Robust Interval Observers for Discrete–time Systems With Input and Output ⋆

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Abstract

For a family of nonlinear discrete-time systems with input, output and uncertain terms, a new *interval observer* is designed. Its main feature is that it is composed of two copies of classical observers. This interval observer applies in the presence of unknown bounded nonlinear terms and disturbances and can be used to achieve asymptotic stability through an appropriate choice of dynamic output feedback. An illustrative example completes the presentation.

Key words: Interval observer; discrete-time; robustness; estimation.

1 Introduction

The interval observer technique is a state estimation approach based on a guaranteed state estimator composed of a dynamic extension with two outputs giving an upper and a lower bound for the solutions of the considered system. Such a method makes it possible to cope with large disturbances and gives componentwise an information on the range of the possible solutions at any time instant. Thus, this approach is fundamentally different from classical techniques of robust stability analysis or design of control laws for systems with disturbances affecting continuous or discrete systems, as presented, for instance, in Konstantopoulos & Antsaklis (1995). To the best of the authors' knowledge, the guaranteed state estimation technique can be traced back to the seminal work Schweppe (1968), but the notion of *interval observer* is more recent. It originates in Gouzé et al. (2000) and has been developed in many directions since state estimation is essential for monitoring, fault detection and control purposes (more explanations can be found in particular in Alcaraz-Gonzalez & Gonzalez-Alvarez (2007)). Some works on interval observers are devoted to various classes of linear systems (Combastel & Raka (2011), Mazenc & Bernard (2010), Mazenc & Bernard (2011), Mazenc, Niculescu & Bernard (2012), Mazenc, Kieffer & Walter (2012)) and others concern some classes of nonlinear systems (Raissi et al. (2012), Raissi et al. (2005), Moisan et al. (2009), Mazenc & Bernard (2013)). Most of these works deal with continuous-time systems only, although such a technique is appealing in the context of discrete-time systems: notice in particular that systems with sampled data often lead to discrete-time systems, as explained for instance in Astrom & Wittenmark (1997), which are frequently affected by disturbances. This motivates the development of robust state estimation techniques, as for example the one based on interval observers. This remark motivated the contributions by Efimov et al. (2012) and Mazenc, Dinh & Niculescu (2012) and motivates the present paper too.

In Efimov et al. (2012), interval observers are constructed for families of time-varying discrete-time systems without inputs and in Mazenc, Dinh & Niculescu (2012), interval observers for two important families of discrete-time systems are proposed. The first is composed of time-invariant nonlinear systems which possess specific stability and monotonicity properties. The second is the general family of the linear time-invariant exponentially stable systems. The authors established that these systems can be transformed into *nonnegative* and *exponentially stable time-invariant* systems (see, for instance, Haddad et al. (2010) for the definition of nonnegative system and Section 2) through linear, possibly time-varying, changes of coordinates. Using this key result, interval observers for linear systems without input and an output have been constructed, under an appropriate detectability assumption. Such constructions are based on some dynamic extension, which is nonnegative

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when the output is identically equal to zero.

The present paper complements the contribution Mazenc, Dinh & Niculescu (2012). We consider a nonlinear system of the form:

$$\begin{cases} x_{k+1} = [\mathcal{A} + \mathcal{A}_d(x_k)]x_k + \mathcal{B}u_k + \Phi(y_k) + \nu_k, \\ y_k = \mathcal{C}x_k, \end{cases} \tag{1}$$

with $k \in \mathbb{N}$, $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times q}$, $\mathcal{C} \in \mathbb{R}^{p\times n}$, where $\mathcal{A}_d : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is an unknown bounded nonlinear function, ν_k is an additive disturbance and $\Phi : \mathbb{R}^p \to \mathbb{R}^n$ is a known nonlinear function. We give some sufficient conditions ensuring that the system admits an interval observer which is exponentially stable when the system is in closed-loop with a feedback depending only on y_k and the values provided by the bounds of the interval observer. The proposed interval observer consists of two copies of Luenberger observers (Luenberger (1971)) with additional terms taking into account the presence of the uncertainties and of Φ endowed with appropriate outputs (the 'bounds' of the interval observer). This result may sound surprising because such observers or their associated error equations do not possess the property of being nonnegative systems, although this property is usually used when constructing interval observers (see, for example, Mazenc, Dinh & Niculescu (2012), Efimov et al. (2012), Mazenc, Kieffer & Walter (2012), Goffaux et al. (2009), Alcaraz-Gonzalez et al. (2002) and the discussions therein). In fact, the notion of nonnegative system will be used as well, but only indirectly to select appropriate initial conditions and upper and lower bounds for the interval observer. It is worth noticing that the idea of taking advantage of interval observers to design stabilizing control laws is not new: in particular, it is used in Efimov et al. (2011) to stabilize nonlinear systems. However, to the best of the authors' knowledge, there do not exist any similar results in the literature to the one presented in this contribution, even for continuous time systems.

The main advantage of the new approach is that it makes it possible to let classical observers play simultaneously the role of observers and interval observers and therefore the introduction of extra dynamics with some nonnegativity property is not explicitly needed. Moreover, since the choice of initial conditions and bounding outputs for the interval observer is not unique, this technique may be used to construct a bundle of interval observers, as done for instance in Bernard & Gouzé (2004), without having to introduce extra dynamics. Thus, better estimates can be obtained without having to consider interval observers of dimension larger than twice the dimension of the studied system. Incidentally, it is worth mentioning that a single Luenberger observer cannot be the dynamics of an interval observer: one can prove that in the fundamental case of an interval observer associated to linear initial conditions and linear bounds, the dimension of the interval observer is necessarily strictly larger than the dimension of the system studied. Due to length limitation, the proof of this result is omitted.

The paper is organized as follows. Notation, definitions and prerequisites are given in Section 2. In Section 3 the main result of the work is presented, i.e. a general construction of interval observer. An illustrative example is given in Section 4. Concluding remarks are drawn in Section 5 and end the paper.

2 Notation, definitions and prerequisites

The notation will be simplified whenever no confusion can arise from the context. Any $k \times n$ matrix, whose entries are all 0 is simply denoted 0. The Euclidean norm of vectors of any dimension and the induced norm of matrices of any dimensions are denoted $|\cdot|$. All the inequalities must be understood *componentwise* (partial order of $\mathbb{R}^r$) *i.e.* $v_a = (v_{a1}, ..., v_{ar})^\top \in \mathbb{R}^r$ and $v_b = (v_{b1}, ..., v_{br})^\top \in \mathbb{R}^r$ are such that $v_a \leq v_b$ if and only if, for all $i \in \{1, ..., r\}$, $v_{ai} \leq v_{bi}$. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive semidefinite (resp. negative semidefinite) if for all vectors $v \in \mathbb{R}^n$, $v^\top Av \geq 0$ (resp. $v^\top Av \leq 0$). Then we denote $A \succeq 0$ (resp. $A \preceq 0$). A matrix $A \in \mathbb{R}^{n\times n}$ is said to be Schur stable if its spectral radius is smaller than 1. For two matrices $A = (a_{ij}) \in \mathbb{R}^{r\times s}$ and $B = (b_{ij}) \in \mathbb{R}^{r\times s}$ of same dimension $\max\{A, B\}$ is the matrix where each entry is $m_{ij} = \max\{a_{ij}, b_{ij}\}$. For a matrix $A \in \mathbb{R}^{r\times s}$, $A^+ = \max\{A, 0\}$, $A^- = \max\{-A, 0\}$. A matrix $A \in \mathbb{R}^{r\times s}$ is said to be *nonnegative* if every entry $a_{i,j}$ of A satisfies $a_{i,j} \geq 0$. A sequence (u_i) is *nonnegative* if for all integer k, u_k is nonnegative. A system $x_{k+1} = f(k, x_k)$ is *nonnegative* if for all integer k_0 and any initial condition $x_{k_0} \geq 0$, the solution x_k satisfies $x_k \geq 0$ for all integer $k \geq k_0$. Let $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then the inequality:

$$|x + y|^2 \leq 2|x|^2 + 2|y|^2 \tag{2}$$

holds. Due to the features of the systems considered in the present work, a slightly different definition of interval observer for discrete-time nonlinear time-varying systems than the one introduced in Mazenc, Dinh & Niculescu (2012) is proposed. A definition of framers is also given. These two notions have been introduced, with slightly different features, in several papers (see, for instance, (Mazenc, Niculescu & Bernard (2012), Gouzé et al. (2000)) to cite only a few).

