being symmetric, reversing the roles of the two players leads of course to the same result.

This means that \( (x = 1 - c, y = 1 - c) \) is a mixed-strategy Nash equilibrium for the multiple access Game.

**Exercise:** Show that for the jamming game a mixed strategy Nash equilibrium dictates each player to play a uniformly random distribution strategy (selecting one of the channels with probability 0.5).
Importance of mixed strategies


The importance of mixed strategies is mainly due to the following theorem.

**Theorem.** Every finite strategic-form game has a mixed strategy equilibrium

**Remark:** For mixed strategies, there is the underlying assumption that players are uncoordinated in their random choice.
NON-COOPERATIVE GAMES

John Nash
(Received October 11, 1950)

Introduction

Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book Theory of Games and Economic Behavior. This book also contains a theory of \(n\)-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

Our theory, in contradistinction, is based on the absence of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.

The notion of an equilibrium point is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing “good strategies.”

In the immediately following sections we shall define equilibrium points and prove that a finite non-cooperative game always has at least one equilibrium point. We shall also introduce the notions of solvability and strong solvability of a non-cooperative game and prove a theorem on the geometrical structure of the set of equilibrium points of a solvable game.

As an example of the application of our theory we include a solution of a simplified three person poker game.
Some useful theorems

**Theorem:** Consider a strategic-form game whose strategy spaces $S_i$ are nonempty compact convex subsets of an Euclidean space. If the payoff functions $u_i$ are continuous in $s$ and quasi-concave in $s_i$, there exists a pure-strategy Nash equilibrium.

This theorem is useful of games with an uncountable number of actions and the payoffs are continuous.
Theorem: Consider a strategic-form game whose strategy spaces $S_i$ are nonempty compact subsets of a metric space. If the payoff functions $u_i$ are continuous then there exists a Nash equilibrium in mixed strategies.

The theorem is useful when the payoff functions are continuous but not necessarily quasi-concave.
To solve a game, one has to investigate the existence of a Nash equilibria.

Three ways have been provided: iterated dominance, pure strategy Nash equilibria, mixed strategy Nash equilibria.

The next steps are:

- To determine the uniqueness or not of the equilibrium.
- When several equilibrium exists, one has to determine the most efficient.

Equilibrium selection intends to determine which one to choose by studying the efficiency of the equilibrium.
In games of incomplete information, the Nash equilibrium definition is refined to reflect uncertainties about player’s types, leading to the concept of Bayesian equilibrium.
Example of the Multiple Access Game

Both players in the multiple access game **know** that three Nash equilibria exist:

- Two pure strategy Nash equilibria \((Q,T)\) and \((T,Q)\).
- One mixed strategy equilibrium.

What should be the final strategy?
A method to identify the desired equilibrium point in a game is to compare strategy profiles using the concept of Pareto-optimality.

**Definition.** The strategy profile $s$ is Pareto-superior to the strategy profile $s'$ if for any player $i \in N$:

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s'_{-i})$$

with strict inequality for at least one player.

The strategy profile $s^*$ is pareto-optimal if there exists no other strategy that is pareto-superior to $s^*$.

**Remark:** One cannot increase the utility of player $i$ without decreasing the utility of at least one other player. In the Nash equilibrium, one can not **UNILATERALLY** change his strategy to increase his utility.
The Nash equilibrium (D,D) is not pareto-optimal. However, the strategy profile (F,F) is pareto-optimal.
Joint packet forwarding

(F,F) and (D,D) are Nash equilibria but only (F,F) is pareto optimal.
(T,Q) and (Q,T) are Nash equilibria and Pareto optimal.

The mixed strategy Nash equilibrium \( (x = 1 - c \text{ and } y = 1 - c) \) results in the expected pay-off \( (0,0) \) and is pareto inferior to the two pure strategy Nash equilibria.

It can be shown that there exists no mixed strategy profile that is Pareto-superior to all pure strategies

**Why?** Any mixed strategy of a player \( i \) is a linear combination of his pure strategies with weights that sum to one.
There exists no pure strategy Nash equilibrium. All pure strategies are pareto-optimal.
Can we cooperate without cooperating?

We only went through one type of equilibria in the case of non-cooperative games.

Nash equilibirum may be very efficient and cooperation is sometimes better off (Pareto equilibrium).

However, we don’t necessarily need cooperation. Coordination is enough (Correlated equilibrium).

Example: (Aumann and Schelling, Nobel prize): People can coordinate rather well without communicating.


- Consider the game where two people are asked to select a positive integer.
- If they choose the same integer, both get an award, otherwise no award is given.
- In such a setting, the majority tends to select the number 1. This number is distinctive, since it is the smallest integer.
Correlated Games

Robert Ysral Aumann, 1930-

He received the Noble prize in economy in 2005 with Thomas Schelling for his contribution to the analysis of conflict and cooperation with Game theoretic tools.
The correlated equilibria are defined in the context where there is an arbitrator who can send (private or public) signals to the players.

These signals allow the players to coordinate their actions and perform joint randomization over strategies.

The arbitrator can be a virtual entity (the players can agree on the first word they hear on the radio) and generate signals that do not depend on the system.

A multi-strategy obtained using the signals is a set of strategies (one strategy for each player which may depend on all the information available to the player including the signal it receives)

It is said to be a correlated equilibrium if no player has an incentive to deviate unilaterally from its part of the multi-strategy.

A special type of "deviation" can be of course to ignore the signals.
Evolutionary Games


Optimality of the Nash Equilibrium


In many cases, the Nash equilibrium solution is not optimal.

Adding more options to the players does not necessarily improve the performance (Braess paradox).

Incentive mechanisms (through punishment,..) can on the contrary push the players towards the desired solution.

Centralized system: With a centralized system, one can have a global perspective with achieves a social optimum. This point is in many respect efficient but unfair.

Decentralized: In this case, players are in competition and one operating point is the Nash equilibrium. This point is in many respect not efficient but the players have no regret.

The Price of Anarchy (PoA), defined as the ratio of the cost function at equilibrium with the social optimum case, measures the price of not having a central coordination in the system.
Suppose we have a system where cost functions $C$ depend on the total demand $x$. 

