

# Practical stabilization of nonlinear systems with state-dependent sampling and retarded inputs

Frédéric Mazenc, Claudio de Persis and Mounir Bekaik

**Abstract**—A solution to the problem of stabilizing nonlinear systems with input with a constant pointwise delay and state-dependent sampling is proposed. It relies on a recursive construction of the sampling instants and on a recent variant of the classical reduction model approach. State feedbacks without distributed terms are obtained. A lower bound on the maximal allowable delay is determined via a Lyapunov-Krasovskii analysis.

## I. INTRODUCTION

The importance of control problems of systems with sampled input and output is a well acknowledge fact in the control community. This importance is clearly explained in particular in [8], [17]. A large body of literature is devoted to these control problems as sampling significantly increases the complexity. A similar comment applies to systems with delay: for more information on systems with delay, see [10]. Consequently, some studies are devoted to systems that are affected both by sampling and delay. Most of them present results that rely on delay-dependent conditions [7], [18] and the very recent paper [9] presents, for a wide family of nonlinear systems, a new control strategy based on a predictor-based compensation of delays that allows to cope with arbitrarily large delays. Related methods have been employed in control systems under communication constraints to cope with sampling, quantization and delays (see e.g. [3], [5], [13]). The stabilizing control laws in [9] are given by expressions that incorporate past values of the controls. However, to the best of our knowledge, no work addresses the problem of the stabilization of a system with delay in the inputs and state-dependent sampling. This problem, which is motivated by the fact that state-dependent sampling may potentially lead to a reduced sampling rate, is the one we address in the present work. Its difficulty arises from the fact that state-dependent sampling precludes the utilization of the classical reduction model approaches (see for instance [14], [1], [11]), even in the linear case. This leads us to use another technique: the main result we propose relies on the recent contribution [16]. Let us briefly recall its main features. The nonlinear system with delay

$$\dot{x}(t) = f(t, x(t)) + f_\tau(t - \tau, x(t - \tau))u(t - \tau), \quad (1)$$

F. Mazenc is with Team INRIA DISCO, CNRS-Supélec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France [Frederic.MAZENC@lss.supelec.fr](mailto:Frederic.MAZENC@lss.supelec.fr)

Claudio de Persis ITM, Faculty of Mathematics and Natural Sciences, University of Groningen, the Netherlands, Tel: +31 50 363 3080, Email: [c.depersis@rug.nl](mailto:c.depersis@rug.nl), and Dipartimento di Informatica e Sistemistica, Sapienza Università di Roma, Via Ariosto 25, 00185 Roma, Italy.

Mounir Bekaik, Team INRIA DISCO, CNRS-Supélec, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France (e-mail: [Mounir.BEKAIK@lss.supelec.fr](mailto:Mounir.BEKAIK@lss.supelec.fr)).

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^p$ ,  $\tau > 0$  is a pointwise delay and  $f$  and  $f_\tau$  are globally Lipschitz functions, is considered. For it, a new strategy of design of stabilizing state feedbacks is presented. It consists of a variant of the reduction model approach. Two of its main desirable features are: (i) it leads to continuous globally asymptotically stabilizing control laws of the form  $u(t, x(t - \tau))$  and (ii) it provides a Lyapunov-Krasovskii functional for the closed-loop system, from which some robustness properties can be derived. Its main limitation is that it applies only if suitable delay dependent conditions are satisfied.

The main result of the present work has both the advantages and the limitation of the main result of [16] we have mentioned. Another advantage of our approach is that it is based on a state-dependent sampling technique that avoids unnecessary samplings by taking into account the difference between the actual value of the sampled control and the value of the desired control. It is inspired by the quantized control problems studied in [4], [6], [12]. One can easily prove that systems of the family (1) cannot be asymptotically stabilized when it is used. However, we will determine control laws that globally *practically* stabilize the origin of the considered system, relative to the sampling period. In other words, we will determine control laws that render attractive a neighborhood of the origin whose size is proportional to the accuracy of the sampling.

The control design and the stability analysis for the closed-loop system we propose can be decomposed into three steps. In a first step, we give the analytic expression of the stabilizing control laws we consider and we describe the sampling procedure we adopt. In a second step, we prove that this control strategy results in a forward-complete system (see [10] for the definition of forward-complete system) for which there is no solution whose corresponding sequence of sampling times accumulate in finite time. In a third step, we prove the stability of the closed-loop system by revisiting the proof of the main result of [16]. This last step is needed because the stability analysis of [16] cannot be directly adapted to the closed-loop system we obtain because the sampling procedure introduces terms that preclude the use of the Lyapunov-Krasovskii functional provided in [16]. From the new Lyapunov-Krasovskii functional we construct, we will determine a lower bound on the maximal delay for which a feedback that is independent of the past values of the inputs and globally practically stabilizes the system can be designed.

To the best of our knowledge, the result we propose is new, even when particularized to the family of the linear

time-invariant systems and it does not seem to us that other techniques of control are more suitable than the one we adopt. We presume that the technique of [6], where we solved the problem of stabilizing a family of nonlinear systems with discontinuous retarded inputs by using a Lyapunov construction of the control laws that is reminiscent of ideas of the construction of [15], can be applied to solve the problem that is investigated in the present study. But we do not think that it would lead to a simpler or less conservative result.

The paper is organized as follows. Section II presents the control problem we consider. In Section III, the sampling strategy and the control law are described and studied. The main result is stated and proved in Section IV. An illustration is given in Section V. Concluding remarks in Section VI end the paper.