Definition. *Consider a discrete-time system:*

$$x_{k+1} = f_1(k, x_k, \nu_k), \tag{3}$$

with $x_k \in \mathbb{R}^n$, with $\nu_k \in \mathbb{R}^d$, with an output $y_k = m(x_k) \in \mathbb{R}^p$, and where f_1 and m are two nonlinear functions. The initial condition at the instant $k_0 \in \mathbb{N}$, $x_{k_0} \in \mathbb{R}^n$ is assumed to be bounded by two known bounds:

$$x_{s,k_0} \leq x_{k_0} \leq x_{l,k_0} \tag{4}$$

and the disturbances ν_k are supposed to be upper and lower bounded by two known sequences ν_k^+, ν_k^-, i.e. for all $k \in \mathbb{N}$, $\nu_k^- \leq \nu_k \leq \nu_k^+$. Then, the dynamical system:

$$z_{k+1} = f_2\left(k, z_k, y_k, \nu_k^+, \nu_k^-\right), \tag{5}$$

associated with the initial condition $z_{k_0} = g(k_0, x_{l,k_0}, x_{s,k_0}) \in \mathbb{R}^{n_z}$ and bounds for the solution x_k:

$$x_k^+ = h^+(k, z_k)\ ,\ x_k^- = h^-(k, z_k), \tag{6}$$

where f_2, g, h^+ and h^- are nonlinear functions, is called (i) a framer for (3) if for any vectors x_{k_0}, x_{s,k_0} and x_{l,k_0} in $\mathbb{R}^n$ satisfying (4), the solutions of (3)-(5) with respectively x_{k_0}, $z_{k_0} = g(k_0, x_{l,k_0}, x_{s,k_0})$ as initial condition at $k = k_0$, denoted respectively x_k and z_k satisfy, for all $k \geq k_0$, the inequalities

$$x_k^- = h^-(k, z_k) \leq x_k \leq h^+(k, z_k) = x_k^+, \tag{7}$$

(ii) a robust interval observer for (3) if, in addition, there exists a function γ of class $\mathcal{K}$ such that, in the particular case where there is a constant $\overline{\nu} > 0$ such that for all $|\nu_k^+| + |\nu_k^-| \leq \overline{\nu}$ for all $k \in \mathbb{N}$, then any solution (x_k, z_k) of (3)-(5) is such that there exists $k_c \in \mathbb{N}$ such that, for all integer $k \geq k_c$, $|h^+(k, z_k) - h^-(k, z_k)| \leq \gamma(\overline{\nu})$.

3 Interval observer

In the sequel, a family of nonlinear systems with uncertain terms is considered. It will be shown that, despite the presence of these uncertainties, one can construct framers which are interval observers when these systems are in closed-loop with stabilizing output feedbacks that depend on the values of the bounds of the interval observer. More precisely, we consider:

$$\left\{\begin{array}{rl} x_{k+1} &= [\mathcal{A} + \mathcal{A}_d(x_k)]x_k + \mathcal{B}u_k + \Phi(y_k) + \nu_k, \\ y_k &= \mathcal{C}x_k, \end{array}\right. \tag{8}$$

with $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^q$ is the input and $y_k \in \mathbb{R}^p$ is the output, where Φ is a nonlinear function, where $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times q}$, $\mathcal{C} \in \mathbb{R}^{p\times n}$, $\mathcal{A}_d(x) \in \mathbb{R}^{n\times n}$ are unknown terms bounded in norm by a known constant and where ν_k are disturbances such that for two known sequences ν_k^+, ν_k^- the inequalities:

$$\nu_k^- \leq \nu_k \leq \nu_k^+ \tag{9}$$

are satisfied for all $k \in \mathbb{N}$.

Some assumptions are introduced.

Assumption 1. *There exists an invertible matrix $\mathcal{R} \in \mathbb{R}^{n\times n}$ such that*

$$\mathcal{R}\mathcal{A}\mathcal{S} = \mathcal{E}, \tag{10}$$

where $\mathcal{E} \in \mathbb{R}^{n\times n}$ is a nonnegative Schur stable matrix and $\mathcal{S} = \mathcal{R}^{-1}$.

Notice for later use that Assumption 1 guarantees the existence of a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$\mathcal{A}^\top Q\mathcal{A} - Q \preceq -I. \tag{11}$$

Assumption 2. *There exist a positive definite and radially unbounded continuous function $\mathfrak{U}$ and a continuous feedback θ such that, along the system*

$$\xi_{k+1} = [\mathcal{A} + \mathcal{A}_d(\xi_k)]\xi_k + \mathcal{B}\theta(\mathcal{C}\xi_k, \xi_k + d_k) + \Phi(\mathcal{C}\xi_k) + s_k, \tag{12}$$

where d_k and s_k are any sequences, the function $\mathfrak{U}$ satisfies, for all $k \in \mathbb{N}$,

$$\mathfrak{U}(\xi_{k+1}) - \mathfrak{U}(\xi_k) \leq -|\xi_k|^2 + \mathfrak{c}|d_k|^2 + \mathfrak{g}|s_k|^2, \tag{13}$$

with $\mathfrak{c} > 0$, $\mathfrak{g} > 0$. Moreover, there are two constants $0 < \mathfrak{u}_s < \mathfrak{u}_l$ such that

$$\mathfrak{u}_s|\xi|^2 \leq \mathfrak{U}(\xi) \leq \mathfrak{u}_l|\xi|^2, \tag{14}$$

for all $\xi \in \mathbb{R}^n$.