![Diagram](image-url)
The cost of participant is given by: \( C(x) = \sum_{a \in A} c_a(x_a)x_a \)

**Definition:** A social optimum is a feasible flow \( x^{so} \) which minimizes \( C(.) \).
Wardrop equilibrium


Definition: A wardrop equilibrium is a flow $x^{NE}$ such that nobody can switch to a path with a smaller travel time.

Characterization: (Smith (1979), Dafermos (1980)) The wardrop equilibrium is characterized by the variational inequality:

$$\sum_{a \in A} c_a(x_a^{NE})x_a^{NE} \leq \sum_{a \in A} c_a(x_a^{NE})x_a$$

for all $x$. One can show that the wardrop equilibrium exists and is unique (Beckmann, Mcguire and Winsten, 1956)
Wardrop equilibrium


Wardrop (1952) postulated that users in a network game select routes of minimal length. In the asymptotic regime, the non-cooperative game becomes a non-atomic one, in which the impact of a single player on the others is negligible.

In the networking game context, the related solution concept is often called Wardrop equilibrium and is often much easier to compute than the original Nash equilibrium.
Price of Anarchy

The price of anarchy measures the impact of a lack of central coordination.

\[ \text{POA} = \frac{C(NE)}{C(SO)} \]

**Theorem:** For affine costs,

\[ \text{POA} \leq \frac{4}{3} \]

Selfishness drives the system close to optimality!
Proof


\[
C(NE) = \sum_{a \in A} c_a(x_a^{NE}) x_a^{NE} \leq \sum_{a \in A} c_a(x_a^{NE}) x_a
\]

\[
= \sum_{a \in A} c_a(x_a)x_a + \sum_{a \in A} \left( c_a(x_a^{NE}) - c_a(x_a) \right) x_a
\]

\[
\leq C(x) + \frac{1}{4} \sum_{a \in A} c_a(x_a^{NE}) x_a^{NE}
\]

\[
= C(x) + \frac{1}{4} C(NE)
\]
General costs


\[
C'(NE) \leq \sum_{a \in A} c_a(x_a)x_a + \sum_{a \in A} \left( c_a(x_a^{NE}) - c_a(x_a) \right) x_a \\
\leq C(x) + \beta(C)C(NE)
\]

Let

\[
\beta(C) = \max_{0 < x < y} \left( \frac{[c_a(x) - c_a(y)]x}{c_a(y)y} \right) \\
= \max \left( \frac{\text{shared area}}{\text{big rectangle}} \right)
\]

**Theorem:** If cost are drawn from a family of continuous costs \( C \),

\[
\text{POA} = \frac{C(NE)}{C(SO)} \leq (1 - \beta(C))^{-1}
\]
Bounds on the price of anarchy

**Theorem.** For polynomials of maximum degree $p$, the price of anarchy is bounded by:

<table>
<thead>
<tr>
<th>degree</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>POA</td>
<td>4/3</td>
<td>1.626</td>
<td>1.896</td>
<td>2.151</td>
<td>...</td>
<td>$\Omega\left(\frac{p}{\ln(p)}\right)$</td>
</tr>
</tbody>
</table>
4. Dynamic Games
In the static game, players move simultaneously without knowing what the other players do.

The case of sequential interaction, the framework falls in the realm of dynamic games.

These games are represented in an extensive form as opposed to the strategic form.
Complete information versus perfect information

A game with perfect information is a game where the players have perfect knowledge of all the previous moves in the game at any moment they have to make a new move.

It concerns only the previous moves. Hence, there is still a game.
Strategic versus extensive

In the strategic form, the game was represented by a matrix.

In the extensive form, the game is represented by a tree.

The root of the tree is the start of the game.

One level of the tree is a stage.

The sequences of moves define a path on the tree.

Each node has of course a unique history.

Each terminal node of the tree defines a potential end of the game called outcome and it is assigned a corresponding pay-off.

Finite-horizon games are considered (a game with a finite number of stages).
A basic technique in CSMA/CA protocol

The two devices $p_1$ and $p_2$ are not perfectly synchronized.

$p_1$ decides to transmit or not.

$p_2$ observes $p_1$ before making his own move.

The strategy of $p_1$ is to transmit (T) or to be quiet (Q).
Representation of the Sequential Multiple Access Game

How many pure Nash equilibria do we have?

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>$p_2$</th>
<th>payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>(-c,-c)</td>
</tr>
<tr>
<td>T</td>
<td>Q</td>
<td>(1-c,0)</td>
</tr>
<tr>
<td>Q</td>
<td>T</td>
<td>(0,1-c)</td>
</tr>
<tr>
<td>Q</td>
<td>Q</td>
<td>(0,0)</td>
</tr>
</tbody>
</table>

How many pure Nash equilibria do we have?
Important Result


**Theorem. (Kuhn, 1953).** Every finite extensive-form game of perfect information has a pure strategy.
Backward induction for Sequential Multiple Access Game

How do we solve the game?

If player $p_2$ plays the strategy $T$ then the best response of player $p_1$ is to play Q.

however, $T$ is not the best strategy of player $p_2$ if player $p_1$ chooses $T$

We can eliminate some possibilities by backward induction.
Backward induction for Sequential Multiple Access Game

Player $p_2$ knows that he has the last move...

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>$p_2$</th>
<th>payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$(-c,-c)$</td>
</tr>
<tr>
<td>$T$</td>
<td>$Q$</td>
<td>$(1-c,0)$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$T$</td>
<td>$(0,1-c)$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$Q$</td>
<td>$(0,0)$</td>
</tr>
</tbody>
</table>

Given all the best moves of $p_2$, player $p_1$ calculates his best moves as well.

It turns out that the backward induction solution is the historical move ($T$) then ($Q$).
Backward induction

The technique of backward induction is similar to the iterated strict dominance technique. It helps reducing the strategy space but becomes very complex for longer extensive form games.

The method provides a technique to identify Stackelberg equilibrium.
Stackelberg equilibrium

**Definition.** The strategy profile $s$ is a Stackelberg equilibrium with player $p_1$ as the leader and player $p_2$ as the follower if player $p_1$ maximizes his payoff subject to the constraint that player $p_2$ chooses according to his best response function.