Assumption 3. *There are continuous functions $\mathfrak{P}$, $\mathfrak{Q}$ and a known constant matrix $\mathfrak{K} \geq 0$ such that, for all $x \in \mathbb{R}^n$,*

$$\mathcal{R}\mathcal{A}_d(x)\mathcal{S} = \mathfrak{P}(x) - \mathfrak{Q}(x), \tag{15}$$

$$0 \leq \mathfrak{P}(x) \leq \mathfrak{K}\ ,\quad 0 \leq \mathfrak{Q}(x) \leq \mathfrak{K}, \tag{16}$$

and

$$|\mathfrak{K}| \leq \frac{1}{2|\mathcal{S}||\mathcal{R}|}\min\left\{\frac{1}{2\sqrt{\mathfrak{q}_2}}, \frac{\sqrt{3}}{\sqrt{5\mathfrak{c}(\mathfrak{q}_1 + 2\mathfrak{q}_2)}}\right\}, \tag{17}$$

with

$$\mathfrak{q}_1 = 2(|Q\mathcal{A}|^2 + |Q|)\ ,\ \mathfrak{q}_2 = 2(5|Q\mathcal{A}|^2 + 3|Q|). \tag{18}$$

The main result of the section stated below is proved in Appendix B

Theorem 1 *Let the system (8) satisfy Assumptions 1 to 3. Then the system:*

$$\left\{\begin{array}{rl} \hat{x}_{k+1}^+ &= \mathcal{A}\hat{x}_k^+ + \mathcal{B}u_k + \Phi(y_k) \\ &\quad +\mathcal{S}\mathfrak{K}[\max\{0, \mathcal{R}\hat{x}_k^+\} - \min\{0, \mathcal{R}\hat{x}_k^-\}] \\ &\quad +\mathcal{S}[\mathcal{R}^+\nu_k^+ - \mathcal{R}^-\nu_k^-], \\ \hat{x}_{k+1}^- &= \mathcal{A}\hat{x}_k^- + \mathcal{B}u_k + \Phi(y_k) \\ &\quad +\mathcal{S}\mathfrak{K}[\min\{0, \mathcal{R}\hat{x}_k^-\} - \max\{0, \mathcal{R}\hat{x}_k^+\}] \\ &\quad +\mathcal{S}[\mathcal{R}^+\nu_k^- - \mathcal{R}^-\nu_k^+], \end{array}\right. \tag{19}$$

associated with the initial conditions

$$\begin{aligned}\hat{x}^+_{k_0} &= \mathcal{S}\left[\mathcal{R}^+ x^+_{k_0} - \mathcal{R}^- x^-_{k_0}\right], \\ \hat{x}^-_{k_0} &= \mathcal{S}\left[\mathcal{R}^+ x^-_{k_0} - \mathcal{R}^- x^+_{k_0}\right],\end{aligned} \tag{20}$$

the bounds for the solutions x_k

$$\begin{aligned}x^+_k &= \mathcal{S}^+\mathcal{R}\hat{x}^+_k - \mathcal{S}^-\mathcal{R}\hat{x}^-_k, \\ x^-_k &= \mathcal{S}^+\mathcal{R}\hat{x}^-_k - \mathcal{S}^-\mathcal{R}\hat{x}^+_k,\end{aligned} \tag{21}$$

is a robust interval observer for the system (8) when in closed-loop with the feedback

$$u(\hat{x}^+_k, y_k) = \theta(y_k, \hat{x}^+_k). \tag{22}$$

Discussion of Theorem 1.

• Assumption 1 ensures the existence of a time-invariant change of coordinates that transforms $\mathcal{A}$ into a nonnegative Schur stable matrix. For the sake of simplicity, only this case is considered, although one can cope with the general case by taking advantage of the time-varying change of coordinates provided in Mazenc, Dinh & Niculescu (2012).

• Assumption 1 implies that the matrix $\mathcal{A}$ is Schur stable. But Assumptions 1, 2, 3 do not imply that, in the absence of $\mathcal{A}_d$, the system is globally or even locally exponentially stabilizable by a static output feedback. The example in Section 4 illustrates this fact. Besides, it is worth mentioning that any system $\xi_{k+1} = A\xi_k + Bu_k$ with output $y_k = C\xi_k$ such that (A, B) is stabilizable and (A, C) is observable satisfies Assumptions 1 to 3. Indeed, the observability of (A, C) ensures that there exists a matrix L such that $A + LC$ is Schur stable with distinct eigenvalues, which implies that Assumption 1 is satisfied (with $\mathcal{A} = A + LC$) and the stabilizability of (A, B) ensures that Assumption 2 is satisfied with a linear stabilizing feedback and finally, in the absence of disturbance, Assumption 3 is always satisfied.

• Assumption 2 is a stabilizability assumption by state feedback for (8). When (8) is a linear system, this assumption is always satisfied under the standard stabilizability assumption. It is also the case if Φ is of class C^1 and such that $\Phi(0) = 0$, the pair $\left(\mathcal{A} + \frac{\partial\Phi}{\partial y}(0)\mathcal{C}, \mathcal{B}\right)$ is stabilizable, the pair (A, C) is detectable and there exists a function Ω such that, for all $y \in \mathbb{R}^p$, $\Phi(y) - \frac{\partial\Phi}{\partial y}(0)y = \mathcal{B}\Omega(y)$. The requirement (14) is often satisfied in practice for systems of the type (12) and this assumption can be relaxed, but at the cost of an increased amount of complexity. So, for the sake of simplicity, this requirement is imposed.

• Assumption 3 is a restriction imposed on the unknown terms. Such a restriction is used to establish the stability of (8)-(19) in closed-loop with (22). The system (19), associated to (20) and (21) is a framer for the system (8) for any input. But it is an interval observer only when the system (8) is in closed-loop with suitably chosen stabilizing feedbacks. Of course feedbacks different from (22) can be used: for instance one can choose $u(\hat{x}^-_k, y_k) = \theta(y_k, \hat{x}^-_k)$. When $\mathcal{A}_d(x)$ is bounded in norm, finding functions $\mathfrak{P}(x)$, $\mathfrak{Q}(x)$ such that Assumption 3 is satisfied is an easy task. Indeed, assume that all the entries of $\mathcal{R}\mathcal{A}_d(x)\mathcal{S}$ are bounded in norm by constants $\mathfrak{p}_{i,j} \geq 0$. Let $\mathfrak{P}(x)$ be the matrix whose entries are $\mathfrak{p}_{i,j}$. Then $\mathcal{R}\mathcal{A}_d(x)\mathcal{S} = \mathfrak{P}(x) - \mathfrak{Q}(x)$ with $\mathfrak{Q}(x) = \mathfrak{P}(x) - \mathcal{R}\mathcal{A}_d(x)\mathcal{S}$. Moreover, $\mathfrak{P}(x) \geq 0$, $\mathfrak{Q}(x) \geq 0$ and $\mathfrak{P}(x) \leq \mathfrak{K}$, $\mathfrak{Q}(x) \leq \mathfrak{K}$ with $\mathfrak{K} = 2\mathfrak{P}(x)$. Conversely, when $\mathfrak{P}$ and $\mathfrak{Q}$ are bounded functions, necessarily $\mathcal{A}_d$ is bounded.

4 Illustrative example

In this section, Theorem 1 is illustrated through the following nonlinear system:

$$\begin{cases} x_{1,k+1} &= x_{1,k} + hx_{2,k}, \\ x_{2,k+1} &= x_{2,k} + hb_1x_{3,k} - h(a_1 + \varsigma)\sin(x_{1,k}) \\ &\quad -ha_2x_{2,k}, \\ x_{3,k+1} &= x_{3,k} + hb_0u_k - ha_3x_{2,k} - ha_4x_{3,k}, \\ y_k &= x_{1,k} \,. \end{cases} \tag{23}$$

This system is the model of electromechanical system described in D.M. Dawson et al. (1994) after discretization. We let $b_0 = 40, b_1 = 15, a_1 = 35, a_2 = \frac{1}{4}, a_3 = 36$, and $a_4 = 200$. These values are close to the numerical values given in D.M. Dawson et al. (1994). To illustrate the robustness of the proposed interval observers, the constant ς is supposed to be unknown and smaller in norm than a known constant $\bar{\varsigma}$. The system rewrites as:

$$\begin{cases} x_{1,k+1} &= hx_{2,k} + y_k, \\ x_{2,k+1} &= g_2x_{2,k} + hb_1x_{3,k} - ha_1\sin(y_k) + \nu_k, \\ x_{3,k+1} &= -ha_3x_{2,k} + g_3x_{3,k} + hb_0u_k, \\ y_k &= x_{1,k}, \end{cases} \tag{24}$$

with $g_2 = 1 - ha_2$, $g_3 = 1 - 200h$ and $\nu_k = -h\varsigma\sin(x_{1,k})$. Using the notation of Section 3 and selecting $h = \frac{1}{200}$, the following choices can be made: $\mathcal{A}_d(x_k) = 0$,

$$\mathcal{A} = \begin{bmatrix} 0 & \frac{1}{200} & 0 \\ 0 & \frac{799}{800} & \frac{3}{40} \\ 0 & -\frac{9}{50} & 0 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{5} \end{bmatrix}, \Phi(y) = \begin{bmatrix} y \\ -\frac{35}{200}\sin(y) \\ 0 \end{bmatrix}.$$

To check that Assumptions 1 to 3 are satisfied, introduce:

$$\mathcal{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{129} & -\frac{10}{129} \\ 0 & \frac{130}{129} & \frac{10}{129} \end{bmatrix}, \mathcal{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -13 & -\frac{1}{10} \end{bmatrix}. \tag{25}$$

Then $\mathcal{S} = \mathcal{R}^{-1}$ and

$$\mathcal{RAS} = \mathcal{E} \quad \text{with} \quad \mathcal{E} = \begin{bmatrix} 0 & \frac{1}{200} & \frac{1}{200} \\ 0 & \frac{1421}{103200} & \frac{647}{103200} \\ 0 & \frac{103}{10320} & \frac{2033}{2064} \end{bmatrix}. \tag{26}$$

The inequality $\mathcal{A}^\top Q \mathcal{A} - Q \preceq -I$ with the symmetric positive definite matrix $Q = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 50 & 4 \\ 0 & 4 & 5 \end{bmatrix}$ is satisfied and $\mathcal{E}$ is a nonnegative Schur stable matrix. Therefore Assumption 1 is satisfied.

The next step consists in determining a stabilizing control law for

$$\left\{ \begin{array}{rcl} \xi_{1,k+1} &=& \xi_{1,k} + h\xi_{2,k}, \\ \xi_{2,k+1} &=& \xi_{2,k} + hb_1\xi_{3,k} - ha_1 \sin(\xi_{1,k}) \\ && -ha_2\xi_{2,k} + s_{2,k}, \\ \xi_{3,k+1} &=& \xi_{3,k} + hb_0\theta(\xi_{1,k}, \xi_k + d_k) - ha_3\xi_{2,k} \\ && -ha_4\xi_{3,k}, \end{array} \right. \tag{27}$$

so that Assumption 2 is satisfied. The coordinates:

$$\alpha_k = \frac{1}{4}\xi_{1,k} + \xi_{2,k} \ , \ \beta_k = 15\xi_{3,k} - 35\sin(\xi_{1,k}) \tag{28}$$

lead to

$$\left\{ \begin{array}{rcl} \xi_{2,k+1} &=& -\frac{799}{800}\xi_{2,k} + \frac{1}{200}\beta_k + s_{2,k}, \\ \alpha_{k+1} &=& \alpha_k + \frac{1}{200}\beta_k + s_{2,k}, \\ \beta_{k+1} &=& 3\theta(\xi_{1,k}, \xi_k + d_k) - \frac{27}{10}\xi_{2,k} \\ && -35\sin\left(\xi_{1,k} + \frac{1}{200}\xi_{2,k}\right). \end{array} \right. \tag{29}$$

Let

$$\theta(y, \xi) \ = \ \frac{9}{10}\xi_2 + \frac{35\sin\left(y + \frac{1}{200}\xi_2\right)}{3} - \frac{199}{600}\left(\frac{1}{4}y + \xi_2\right). \tag{30}$$

Then, the system (29) in closed-loop with $\theta(\xi_{1,k}, \xi_k + d_k)$ is

$$\left\{ \begin{array}{rcl} \xi_{2,k+1} &=& -\frac{799}{800}\xi_{2,k} + \frac{1}{200}\beta_k + s_{2,k}, \\ \alpha_{k+1} &=& \alpha_k + \frac{1}{200}\beta_k + s_{2,k}, \\ \beta_{k+1} &=& -\frac{199}{200}\alpha_k + R(\xi_k, d_k), \end{array} \right. \tag{31}$$

with

$$\begin{array}{rcl} R(\xi_k, d_k) &=& \frac{341}{200}d_k + 35\sin\left(\xi_{1,k} + \frac{\xi_{2,k}}{200} + \frac{d_k}{200}\right) \\ && -35\sin\left(\xi_{1,k} + \frac{\xi_{2,k}}{200}\right). \end{array}$$

Consider the quadratic function

$$\mathfrak{W}(\alpha, \beta) = (\alpha + \beta)^2 + 2\left(\frac{199}{20}\alpha + \frac{1}{20}\beta\right)^2. \tag{32}$$

Then, with the simplifying notation $\Delta\mathfrak{W}_k = \mathfrak{W}(\alpha_{k+1}, \beta_{k+1}) - \mathfrak{W}(\alpha_k, \beta_k)$, one can prove through simple calculations that there are constants $\mathfrak{f}_1 > 0$, $\mathfrak{f}_2 > 0$ such that $\Delta\mathfrak{W}_k \leq -\mathfrak{f}_1\alpha_k^2 - \mathfrak{f}_2\beta_k^2$. Since the function sinus is globally Lipschitz, we deduce easily that there is a constant $\mathfrak{f}_3 > 0$ such that, when s_k and d_k are present, the inequality:

$$\Delta\mathfrak{W}_k \leq -\mathfrak{f}_1\alpha_k^2 - \mathfrak{f}_2\beta_k^2 + \mathfrak{f}_3[d_k^2 + s_k^2] \tag{33}$$

is satisfied. On the other hand, using the simplifying notation $\Delta\Xi_k = \xi_{2,k+1}^2 - \xi_{2,k}^2$, simple calculations give

$$\Delta\Xi_k \ \leq \ -\frac{799}{640000}\xi_{2,k}^2 + \frac{639201}{32000000}\beta_k^2. \tag{34}$$

It follows that there are constants $\mathfrak{f}_4 > 0$, $\mathfrak{f}_5 > 0$ such that

$$\mathfrak{R}(\alpha, \beta, \xi_2) = \mathfrak{f}_4\xi_2^2 + \mathfrak{W}(\alpha, \beta) \tag{35}$$

satisfies

$$\Delta\mathfrak{R}_k \ \leq \ -\mathfrak{f}_5\xi_{2,k}^2 - \mathfrak{f}_1\alpha_k^2 - \mathfrak{f}_2\beta_k^2 + \mathfrak{f}_3[d_k^2 + s_k^2], \tag{36}$$

with $\Delta\mathfrak{R}_k = \mathfrak{R}(\alpha_{k+1}, \beta_{k+1}, \xi_{2,k+1}) - \mathfrak{R}(\alpha_k, \beta_k, \xi_{2,k})$. Then, one can easily prove that there is $\mathfrak{f}_6 > 0$ such that Assumption 2 is satisfied with $\mathfrak{U}(\xi) = \mathfrak{f}_6\mathfrak{R}(a_2\xi_1 + \xi_2, b_1\xi_3 - a_1\sin(\xi_1), \xi_2)$.