**Application:**

- If $p_1$ chooses $T$ then the best response of $p_2$ is to play $Q$ (payoff of $1-c$).
- If $p_1$ chooses $Q$, then the best response of $p_2$ is $T$ (payoff of 0 for $p_1$).

$p_1$ will therefore choose $T$ which is the Stackelberg equilibria.
Exercise

Discuss the extensive form for the other three cases.
Summary

Except for the case of repeated games (players interact several times), we have now the main tools to analyze different non-cooperative schemes in wireless networks.

We will deal with repeated games if time is left..
5. Optimization Results for Game theory
Definition. A norm $T(H)$ on the space of $N \times N$ matrices satisfies the following properties:

(1) $T(H) \geq 0$ with equality if and only if $H = 0$ is the all zero matrix.

(2) For any two matrices $H_1$ and $H_2$,

$$T(H_1 + H_2) \leq T(H_1) + T(H_2)$$

(3) For any scalar $\alpha$ and matrix $H$,

$$T(\alpha H) = |\alpha| T(H)$$
For an $N \times N$ complex matrix $W = (w_{ij})$, the Hilbert-Schmidt norm (or Schur norm or Euclidean norm or Frobenius norm) of $W$ is defined as:

$$|| W || = \sqrt{\sum_{ij} |w_{ij}|^2}$$
An affine function consists of a linear transformation followed by a translation:

\[ x \rightarrow Ax + b \]
A subset $X$ of a Euclidean space is compact if any sequence in $X$ has a subsequence that converge to a limit point in $X$.

The definition of compactness for more general topological spaces uses the notion of "cover", which is a collection of open sets whose union includes the set $X$.

$X$ is compact if any cover has a finite subcover.

A subset of Euclidean space $\mathbb{R}^n$ is called compact if it is closed and bounded.
A set $X$ in a linear vector space is convex if, for any $x$ and $x'$ belonging to $X$ and any $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)x'$ belongs to $X$.

A real-valued function $f$ defined on an interval is called convex if for any two points $x$ and $y$ in its domain $C$ and any $t$ in $[0,1]$, we have

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

A differentiable function of one variable is convex on an interval if and only if its derivative is monotonically non-decreasing on that interval.

The convex hull for a set of points $X$ in a real vector space $V$ is the minimal convex set containing $X$. 

Convex
A function defined on a convex subset $S$ of a real vector space is quasiconvex if whenever $x, y \in S$ and $\lambda \in [0, 1]$ then

$$f(\lambda x + (1 - \lambda y)) \leq \max(f(x), f(y))$$

A quasiconcave function is a function whose negative is quasiconvex.
Convex optimization


Given a real vector space $X$ together with a convex, real-valued function $f$ defined on a convex subset $L$ of $X$, the problem of convex optimization is to find the point $x^*$ in $L$ such that:

$$f(x^*) \leq f(x)$$

for all $x \in X$

**Results:** The following statements hold for convex optimization:

- the set of all (global) minima is convex.
- if the function is strictly convex, then there exists at most one minimum.
- if a local minimum exists, then it is a global minimum.
Karush-Kuhn-Tucker: sufficient conditions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be the function to be minimized under the continuously differentiable constraints $g_i : \mathbb{R}^n \to \mathbb{R}$, $(i = 1, \ldots, m)$ and $h_j : \mathbb{R}^n \to \mathbb{R}$, $(j = 1, \ldots, l)$. If $x^*$ is a local minimum, then there exists constants $\mu_i(i = 1, \ldots, m)$ and $\lambda_i(i = 1, \ldots, l)$ such as

$$f'(x^*) - \sum_{i=1}^{m} \mu_i g'_i(x^*) - \sum_{j=1}^{l} \lambda_j h'_j(x^*) = 0$$

with

$$\mu_i \geq 0 (i = 1, \ldots, m)$$

$$g_i(x^*) \geq 0, (i = 1, \ldots, m)$$

$$h_j(x^*) = 0, (j = 1, \ldots, l)$$

$$\mu_i g_i(x^*) = 0, (i = 1, \ldots, m)$$
Karush-Kuhn-Tucker: necessary conditions

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be convex and the constraints \( g_i : \mathbb{R}^n \to \mathbb{R}, \ (i = 1, \ldots, m) \) and \( h_j : \mathbb{R}^n \to \mathbb{R}, \ (j = 1, \ldots, l) \) be respectively concave and affine functions. If there exists constants \( \mu_i (i = 1, \ldots, m) \) and \( \lambda_i (i = 1, \ldots, l) \) such as

\[
f'(x^*) - \sum_{i=1}^{m} \mu_i g'_i(x^*) - \sum_{j=1}^{l} \lambda_j h'_j(x^*) = 0
\]

with

\[
\mu_i \geq 0 (i = 1, \ldots, m) \\
g_i(x^*) \geq 0, \ (i = 1, \ldots, m) \\
h_j(x^*) = 0, \ (j = 1, \ldots, l) \\
\mu_i g_i(x^*) = 0, \ (i = 1, \ldots, m)
\]

then the point \( x^* \) is a global minimum.
6. Wireless Communication metrics for Game theory
Model representing multiple-antennas, CDMA, OFDM, ad-hoc networks with cooperation,...

\[ y = W \begin{bmatrix} P \end{bmatrix} s + n \]

\[ \begin{array}{c|c|c|c|c} y & W & P & s & n \\ \hline \text{Received signal} & N \times 1 & \text{MIMO matrix} & N \times K & \text{Power} & K \times K & \text{emitted signal} & K \times 1 & \text{AWGN} & \sim \mathcal{N}(0, \sigma^2 I_N) \end{array} \]

Let

\[ W = \begin{bmatrix} u & U \end{bmatrix}, \quad s = \begin{bmatrix} s_1 \\ x \end{bmatrix} \]

The goal is to detect \( s \).
Shannon Capacity

Mutual information $M$ between input and output:

\[
M(s; (y, W)) = M(s; W) + M(s; y | W)
\]
\[
= M(s; y | W)
\]
\[
= H(y | W) - H(y | s, W)
\]
\[
= H(y | W) - H(n)
\]