Now, observe that Assumption 3 is satisfied since $\mathcal{A}_d = 0$. It follows that Theorem 1 applies to (24). Consequently, the dynamic extension

$$\left\{ \begin{array}{rcl} \hat{x}_{k+1}^+ &=& \mathcal{A}\hat{x}_k^+ + \mathcal{B}u_k + \Phi(y_k) + \mathcal{S}[\mathcal{R}^+\nu_k^+ - \mathcal{R}^-\nu_k^-], \\ \hat{x}_{k+1}^- &=& \mathcal{A}\hat{x}_k^- + \mathcal{B}u_k + \Phi(y_k) + \mathcal{S}[\mathcal{R}^+\nu_k^- - \mathcal{R}^-\nu_k^+], \end{array} \right. \tag{37}$$

with $\nu_k^+ = -\nu_k^- = \frac{\bar{\varsigma}}{200}[0 \quad 1 \quad 0]^\top$, associated with the initial conditions

$$\hat{x}_{k_0}^+ = \mathfrak{M}_1 x_{k_0}^+ + \mathfrak{M}_2 x_{k_0}^- \ , \ \hat{x}_{k_0}^- = \mathfrak{M}_1 x_{k_0}^- + \mathfrak{M}_2 x_{k_0}^+, \tag{38}$$

with $\mathfrak{M}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{130}{129} & \frac{10}{129} \\ 0 & -\frac{13}{129} & -\frac{1}{129} \end{bmatrix}$, $\mathfrak{M}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{129} & -\frac{10}{129} \\ 0 & \frac{13}{129} & \frac{130}{129} \end{bmatrix}$, and the bounds

$$x_k^+ = \mathfrak{N}_1\hat{x}_k^+ + \mathfrak{N}_2\hat{x}_k^- \ , \ x_k^- = \mathfrak{N}_1\hat{x}_k^- + \mathfrak{N}_2\hat{x}_k^+, \tag{39}$$

with $\mathfrak{N}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathfrak{N}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a robust interval observer for the system (24) when in closed-loop with

$$u(\hat{x}_k^+, y_k) = \tfrac{341}{600}\hat{x}_2^+ + \tfrac{35}{3}\sin\left(y + \tfrac{1}{200}\hat{x}_2^+\right) - \tfrac{199}{2400}y.$$

Fig. 1 below illustrates the result in the case where there is no disturbances ($\varsigma = 0$). A trajectory with $k_0 = 0, x_{k_0} = [20, 10, 5]^\top, x_{k_0}^+ = [23, 13, 8]^\top, x_{k_0}^- = [17, 7, 2]^\top, \hat{x}_{k_0}^+ = [23, \frac{581}{43}, \frac{58}{43}]^\top, \hat{x}_{k_0}^- = [17, \frac{279}{43}, \frac{372}{43}]^\top$ as initial conditions is simulated.
Fig. 2 below shows the solution with the same initial conditions in the case where the system is affected by disturbances $\varsigma = 8$.

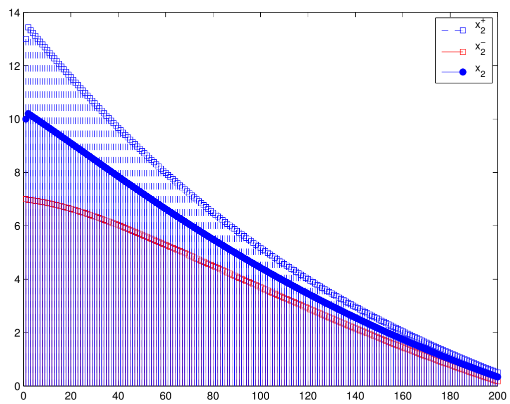

Fig. 1. Evolution with number of iterations of the state component x_2 and its bounds x_2^+, x_2^- without uncertainty

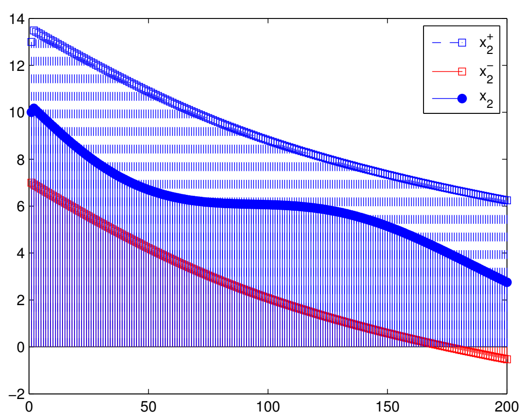

Fig. 2. Evolution with number of iterations of the state component x_2 and its bounds x_2^+, x_2^- with the uncertainties

5 Conclusion

A new technique of construction of stable interval observers for discrete-time nonlinear time-invariant systems with uncertainties has been developed. A key advantage of this approach is the simplicity of the dynamics of the proposed interval observer: basically, it is composed of two copies of a classical Luenberger observer with extra terms whose presence is due to the uncertain terms. Much remains to be done. Other types of robustness properties, such as robustness with respect to noises in the measurements, may be the subject of further studies. Extensions of the results to families of discrete-time time-varying systems with delays and their adaptation to the continuous-time case can be expected too.
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A Technical lemmas

The following result is a direct consequence of (Haddad et al., 2010, Chapt. 5, Proposition 5.6).

Lemma 1 *The system $z_{k+1} = \mathcal{A}z_k$ where $z_k \in \mathbb{R}^g$, $\mathcal{A} \in \mathbb{R}^{g\times g}$ is nonnegative if and only if the matrix $\mathcal{A}$ is nonnegative.*

Lemma 2 *Let $\xi_1 \in \mathbb{R}^g$, $\xi_2 \in \mathbb{R}^g$, $\xi_3 \in \mathbb{R}^g$ be vectors such that the inequalities*

$$\xi_1 \le \xi_2 \le \xi_3 \tag{A.1}$$

are satisfied. Let $M \in \mathbb{R}^{g\times g}$ be a constant matrix. Then the inequalities

$$M^+\xi_1 - M^-\xi_3 \le M\xi_2 \le M^+\xi_3 - M^-\xi_1 \tag{A.2}$$

are satisfied.

Proof. Since $M^+ \ge 0$ and $M^- \ge 0$, the inequalities $M^+\xi_1 \le M^+\xi_2 \le M^+\xi_3$, $-M^-\xi_3 \le -M^-\xi_2 \le -M^-\xi_1$ are satisfied. Since $M = M^+ - M^-$, it follows that the inequalities (A.2) are satisfied.

Lemma 3 *Let $\xi_{k+1} = \mathcal{A}\xi_k + \mu_k$, where $\xi_k \in \mathbb{R}^n$, $\mu_k \in \mathbb{R}^n$, $\mathcal{A} \in \mathbb{R}^{n\times n}$ is a matrix satisfying Assumption 1. Then $U(\xi) = \xi^\top Q\xi$ satisfies*

$$U(\xi_{k+1}) - U(\xi_k) \le -|\xi_k|^2 + 2\xi_k^\top \mathcal{A}^\top Q\mu_k + \mu_k^\top Q\mu_k. \tag{A.3}$$

Proof. The equality

$$\begin{aligned} U(\xi_{k+1}) - U(\xi_k) &= \xi_k^\top [\mathcal{A}^\top Q\mathcal{A} - \mathcal{A}]\xi_k \\ &+2\xi_k^\top \mathcal{A}^\top Q\mu_k + \mu_k^\top Q\mu_k \end{aligned} \tag{A.4}$$

and the inequality (11) lead to (A.3).

B Proof of Theorem 1

The proof splits up into two parts. The first establishes that the system (19) with suitable initial conditions and bounds is a framer for the system (8) for any sequence of inputs u_k. The second is devoted to the stability analysis of the systems (8)-(19) in closed-loop with the dynamic output feedback (22).

1. *Property of framer.*

Let us prove that (19) is a framer for the system (8) with the initial conditions (20) and the bounds for the solutions x_k given in (21). For an initial instant $k_0 \in \mathbb{N}$, consider vectors $x_{k_0}, x_{l,k_0}, x_{s,k_0}$ in $\mathbb{R}^n$ such that $x_{s,k_0} \le x_{k_0} \le x_{l,k_0}$. Then, by Lemma 2,

$$\mathcal{R}\hat{x}^-_{k_0} \le \mathcal{R}x_{k_0} \le \mathcal{R}\hat{x}^+_{k_0}, \tag{B.1}$$

where $\hat{x}^+_{k_0}$, $\hat{x}^-_{k_0}$ are the vectors defined in (20). Next, observe that the solutions x_k, $\hat{x}^+_k$, $\hat{x}^-_k$ of the systems (8) and (19) with the initial conditions $x_{k_0}, \hat{x}^+_{k_0}, \hat{x}^-_{k_0}$ selected above satisfy

$$\begin{cases} x_{k+1} = [\mathcal{A}+\mathcal{A}_d(x_k)]x_k + \mathcal{B}u_k + \Phi(y_k) + \nu_k, \\ \hat{x}^+_{k+1} = \mathcal{A}\hat{x}^+_k + \mathcal{B}u_k + \Phi(y_k) + \mathcal{S}\rho_1(\nu^+_k,\nu^-_k) \\ \qquad +\mathcal{S}\mathfrak{K}[\max\{0,\mathcal{R}\hat{x}^+_k\} - \min\{0,\mathcal{R}\hat{x}^-_k\}] \\ \hat{x}^-_{k+1} = \mathcal{A}\hat{x}^-_k + \mathcal{B}u_k + \Phi(y_k) + \mathcal{S}\rho_2(\nu^+_k,\nu^-_k) \\ \qquad +\mathcal{S}\mathfrak{K}[\min\{0,\mathcal{R}\hat{x}^-_k\} - \max\{0,\mathcal{R}\hat{x}^+_k\}], \end{cases} \tag{B.2}$$

with $\rho_1(\nu^+_k,\nu^-_k) = \mathcal{R}^+\nu^+_k - \mathcal{R}^-\nu^-_k$, $\rho_2(\nu^+_k,\nu^-_k) = \mathcal{R}^+\nu^-_k - \mathcal{R}^-\nu^+_k$. It follows that

$$\begin{cases} \mathcal{R}x_{k+1} = \mathcal{R}\mathcal{A}x_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) \\ \qquad +\mathcal{R}\mathcal{A}_d(x_k)x_k + \mathcal{R}\nu_k\ , \\ \mathcal{R}\hat{x}^+_{k+1} = \mathcal{R}\mathcal{A}\hat{x}^+_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) \\ \qquad +\mathfrak{K}[\max\{0,\mathcal{R}\hat{x}^+_k\} - \min\{0,\mathcal{R}\hat{x}^-_k\}] \\ \qquad +\rho_1(\nu^+_k,\nu^-_k) \\ \mathcal{R}\hat{x}^-_{k+1} = \mathcal{R}\mathcal{A}\hat{x}^-_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) \\ \qquad +\mathfrak{K}[\min\{0,\mathcal{R}\hat{x}^-_k\} - \max\{0,\mathcal{R}\hat{x}^+_k\}] \\ \qquad +\rho_2(\nu^+_k,\nu^-_k). \end{cases} \tag{B.3}$$

Since Assumption 1 guarantees that $\mathcal{R}\mathcal{A} = \mathcal{E}\mathcal{R}$ with the notation $z_k = \mathcal{R}x_k$, $\hat{z}^+_k = \mathcal{R}\hat{x}^+_k$, $\hat{z}^-_k = \mathcal{R}\hat{x}^-_k$, we obtain:

$$\begin{cases} z_{k+1} = \mathcal{E}z_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) \\ \qquad +\mathcal{R}\mathcal{A}_d(x_k)\mathcal{S}z_k + \mathcal{R}\nu_k, \\ \hat{z}^+_{k+1} = \mathcal{E}\hat{z}^+_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) \\ \qquad +\mathfrak{K}[\max\{0,\hat{z}^+_k\} - \min\{0,\hat{z}^-_k\}] \\ \qquad +\rho_1(\nu^+_k,\nu^-_k) \\ \hat{z}^-_{k+1} = \mathcal{E}\mathcal{R}\hat{x}^-_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) \\ \qquad +\mathfrak{K}[\min\{0,\hat{z}^-_k\} - \max\{0,\hat{z}^+_k\}] \\ \qquad +\rho_2(\nu^+_k,\nu^-_k). \end{cases} \tag{B.4}$$

According to Assumption 3, the z_k-subsystem rewrites as

$$\begin{aligned} z_{k+1} &= \mathcal{E}z_k + \mathcal{R}\mathcal{B}u_k + \mathcal{R}\Phi(y_k) \\ &\quad +[\mathfrak{P}(x_k) - \mathfrak{Q}(x_k)]z_k + \mathcal{R}\nu_k. \end{aligned} \tag{B.5}$$

Let $w^+_k = \hat{z}^+_k - z_k$, $w^-_k = z_k - \hat{z}^-_k$. Then, from (B.4) and (B.5), it follows that

$$\begin{cases} w^+_{k+1} = \mathcal{E}w^+_k + \mathfrak{K}[\max\{0,\hat{z}^+_k\} - \min\{0,\hat{z}^-_k\}] \\ \qquad -\mathfrak{P}(x_k)z_k + \mathfrak{Q}(x_k)z_k \\ \qquad +\mathcal{R}^+\nu^+_k - \mathcal{R}^-\nu^-_k - \mathcal{R}\nu_k \\ w^-_{k+1} = \mathcal{E}w^-_k - \mathfrak{K}[\min\{0,\hat{z}^-_k\} - \max\{0,\hat{z}^+_k\}] \\ \qquad +\mathfrak{P}(x_k)z_k - \mathfrak{Q}(x_k)z_k \\ \qquad +\mathcal{R}\nu_k - \mathcal{R}^+\nu^-_k + \mathcal{R}^-\nu^+_k. \end{cases}$$

Next, by reorganizing the terms, we obtain

$$\begin{cases} w^+_{k+1} = \mathcal{E}w^+_k + \mathfrak{K}\max\{0,\hat{z}^+_k\} - \mathfrak{P}(x_k)z_k \\ \qquad +\mathfrak{Q}(x_k)z_k - \mathfrak{K}\min\{0,\hat{z}^-_k\} \\ \qquad +\mathcal{R}^+(\nu^+_k - \nu_k) + \mathcal{R}^-(\nu_k - \nu^-_k) \\ w^-_{k+1} = \mathcal{E}w^-_k + \mathfrak{P}(x_k)z_k - \mathfrak{K}\min\{0,\hat{z}^-_k\} \\ \qquad +\mathfrak{K}\max\{0,\hat{z}^+_k\} - \mathfrak{Q}(x_k)z_k \\ \qquad +\mathcal{R}^+(\nu_k - \nu^-_k) + \mathcal{R}^-(\nu^+_k - \nu_k). \end{cases} \tag{B.6}$$