The differential entropy of a complex Gaussian vector $x$ with covariance $\mathbf{R}$ is given by $\log_2 \det(\pi e \mathbf{R})$. 
In the case of Gaussian independent entries, since

\[ E(yy^H) = \sigma^2 I_N + WPW^H \]
\[ E(nn^H) = \sigma^2 I_N \]

The mutual information per dimension is:

\[
C_N = \frac{1}{N} \left( H(y | W) - H(n) \right) \\
= \frac{1}{N} \left( \log_2 \det(\pi e(\sigma^2 I_N + WPW^H)) - \log_2 \det(\pi e\sigma^2 I_N) \right) \\
= \frac{1}{N} \left( \log_2 \det(I_N + \frac{1}{\sigma^2} WPW^H) \right)
\]
Model example:

\[ y = WP^\frac{1}{2}s + n \]

\[ = up_1^\frac{1}{2}s_1 + UP^\frac{1}{2}x + n \]

\[ = us_1 + n' \]

\[ \mathbb{E}(n'n'^H) = (UP_{-1}U^H + \sigma^2I) = Q\Lambda Q^H \]

Whitening filter:

\[ \tilde{y} = \Lambda^{-\frac{1}{2}}Q^Hy = \Lambda^{-\frac{1}{2}}Q^Hy_p^{\frac{1}{2}}s_1 + \Lambda^{-\frac{1}{2}}Q^Hn' \]

\[ = gp_1^\frac{1}{2}s_1 + b \]

\( b \) is a white Gaussian noise.
\[
\tilde{y} = \Lambda^{-\frac{1}{2}}Q^Hu p_1^\frac{1}{2}s_1 + b
\]

Define \( g = \Lambda^{-\frac{1}{2}}Q^Hu \)

The output SINR is maximized with:

\[
g^H\tilde{y} = g^Hg p_1^\frac{1}{2}s_1 + g^Hb
\]

As a consequence, the receiver is:

\[
g^H\Lambda^{-\frac{1}{2}}Q^H = u^H \left( Q\Lambda^{-1}Q^H \right) = u^H \left( UP_{-1}U^H + \sigma^2I_N \right)^{-1}
\]

**Remark:** The usual MMSE receiver is the unbiased one:

\[
u^H \left( WW^H + \sigma^2I_N \right)^{-1} = \frac{1}{1 + u^H \left( UU^H + \sigma^2I_N \right)^{-1}}u^H \left( UU^H + \sigma^2I_N \right)^{-1}
\]
After MMSE filtering, we obtain:

$$g^H \tilde{y} = g^H g \frac{1}{2} s_1 + g^H b$$

with $g = \Lambda^{-\frac{1}{2}} Q^H u$

Signal to Interference plus Noise Ratio (SINR):

$$\beta_N = \frac{(g^H g)^2 p_1 \mathbb{E}(|s_1|^2)}{g^H g} = g^H g = p_1 u^H \left( UP_{-1} U^H + \sigma^2 I_N \right)^{-1} u$$
PART II
1. Power Allocation Games
System features

- Multiple access channel (MAC): one base station (BS), $K$ mobile stations (MS)
- Multi-antenna mobile and base stations (MIMO)
- Fast fading links
- Global CSIR + global CDIT
- Receiver: successive interference cancellation (SIC)
- Existence of a coordination signal
  - For $K = 2$, $S \in \{1, 2\}$. If $S = 1$ (resp. $S = 2$) user 1 (resp. 2) is decoded last.
    Notation: $p = \Pr[S = 2] = p$, $\Pr[S = 1] = 1 - p = \bar{p}$
  - The BS is assumed to follow the coordination signal
Game (general) features

- Players: mobile stations $k \in \{1, \ldots, K\}$
- Strategy of a player: his precoding matrix/matrices
- Utility: individual Shannon transmission rate
- Type of game:
  - Power allocation game
  - (a) Static game
  - (b) Non-cooperative game (selfish users)
  - (c) With complete information
  - (d) With rational users

Questions

- Can we predict what the players are going to do?
- Is there an equilibrium?
- Uniqueness?
- Efficiency?

...by the way why is it a game?
Mathematical description of the game [belmega08a]

- **Interaction/signal model:**
  \[ y^{(s)}(\tau) = \sum_{k=1}^{2} H_k(\tau) x_k^{(s)}(\tau) + z^{(s)}(\tau) \]

- **Channel matrices:** \( h_k(i, j) \) i.i.d (complex Gaussian RVs)

- **Strategies:**
  \[ Q_k = (Q_k^{(1)}, Q_k^{(2)}) = (\alpha_k^{(1)} P_k I, \alpha_k^{(2)} P_k I) \] [lasaulce07] with
  \[ Q_k^{(s)} = E \left[ x_k^{(s)} x_k^{(s), H} \right] \]
  \[ \alpha_k^{(1)} \in \left[ 0, \frac{1}{\bar{p}_k} \right] \]

- **Strategy sets:**
  \[ p \text{Tr}(Q_k^{(1)}) + \bar{p} \text{Tr}(Q_k^{(2)}) \leq n_t P_k \rightarrow p \alpha_k^{(1)} + \bar{p} \alpha_k^{(2)} \leq 1 \]. Remark: space-time power constraint versus spatial power constraint. Game \( \neq \) optimization problem.

- **Utility:**
  \[ u_k(Q_1^{(1)}, Q_1^{(2)}, Q_2^{(1)}, Q_2^{(2)}) = p R_k^{(1)}(Q_1^{(1)}, Q_2^{(1)}) + (1 - p) R_k^{(2)}(Q_1^{(2)}, Q_2^{(2)}) \]
  \[ R_k^{(1)}(Q_1^{(1)}, Q_2^{(1)}) = E \log |I + \rho H_1 Q_1^{(1)} H_1^H|, \quad R_k^{(2)}(Q_1^{(2)}, Q_2^{(2)}) = E \log |I + \rho H_1 Q_1^{(1)} H_1^H + \rho H_2 Q_2^{(2)} H_2^H - E \log |I + \rho H_1 Q_1^{(1)} H_1^H|, \text{etc.} \]

Is there a pure Nash equilibrium in this game?
Existence of a pure Nash equilibrium

- Theorem 1 [rosen65]. Assumptions: (1) the strategy sets are compact and convex; (2) each player's utility is continuous in his strategy and those of the others; (3) each player's utility is concave in his strategy. Conclusion: there exists at least one pure Nash equilibrium.

Question for the audience: why not considering mixed Nash equilibria?

Is there only one pure Nash equilibrium?
Uniqueness of the Nash equilibrium

Theorem 2 [rosen65]. Assumptions: (1)–(3); (4) the diagonal strict concavity condition is met i.e. $\mathcal{D} > 0$ with

$$
\mathcal{D} = (\alpha''_1 - \alpha'_1) \left[ \frac{\partial u_1}{\partial \alpha_1}(\alpha'_1, \alpha'_2) - \frac{\partial u_1}{\partial \alpha_1}(\alpha''_1, \alpha''_2) \right] + (\alpha''_2 - \alpha'_2) \left[ \frac{\partial u_2}{\partial \alpha_2}(\alpha'_1, \alpha'_2) - \frac{\partial u_2}{\partial \alpha_2}(\alpha''_1, \alpha''_2) \right] > 0
$$

for all $(\alpha'_1, \alpha'_2) \in \mathcal{A}_1^2$ and $(\alpha''_1, \alpha''_2) \in \mathcal{A}_2^2$ such that either $\alpha'_1 \neq \alpha''_1$ or $\alpha'_2 \neq \alpha''_2$.

Conclusion: there is a unique pure NE.

In our context proving Theorem 2 amounts to proving the following lemma.

Lemma 1 [belmega08b]. Assumptions: let $A'$, $A''$, $B'$ and $B''$ be Hermitian and non-negative matrices such that either $A' \neq A''$ or $B' \neq B''$. Conclusion: $\text{Tr}(M) > 0$ where

$$
M = (A'' - A') \left[ (I + A')^{-1} - (I + A'')^{-1} \right] + (B'' - B') \left[ (I + B' + A')^{-1} - (I + B'' + A'')^{-1} \right].
$$
Shannon Rate-Efficient Power Allocation Games (6)

Determination of the Nash equilibrium

- Optimization problem (say for user 1): under the power constraint compute

\[
\arg \max_{Q_1^{(1)}, Q_1^{(2)}, Q_2^{(1)}, Q_2^{(2)}} u_1(Q_1^{(1)}, Q_1^{(2)} Q_2^{(1)}, Q_2^{(2)}) = \\
\arg \max_{Q_1^{(1)}, Q_1^{(2)}} \left\{ p\mathbb{E} \log |I + \rho H_1 Q_1^{(1)} H_1^H| + \bar{p}\mathbb{E} \log |I + \rho H_1 Q_2^{(2)} H_1^H + \rho H_2 Q_2^{(2)} H_2^H| \\
- \bar{p}\mathbb{E} \log |I + \rho H_1 Q_1^{(1)} H_1^H| \right\}
\]

- Problem 1: When implementable, usual numerical optimization techniques are generally not very attractive because they rely on computationally-intensive Monte-Carlo simulations (see e.g. [vu05])

- Problem 2: physical interpretations are not easy to make

Solution: find an approximation. Good news: when the system sizes increase quantities like \( \frac{1}{nr} \log |I + \rho H_1 Q_1^{(1)} H_1^H| \) converge a.s. for usual channel models.
Determination of the Nash equilibrium [belmega08b]

- Large system approximation of the utilities [sylverstein95, tulino05]

Assumptions: \( n_t \to \infty \), \( n_r \to \infty \), and \( \lim_{n_t \to \infty, n_r \to \infty} \frac{n_t}{n_r} = c < \infty \)

Result:

\[
\begin{align*}
\tilde{R}_1^{(1)} &= n_t \log_2 (1 + \rho \alpha_1 \gamma_1) + n_r \log_2 (c \gamma_1) - n_t \gamma_1 (c \gamma_1 - 1) \log_2 e \\
\tilde{R}_2^{(1)} &= n_t \log_2 (1 + \rho \alpha_1 \gamma_1) + n_t \log_2 \left[ 1 + 2 \rho_2 \frac{1 - (1 - p) \alpha_2 \gamma_2}{p} \right] + n_r \log_2 (2 c \gamma_2) \\
&- 4 n_t \gamma_2 (c \gamma_2 - 1) \log_2 e - \tilde{R}_1^{(1)}
\end{align*}
\]

with \( \gamma_1 = \frac{n_r}{n_t} \frac{1}{1 + \rho \frac{\alpha_1 P_1}{1 + \rho \alpha_1 P_1 \gamma_1}} \)

\( \gamma_2 = \frac{n_r}{n_t} \frac{1}{1 + \rho \left\{ \frac{\alpha_1 P_1}{1 + 2 \rho \alpha_1 P_1 \gamma_2} + \frac{[1 - (1 - p) \alpha_2] P_2}{p + 2 \rho [1 - (1 - p) \alpha_2] P_2 \gamma_2} \right\}} \).

- Power allocation at the NE
  (a) Kuhn-Tucker optimality conditions
  (b) Fixed-point method + iterative algorithm

...what about the approximated game? See [dumont07]
Sum-rate efficiency of the Nash equilibrium [belmega08b]
Sum-rate efficiency of the Nash equilibrium: observations

- Sum-rate vs $p$: $\forall p \in [0, 1]$, the gap between the sum-rate of a decentralized MIMO MAC with selfish users and the sum-capacity of the equivalent virtual MIMO network (EVMN) is very small.
- Remark. The social optimum corresponds to a global optimization. Does it coincide with the sum-capacity?
- Sum-rate vs transmit power ($P_1 = P_2 = P$): same conclusion.
- **SIC vs SUD**: the network sum-rate when the BS implements a SIC is much better than that obtained when SUD is implemented, regardless of the distribution of the coordination signal. This comparison makes especially sense for the point $p = \frac{1}{2}$ (fair comparison).
2. Power Control Games
Energy-Efficient Power Control Games (1)

System features

- MAC, single-antenna BS and MSs
- Slow and flat fading channels (for MC-CDMA see [meshkati06])
- Global CSIR + local CSIT

Review of the non-cooperative game features [goodman00]