Now, we prove by induction that for all $k \geq k_0$,

$$\hat{z}^-_k \leq z_k \leq \hat{z}^+_k. \tag{B.7}$$

According to (B.1), the property is satisfied at the instant k_0. Assume that there exists $j > k_0$ such that, for all $i \in \{k_0, ..., j-1\}$, $\hat{z}^-_i \leq z_i \leq \hat{z}^+_i$. Since, for all $x \in \mathbb{R}^n$, $0 \leq \mathfrak{P}(x) \leq \mathfrak{K}$, it follows that, for all $i \in \{k_0, ..., j-1\}$,

$$0 \leq \mathfrak{P}(x_i)\max\{0,z_i\} \leq \mathfrak{K}\max\{0,z_i\} \leq \mathfrak{K}\max\{0,\hat{z}^+_i\},$$

$$0 \leq -\mathfrak{P}(x_i)\min\{0,z_i\} \leq -\mathfrak{K}\min\{0,z_i\} \leq -\mathfrak{K}\min\{0,\hat{z}^-_i\}.$$

Therefore $\mathfrak{K}\min\{0,\hat{z}^-_i\} \leq \mathfrak{P}(x_i)z_i \leq \mathfrak{K}\max\{0,\hat{z}^+_i\}$. Similarly, $\mathfrak{K}\min\{0,\hat{z}^-_i\} \leq \mathfrak{Q}(x_i)z_i \leq \mathfrak{K}\max\{0,\hat{z}^+_i\}$. Consequently,

$$\mathfrak{K}\max\{0,\hat{z}^+_i\} - \mathfrak{P}(x_i)z_i + \mathfrak{Q}(x_i)z_i - \mathfrak{K}\min\{0,\hat{z}^-_i\} \geq 0 \tag{B.8}$$

and

$$\mathfrak{P}(x_i)z_i - \mathfrak{K}\min\{0,\hat{z}^-_i\} + \mathfrak{K}\max\{0,\hat{z}^+_i\} - \mathfrak{Q}(x_i)z_i \geq 0. \tag{B.9}$$

From (B.8), (B.9), the inequalities $\nu_k - \nu^-_k \geq 0$, $\nu^+_k - \nu_k \geq 0$ and (B.6), we deduce that $w^+_j \geq 0$ and $w^-_j \geq 0$ and therefore

$$\hat{z}^-_j \leq z_j \leq \hat{z}^+_j.$$

Therefore the induction assumption is satisfied at the step $j+1$. We deduce that, for all $k \geq k_0$, the inequalities $\mathcal{R}\hat{x}^-_k \leq \mathcal{R}x_k \leq \mathcal{R}\hat{x}^+_k$ hold.

From Lemma 2, it follows that, for all $k \geq k_0$,

$$\mathcal{S}^+\mathcal{R}\hat{x}^-_k - \mathcal{S}^-\mathcal{R}\hat{x}^-_k \leq x_k \leq \mathcal{S}^+\mathcal{R}\hat{x}^+_k - \mathcal{S}^-\mathcal{R}\hat{x}^-_k. \tag{B.10}$$

Thus, for all integer $k \geq k_0$,

$$x^-_k \ \leq \ x_k \ \leq \ x^+_k. \tag{B.11}$$

2. *Stability of the system (8) - (19) - (22).*

In the absence of additive disturbances, the closed loop system is:

$$\begin{cases} \hat{x}_{k+1}^{+} = \mathcal{A}\hat{x}_k^{+} + \mathcal{S}\mathfrak{K}[\max\{0,\mathcal{R}\hat{x}_k^{+}\} - \min\{0,\mathcal{R}\hat{x}_k^{-}\}] \\ \qquad\quad +\mathcal{B}\theta(y_k,\hat{x}_k^{+}) + \Phi(y_k) \\ x_{k+1} = [\mathcal{A} + \mathcal{A}_d(x_k)]x_k + \mathcal{B}\theta(y_k,\hat{x}_k^{+}) + \Phi(y_k) \\ \hat{x}_{k+1}^{-} = \mathcal{A}\hat{x}_k^{-} + \mathcal{S}\mathfrak{K}[\min\{0,\mathcal{R}\hat{x}_k^{-}\} - \max\{0,\mathcal{R}\hat{x}_k^{+}\}] \\ \qquad\quad +\mathcal{B}\theta(y_k,\hat{x}_k^{+}) + \Phi(y_k). \end{cases} \tag{B.12}$$

Let $p_k = \hat{x}_k^{+} - x_k$ and $q_k = \hat{x}_k^{+} - \hat{x}_k^{-}$. Then

$$\begin{cases} q_{k+1} = \mathcal{A}q_k + 2\mathfrak{F}(r_k) \\ p_{k+1} = \mathcal{A}p_k + \mathfrak{G}(r_k) \\ x_{k+1} = [\mathcal{A} + \mathcal{A}_d(x_k)]x_k + \mathcal{B}\theta(y_k, x_k + p_k) \\ \qquad\quad +\Phi(y_k), \end{cases} \tag{B.13}$$

with $r_k = (p_k, q_k, x_k)$, $\mathfrak{F}(r_k) = \mathcal{S}\mathfrak{K}[\max\{0,\mathcal{R}(p_k+x_k)\} - \min\{0,\mathcal{R}(p_k + x_k - q_k)\}]$, $\mathfrak{G}(r_k) = -\mathcal{A}_d(x_k)x_k + \mathfrak{F}(r_k)$. To establish the asymptotic stability of the system (B.13), consider first the positive definite quadratic function

$$\mathcal{V}(p,q) = p^\top Qp + q^\top Qq. \tag{B.14}$$

From Lemma 3, it follows that

$$\begin{aligned} \Delta\mathcal{V}_k \le\ & -|p_k|^2 - |q_k|^2 + \mathfrak{G}(r_k)^\top Q\mathfrak{G}(r_k) \\ & +2p_k^\top\mathcal{A}^\top Q\mathfrak{G}(r_k) + 4\mathfrak{F}(r_k)^\top Q\mathfrak{F}(r_k) \\ & +4q_k^\top\mathcal{A}^\top Q\mathfrak{F}(r_k) \end{aligned} \tag{B.15}$$

with the simplifying notation $\Delta\mathcal{V}_k = \mathcal{V}(p_{k+1}, q_{k+1}) - \mathcal{V}(p_k, q_k)$. The triangle inequality gives

$$\begin{aligned} \Delta\mathcal{V}_k \le\ & -\tfrac{1}{2}|p_k|^2 - \tfrac{1}{2}|q_k|^2 + \mathfrak{G}(r_k)^\top Q\mathfrak{G}(r_k) \\ & +|Q\mathcal{A}|^2|\mathfrak{G}(r_k))|^2 + 4\mathfrak{F}(r_k)^\top Q\mathfrak{F}(r_k) \\ & +8|Q\mathcal{A}|^2|\mathfrak{F}(r_k)|^2. \end{aligned} \tag{B.16}$$