- Strategy of a player (mobile station): his transmit power level
- Utility: $\forall i \in \{1, \ldots, K\}$, $u_i(p_1, \ldots, p_K) = \frac{T_i}{p_i} = \frac{R_i f(SINR_i)}{p_i}$ (bit/J)
- Type of game: power control game, (a)–(d) (see Game 1)
Energy-Efficient Power Control Games (2)

About the Nash equilibrium

- Signal model: \( Y = \sum_{i=1}^{K} h_i X_i + Z \) where \( Z \sim \mathbb{CN}(0, \sigma^2) \).
- Single-user decoding is assumed at the BS:
  \[
  SINR_i = \frac{p_i |h_i|^2}{\sum_{j \neq i} p_j |h_j|^2 + \sigma^2}
  \]
- Transmit powers at the NE (for the non-trivial case where \( p_{i}^{NE} < P_{\text{max}} \)):
  \[
  p_{i}^{NE} = \frac{\sigma^2 \beta^*}{|h_i|^2 1 - (K - 1) \beta^*} \text{ where } \beta^* f'(\beta^*) = f(\beta^*)
  \]
- Problems
  - (A) Existence of a non-trivial equilibrium. Solution: CDMA. The denominator of \( p_{i}^{NE} \) becomes \( |h_i|^2 \left[ 1 - \frac{(K-1)\beta^*}{N} \right] \).
  - (B) The NE can be inefficient [goodman00]
- Solution 1 to (B): pricing [saraydar02]
- Solution 2 to (B): introduce hierarchy [hayel08]
Energy-Efficient Power Control Games (3)

General objective of the hierarchical approach

Split the intelligence between the BS and MSs in order to find a trade-off between the global network performance reached at the equilibrium and the amount of signaling needed to make it work.

1. Introducing hierarchy by choosing a leader [hayel08]

- One BS which implements single-user decoding (SUD)
- $K$ mobile stations
- One MS is chosen to be the game leader
- $K - 1$ followers who know what the leader has played (cognitive radio, broadcasting signal from the BS, etc)
Energy-Efficient Power Control Games (4)

Definition (Stackelberg equilibrium, 2-user case). A strategy profile \((p_1^{SE}, p_2^{SE})\) is called a (pure) Stackelberg equilibrium if \(p_1^{SE}\) maximizes the single-variable utility of the leader and \(p_2^{SE} \in \text{BR}_2(p_1)\). Mathematically:

\[
p_1^{SE} = \arg \max_{p_1} u_1(p_1, p_2(p_1))
\]
\[
p_2^{SE} = \arg \max_{p_2} u_2(p_1^{SE}, p_2).
\]

NB: the notation BR stands for best response.

1.a. Questions

- Existence, uniqueness of a Stackelberg equilibrium (SE)?
- From a user point of view, is it better to be chosen to be a leader or a follower?
- With respect to the non-cooperative game what is the gain brought by introducing hierarchy?
- Do all the players benefit from this?
1.b. Answers

Proposition 1 (existence, uniqueness). Sufficient conditions: $\frac{f''(0)}{f'(0)} \geq 2\frac{(K-1)\beta^*}{1-(K-2)\beta^*}$ and

\[ \phi(x) = x \left[ 1 - \frac{(K-1)\beta^*}{1-(K-2)\beta^*} x \right] f'(x) - f(x) \text{ has a single maximum in } ]0, \gamma^*[ \text{, where } \gamma^* \text{ is the positive solution of the equation } \phi(x) = 0. \]

\[ p_{SE}^i = \frac{\sigma^2}{|h_i|^2} \frac{\gamma^*(1+\beta^*)}{1-(K-1)\gamma^*\beta^*-(K-2)\beta^*}, \]

\[ \forall j \neq i, \quad p_{SE}^j = \frac{\sigma^2}{|h_j|^2} \frac{\beta^*(1+\gamma^*)}{1-(K-1)\gamma^*\beta^*-(K-2)\beta^*}. \]

Comments: handmade proof; $f(x) = (1 - e^{-x})^M$ passed the test.
1.b. Answers (continued)

**Proposition 2.** Every user has always a better utility by being chosen as a follower than a leader.

**Proposition 3.** Both the leader and followers improve their utilities with respect to the non-cooperative setting.

**Comments**

- **All the players** benefit from hierarchy ≠ duopoly in economics [hamilton90]
- All the users not only obtain a better energy-efficiency in the proposed Stackelberg game but also transmit with a lower power than in the non-cooperative game
2. Introducing hierarchy by implementing SIC at the BS [hayel08]

- The BS implements successive interference cancellation
- There exists a coordination signal indicating the decoding order to the MSs
- Always with global CSIR and local CSIT

2.a. Questions

- Existence, uniqueness of an equilibrium (NE/SE)?
- For a given user, is it better to live in the SUD-based game or in the SIC-based one?
2.b. Answers

Existence and uniqueness of a Nash equilibrium

Proposition 4. Notation: the user who is decoded with rank $K - i + 1$ is assigned index $i$.

The unique NE is given by: $\forall i \in \{1, ..., K\}$, $p_i^{(SIC)} = \frac{\sigma^2}{|h_i|^2} \beta^*(1 + \beta^*)^{i-1}$ with $p_i^{(SIC)} \leq p_i^{\max}$. The proof is based on following two theorems.

Theorem 3 (see e.g. [fundenberg91]). Assumptions: (1) the strategy sets are compact and convex; (2) each player’s utility is continuous in his strategy and those of the others; (3) each player’s utility is quasi-concave in his strategy. Conclusion: there exists at least one pure Nash equilibrium.
Existence and uniqueness of a Nash equilibrium (cont’t)

Theorem 4 [yates95] Assumptions: the best response (BR) correspondence

$$BR(p) = \left( BR_1(p), \ldots, BR_K(p) \right)$$ is a standard function (positive, monotonous and scalable). Conclusion: the NE is unique.

- Positivity: $BR_i(p) > 0$
- Monotonicity: $p \geq p' \Rightarrow BR(p) \geq BR(p')$
- Scalability $\forall \alpha > 1, \alpha BR(p) > BR(\alpha p)$
1.b. Answers (continued)

Proposition 5 (SIC vs SUD). Let \( d_i \) be the decoding rank for user \( i \in \{1, \ldots, K\} \). Also define the critical decoding rank \( d_c \) as follows:

\[
d_c = \max \left\{ 1, \left[ K - \frac{\ln \left[ \frac{1}{1-(K-1)\beta^*} \right]}{\ln(1 + \beta^*)} \right] \right\}.
\]

If the decoding rank of a user is greater than or equal to \( d_c \) then his utility is better when the receiver implements SIC instead of SUD, it is worse otherwise.

Comments

- In the 2-user case users 1 and 2 will always perform better in the noncooperative game with SIC, independently of the value of \( \beta^* \)
- But in general, only the users who have a decoding rank greater than or equal to \( d_c \) will have a better utility in the SIC-based game
- If \( \beta^* \rightarrow 0 \) only users 1 and 2 will perform better whereas all users will improve their utilities in the SIC-based game with random CDMA if \( \frac{K-1}{N} \beta^* \rightarrow 1 \), which occurs, for example, when the load is small
3. Network performance analysis [hayel08]

- Now, the BS becomes a player (say the super-leader) who wants to maximize the overall network performance
- Issues
  - In the Stackelberg formulation with SUD: how to choose the best leader
  - In the hierarchical game with SIC: how to choose the best decoding order

The answers to these questions depend on your vision of the world.

3.a. Performance metrics

- Social welfare [arrow63]:
  \[ w = \sum_{i=1}^{K} u_i = \sum_{i=1}^{K} \frac{T_i}{p_i} \]

- Equivalent virtual MIMO network (EVMN) energy-efficiency [meshkati06]:
  \[ v = \frac{\sum_{i=1}^{K} T_i}{\sum_{i=1}^{K} p_i} \]
3.b. Result examples

Proposition 6 (Best decoding order for $w$). Assume a non-cooperative game with SIC. The best decoding order in the sense of the social welfare is to decode the users in the increasing order of their energy weighted by the coding rate $R_i |h_i|^2$.

Proposition 7 (Best decoding order for $v$). Assume a non-cooperative game with SIC. The best decoding order in the sense of the equivalent virtual MIMO network energy-efficiency is to decode the users in the decreasing order of their signal-to-noise ratio (SNR) $\frac{|h_i|^2}{\sigma^2}$.

Comments

- With $W$ ($w$) as a super-leader: the rich get richer
- With Vladimir ($v$) as a super-leader: the poor get richer (Vladimir is more social than W!)
- Network performance measure: difference with Shannon transmission rate based utilities
3. Handover Games
(a) Fixed network infrastructure

- Network of $S$ parallel multiple access channels
- $S \geq 2$ base stations with non-overlapping bands of frequency
- The BSs are connected through perfect communication links
- BS characteristics
  - Noise power level
  - Bandwidth
  - Number or receive antennas
Handover Games (2)

System Model (continued)

(b) Mobile stations

- $K \geq 2$ mobile stations equipped with a cognitive radio
- Each MS can sense the quality of its links with the different BSs
- Each MS can share its transmit power between the BSs
- Users are selfish and free to choose their power allocation in order to maximize their individual transmission rate
  - Hard handover: best BS selection
  - Soft handover: optimal sharing between the different BSs
(Hard) Handover Games (3)

Game description for a simple model

- Interaction model: \( \forall s \in \{1, \ldots, S\}, y_s = \sum_{k=1}^{K_s} d_{s,k} + z_s \) with \( E|d_{s,k}|^2 = P, E|z_s|^2 = N_s \)
- User’s strategy: \( \forall k \in \{1, \ldots, K\}, P_k = (P_{1,k}, \ldots, P_{S,k}) \)
- Strategy set: \( B = \{e_1, \ldots, e_S\} \) (canonical basis of \( \mathbb{R}^S \))
- User’s utility (SUD): \( u_k(P_1, \ldots, P_K) = \log \left[ 1 + \frac{P}{N_s + (K_s - 1)P} \right] \)

Existence of a Nash equilibrium

- In general there is no pure Nash equilibrium
- However,
  - by assuming a population of users (non-atomic games): there are even several pure NE (see congestion games [milchtaich96][roughgarden04])
  - or by allowing mixed strategies, there is at least one mixed NE [nash50]
Nash/Wardrop equilibrium for a dense network [belmega08c]

- Fraction of users in BS $s$: $x_s = \frac{K_s}{K}$ with $K >> 1$
- Utility of user $k$ if he connects to BS $s$:
  \[
  u_k^{(s)}(x_1, ..., x_S) = \log_2 \left[ 1 + \frac{P}{N_s + (Kx_s - 1)P} \right]
  \]
- At a Nash equilibrium, all the users have the same utility:
  \[
  x_s^{\text{NE,SUD}} = \max \left\{ 0, \frac{1}{S} + \frac{1}{S} \sum_{j=1}^{S} \frac{N_j - N_s}{KP} \right\}
  \]
- Resulting network sum-rate
  \[
  R_{\text{sum}}^{\text{SUD}}(x^{\text{NE,SUD}}) = \sum_{s=1}^{S} Kx_s^{\text{NE,SUD}} \log_2 \left[ 1 + \frac{P}{N_s + (Kx_s^{\text{NE,SUD}} - 1)P} \right]
  \]
- Special cases: (1) $N_1 = ... = N_S$; (2) $KP >> N_s$. 
(Hard) Handover Games (5)

Illustration of the NE sum-rate efficiency

with $POA \triangleq \frac{R_{\text{sum}}^C(x^*, C)}{R_{\text{sum}}(x^{\text{NE}})} \geq 1$
Introducing a pricing technique to improve the NE sum-rate efficiency

Modified game (spontaneous NE $\rightarrow$ stimulated NE)