It follows that

$$\begin{aligned} \Delta\mathcal{V}_k \le\ & -\tfrac{1}{2}|p_k|^2 - \tfrac{1}{2}|q_k|^2 \\ & +(|Q\mathcal{A}|^2 + |Q|)|\mathcal{A}_d(x_k)x_k - \mathfrak{F}(r_k)|^2 \\ & +4(2|Q\mathcal{A}|^2 + |Q|)|\mathfrak{F}(r_k)|^2 \\ \le\ & -\tfrac{1}{2}|p_k|^2 - \tfrac{1}{2}|q_k|^2 \\ & +\mathfrak{q}_1|\mathcal{A}_d(x_k)|^2|x_k|^2 + \mathfrak{q}_2|\mathfrak{F}(r_k)|^2, \end{aligned} \tag{B.17}$$

with $\mathfrak{q}_1$, $\mathfrak{q}_2$ defined in (18). Now, observe that Assumption 3 implies that, for all $x \in \mathbb{R}^n$, $|\mathfrak{P}(x) - \mathfrak{Q}(x)| \le |\mathfrak{K}|$. It follows that

$$|\mathcal{A}_d(x)| \le \mathfrak{q}_3|\mathfrak{K}|, \tag{B.18}$$

with $\mathfrak{q}_3 = |\mathcal{S}||\mathcal{R}|$. Besides, for all $r \in \mathbb{R}^{3n}$

$$|\mathfrak{F}(r)| \le |\mathcal{S}||\mathcal{R}||\mathfrak{K}|[|p + x| + |q|]. \tag{B.19}$$

Therefore

$$|\mathfrak{F}(r_k)|^2 \le 2\mathfrak{q}_3^2|\mathfrak{K}|^2[(|p_k| + |q_k|)^2 + |x_k|^2]. \tag{B.20}$$

It follows that

$$\begin{aligned} \Delta\mathcal{V}_k \le\ & -\tfrac{1}{2}|p_k|^2 - \tfrac{1}{2}|q_k|^2 + \mathfrak{q}_1\mathfrak{q}_3^2|\mathfrak{K}|^2|x_k|^2 \\ & +2\mathfrak{q}_2\mathfrak{q}_3^2|\mathfrak{K}|^2[(|p_k| + |q_k|)^2 + |x_k|^2] \\ \le\ & -\tfrac{1}{2}|p_k|^2 - \tfrac{1}{2}|q_k|^2 + \mathfrak{q}_4|\mathfrak{K}|^2|x_k|^2 \\ & +4\mathfrak{q}_2\mathfrak{q}_3^2|\mathfrak{K}|^2(|p_k|^2 + |q_k|^2), \end{aligned} \tag{B.21}$$

with $\mathfrak{q}_4 = \mathfrak{q}_1\mathfrak{q}_3^2 + 2\mathfrak{q}_2\mathfrak{q}_3^2$. Since the inequality $|\mathfrak{K}| \le \frac{1}{2|\mathcal{S}||\mathcal{R}|}\frac{1}{2\sqrt{\mathfrak{q}_2}}$ in Assumption 3 implies $|\mathfrak{K}|^2 \le \frac{1}{16\mathfrak{q}_2\mathfrak{q}_3^2}$, the inequality

$$\Delta\mathcal{V}_k \le -\tfrac{1}{4}|p_k|^2 - \tfrac{1}{4}|q_k|^2 + \mathfrak{q}_4|\mathfrak{K}|^2|x_k|^2 \tag{B.22}$$

holds. Now, observe that x_k satisfies:

$$x_{k+1} = [\mathcal{A} + \mathcal{A}_d(x_k)]x_k + \mathcal{B}\theta(\mathcal{C}x_k, x_k + p_k) + \Phi(\mathcal{C}x_k).$$

From Assumption 2, it straightforwardly follows that

$$\mathfrak{U}(x_{k+1}) - \mathfrak{U}(x_k) \le -|x_k|^2 + \mathfrak{c}|p_k|^2. \tag{B.23}$$

The inequalities (B.22) and (B.23) lead us to consider

$$\mathcal{W}(r) = \mathfrak{U}(x) + 5\mathfrak{c}\mathcal{V}(p,q). \tag{B.24}$$

This function is positive definite and radially unbounded and

$$\Delta\mathcal{W}_k \le \left(-1 + 5\mathfrak{c}\mathfrak{q}_4|\mathfrak{K}|^2\right)|x_k|^2 - \frac{\mathfrak{c}}{4}|p_k|^2 - \frac{5\mathfrak{c}}{4}|q_k|^2, \tag{B.25}$$

with $\Delta\mathcal{W}_k = \mathcal{W}(r_{k+1}) - \mathcal{W}(r_k)$. Finally, observe that the inequality $\mathfrak{K} \le \frac{1}{2|\mathcal{S}||\mathcal{R}|}\frac{\sqrt{3}}{\sqrt{5\mathfrak{c}(\mathfrak{q}_1+2\mathfrak{q}_2)}}$ in Assumption 3 implies that

$$|\mathfrak{K}|^2 \le \frac{3}{20\mathfrak{c}\mathfrak{q}_4}, \tag{B.26}$$

which leads to the inequality

$$\Delta\mathcal{W}_k \le -\frac{1}{4}|x_k|^2 - \frac{\mathfrak{c}}{4}|p_k|^2 - \frac{5\mathfrak{c}}{4}|q_k|^2. \tag{B.27}$$

Now, bearing in mind Assumption 2, we deduce easily that, when the disturbances are present, there is a constant $\mathfrak{n} > 0$ such that

$$\Delta\mathcal{W}_k \le -\tfrac{1}{4}|x_k|^2 - \tfrac{\mathfrak{c}}{4}|p_k|^2 - \tfrac{5\mathfrak{c}}{4}|q_k|^2 \\ +\mathfrak{n}(|x_k|+|p_k|+|q_k|)(|\nu_k^+|+|\nu_k^-|) + \mathfrak{g}|\nu_k|^2. \tag{B.28}$$

The triangle inequality implies that

$$\Delta\mathcal{W}_k \le -\tfrac{1}{8}|x_k|^2 - \tfrac{\mathfrak{c}}{8}|p_k|^2 - \tfrac{\mathfrak{c}}{4}|q_k|^2 \\ + \left[2+\tfrac{3}{\mathfrak{c}}\right]\mathfrak{n}^2(|\nu_k^+|+|\nu_k^-|)^2 + \mathfrak{g}|\nu_k|^2. \tag{B.29}$$

The inequality (14) implies that there is a constant $\mathfrak{w}$ such that

$$\frac{1}{8}|x_k|^2 + \frac{\mathfrak{c}}{8}|p_k|^2 + \frac{\mathfrak{c}}{4}|q_k|^2 \ge \mathfrak{w}\mathcal{W}(r_k)$$

and since $|\nu_k| \le |\nu_k^+| + |\nu_k^-|$, there is a constant $\mathfrak{y} > 0$ such that

$$\Delta\mathcal{W}_k \le -\mathfrak{w}\mathcal{W}(r_k) + \mathfrak{y}(|\nu_k^+|+|\nu_k^-|)^2. \tag{B.30}$$

Assume that the sequence $|\nu_k^+| + |\nu_k^-|$ is bounded by a constant $\overline{\nu} > 0$. By using (B.30) one can prove that there is an integer k_0 such that, for all integer $k \ge k_0$, $\mathcal{W}(r_k) \le 2\frac{\mathfrak{y}}{\mathfrak{w}}\overline{\nu}^2$. This inequality and (14) allow us to conclude.