- System designer standpoint: maximize the overall network performance $\max_x R_{\text{sum}}(x)$
- New utility (cost function) for user $k$ if he connects to BS $s$: $c_{s,k}(x) = p(\tau_{s,k}(x)) + \beta_s$
  where
  - $\tau_{s,k}(x) = \frac{n_k}{u_{s,k}(x)}$ the time user $k$ is connected to BS $s$ for sending $n_k$ bits
  - $p(.)$ the price per unit of connection time (positive, strictly increasing);
- The system designer chooses $\beta_s$ such that $\tilde{x}^{\text{NE}} = x^*$
  - If $\beta_s > 0$ the system owner will charge more the users that connect to BS $s$ (tax or punishment)
  - If $\beta_s < 0$ the system owner will charge less the users that connect to BS $s$ (subsidies or reward)
- For $\beta_s = -p(\tau_{s,k}(x_s^*)) + p(\tau_{s,k}(x_s^{\text{NE}}))$ the new NE is the desired state
- We have that:

$$1 \leq \widetilde{POA} \leq POA$$
Simulation example: performance improvement brought by introducing pricing

![Simulation graph showing achievable rates for different scenarios.](image)
(Soft) Handover Games (8)

About the soft handover case

- User’s strategy: \( \forall k \in \{1, \ldots, K\}, P_k = (P_{1,k}, \ldots, P_{S,k}) \)
- Strategy set: \( P_k \in [0, P]^S \)
- Does the case of soft handover correspond to the case of mixed strategies?
- ... by the way, is hard handover just a special case of soft handover?

Soft handover vs Hard handover

![Achievable rates graph](image_url)
4. Medium Access Control Games
Medium access control games (1)

System model [altman06]

- Multiple access channel: one BS, $K$ MSs
- BS: it does not schedule the (uplink) transmissions
- Notation: let $\mathcal{N} = \{0, 1\}^K$ be the set of all $2^K$ subsets of $\{1, \ldots, K\}$. On each time slot, a random subset of mobiles $z(t) \in \mathcal{N}$ is active (discrete time system).
- Example: for a subset of $\text{size}(z(t)) = 3$ MSs for which user 1 is not active and users 2 and 3 are active: $z = (011)$. Number of active MSs at time $t$: $w(z) = 2$.
- Probability of being active: let $p_z$, $z \in \mathcal{N}$ be the probability that the subset $z$ of mobiles is active on a time slot
- Access protocol: slotted ALOHA
  - All MSs are synchronized
  - Collision: two or more MSs simultaneously attempt to send a packet then all transmitted packets are lost (erasure)
  - When a collision occurs each concerned user wait a random number of time slots before retransmitting
- Each MS has always a packet to transmit (saturation assumption)
- There is a power constraint. If we denote by $q_i$ the probability that MS $i$ sends a packet when active we have that: $q_i \leq q_{i,\text{max}}$
Non-cooperative game description

- Players: MSs.
- User’s strategy: \( q_i \).
- Set of strategy \( Q_i = [0, q_i^{\text{max}}] \).
- User’s utility (individual throughput conditioned on being active):

\[
\Theta_i^{\text{act}}(q_1, \ldots, q_K) = \mathbb{E}_Z[\Pr[z = e_i]] \\
= \mathbb{E}_Z \left[ q_i \prod_{j \in \{1, \ldots, \text{size}(z)\}, j \neq i} (1 - q_j) \right] \\
= q_i \sum_{z \in \mathcal{N}} p_z \prod_{j \in \{1, \ldots, \text{size}(z)\}, j \neq i} (1 - q_j)
\]

Nash equilibrium

\( T > 0 \Rightarrow \) there is a unique NE: \( \forall i \in \{1, \ldots, K\}, q_i = q_i^{\text{max}} \)
Medium access control games (3)

Coordination mechanism [altman06]

- The set of users is partitioned into $M$ groups $G_1, \ldots, G_M$. User $i$ belongs to $G_j$ iff $i = j - 1 \mod M$.
- Coordination signal: at each time slot $t$, the BS broadcasts a signal to all MSs in the form of a uniformly distributed discrete random variable $S \in \{1, \ldots, M\}$, with $K = nM$. The RV $S$ is assumed to be independent of $Z$.
- Correlated strategy: $x_i = (p_i, q_i)$ where on time slot $t$, an active mobile $i$ transmits a packet with probability $p_i$ if and only if $i \in G_{S(t)}$. Otherwise it transmits with probability $q_i$. Define $\mathcal{X}_i$ be the set of feasible strategies for user $i$. 

Correlated equilibrium (CE)

- A multi-strategy \( x = (x_1, \ldots, x_K) \) is said to be a correlated equilibrium if

\[
\forall i \in \{1, \ldots, K\}, \forall x'_i \in X_i, \Theta_i^{\text{act}}(x) \geq \Theta_i^{\text{act}}(x'_i, x_{-i})
\]

- Finding a correlated equilibrium becomes rapidly an intractable problem as the number of users grows → we consider the special case of a symmetric channel where \( \forall i \in \{1, \ldots, K\}, q_{i}^{\text{max}} = q^{\text{max}} \rightarrow \text{symmetric CE} \)

- It can be shown that \( x^* = (p, q) \) with

\[
q = \max \left\{ 0, \frac{M q_{i}^{\text{max}} - 1}{M - 1} \right\} \min \left\{ 1, \frac{M q_{i}^{\text{max}}}{M - 1} \right\}
\]

\( q \) correspond to CE (see [altman06]), and the corresponding \( p \) is obtained by saturating the power constraint

\[
\frac{p}{M} + \left( 1 - \frac{1}{M} \right) q = q^{\text{max}}
\]
Simulation example: system throughput vs probability of being active

Power constraint $q^{\text{max}} = 0.25$, $K = 6$

The coordination mechanism allows one to obtain higher values in the non-cooperative case.
5. Concluding Remarks
Concluding Remarks (1)

On what we have learned

- Equilibrium
  - Variety: NE, SE, WE, CE,...
  - Existence, uniqueness: Nash, Rosen, Debreu-Fan-Glicksberg, Yates, with hands,...
  - Determination: how to find it (optimization problem, approximation, ...)
  - Efficiency: how to measure it, how to improve it (hierarchy, pricing, coordination signal, ...)

- Interesting (sometimes surprising) results for distributed wireless networks
  - POA
  - Leader/follower
  - SUD vs SIC in energy-efficient PC games
  - etc.
On what we have assumed

- Complete information
- CSIR, CSIT
- Scalability
- Rationality
THANK YOU!
6. References and Reading
References for Part II


References for Part II


Reading