*Notation and definitions.* • Denote  $|\cdot|$  the Euclidean norm of matrices and vectors of any dimension. • Given  $\phi : I \rightarrow \mathbb{R}^p$  defined on an interval  $I$ , denote its (essential) supremum over  $I$  by  $|\phi|_I$ . • Let  $p$  be any positive integer and  $\tau$  be a positive real number. We denote  $C_{\text{in}}$  the set of all  $\mathbb{R}^p$ -valued Lipschitz functions defined on a given interval  $[-\tau, 0]$ . • For a continuous function  $\xi : [-\tau, +\infty) \rightarrow \mathbb{R}^k$ , for all  $t \geq 0$ , the function  $\xi_t$  defined by  $\xi_t(m) = \xi(t+m)$  for all  $m \in [-\tau, 0]$  is sometimes called translation operator. • For any real number  $\Delta > 0$ , let  $\Delta = \{i\Delta, i \in \mathbb{Z}\}$ . • The notations will be simplified whenever no confusion can arise from the context.

## II. PARTICULAR CONTROL PROBLEM

We consider the nonlinear time-varying system

$$\dot{x}(t) = f(t, x(t)) + h(t - \tau, x(t - \tau))u(t_k - \tau), \quad t \in [t_k, t_{k+1}), \quad (2)$$

where  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}^p$  is the input, with  $p = 1$  for the sake of simplicity,  $\tau > 0$  is a constant delay and  $f$  and  $h$  are nonlinear Lipschitz system functions. The times  $t_k$ , with  $k \in \mathbb{N} \cup \{0\}$  and  $t_0 := 0$ , are the times at which a new zero-order control value is applied. The notation  $u(t_k - \tau)$  refers to the fact that the (feedback) control value applied during the interval  $[t_k, t_{k+1})$  depends on the state available at time  $t_k - \tau$ . The state is sampled at times  $t_k - \tau$  and the corresponding control value is applied  $\tau$  units of time later. Thus, the delay  $\tau$  in  $u(t_k - \tau)$  takes into account the difference between the time the control law is computed and the time it is actually applied (for instance due to the presence of a delayed communication channel). Having the same delay  $\tau$  in  $u$  (due to computation and transmission) and  $h$  (due to the nature of the process to control) may be unrealistic. However, we observe that in those cases where the delay in  $h$  is larger than the one in  $u$ , one can always introduce an artificial delay in  $u$  to match the difference. Although this may cause some loss of performance, it allows us to deal with the two sources of delays in a unified manner.

The problem we consider is to design a sequence of sampling times and a zero-order control law  $u$  that result in a practically stable closed-loop system. We solve it under the following assumptions:

**Assumption 1.** *There exist a continuously differentiable function  $g(t, x)$  and a constant Hurwitz matrix  $L$  such that,*

for all  $x \in \mathbb{R}^n$  and  $t \geq -\tau$ , the equality

$$\bar{f}(t, x) = e^{-L\tau}h(t, x)g(t, x), \quad (3)$$

where

$$\bar{f}(t, x) = f(t, x) - Lx, \quad (4)$$

is satisfied.

**Assumption 2.** *There exists a real number  $f_0 > 0$  such that, for all  $x \in \mathbb{R}^n$ ,*

$$\sup_{t \geq -\tau} |f(t, x)| \leq f_0|x|, \quad (5)$$

$$\sup_{t \geq -\tau} |\bar{f}(t, x)| \leq f_0|x|, \quad (6)$$

$$\sup_{t \geq -\tau} |h(t, x)| \leq f_0(|x| + 1). \quad (7)$$

Observe for later use that Assumption 1 implies that there is a symmetric and positive definite matrix  $P \in \mathbb{R}^{n \times n}$  and a positive real number  $c$  such that the inequality

$$PL + L^T P \leq -cP \quad (8)$$

holds. Let  $q_p$  denote the smallest eigenvalue of  $P$ . Notice for later use that  $q_p > 0$ . Let us introduce the constant

$$r_p = \frac{q_p}{|P|}. \quad (9)$$

In the next section we define sequences of sampling times and control values leading to practically stable systems.

## III. STATE-DEPENDENT SAMPLING AND CONTROL GENERATION

The sequence of sampling times is iteratively designed relying on the initial condition of the system and its response. The control value is initialized as a function of the initial condition of the system and when a function of the state exceeds certain thresholds, which depend on the current control value, a new sampling time and a new control value are determined. We describe the process rigorously below.

Let  $\Delta$  be an arbitrary positive constant and  $\varphi \in C_{\text{in}}$  be the initial condition of a solution  $x$  of the system (2) such that  $x(t) = \varphi(t)$  for all  $t \in [t_0 - \tau, t_0]$ , where  $t_0 = 0$ . Let

$$\mathcal{B} = \max_{m \in [-\tau, 0]} |\varphi(m)| \quad (10)$$

and  $\mathcal{C}$  be a constant such that, for all  $(t_1, t_2) \in [-\tau, 0] \times [-\tau, 0]$ ,

$$|\varphi(t_1) - \varphi(t_2)| \leq \mathcal{C}|t_1 - t_2|. \quad (11)$$

First of all, identify that number  $\mu_0 \in \Delta\mathbb{Z}$  such that

$$\mu_0 - \frac{\Delta}{2} \leq -g(t_0 - \tau, \varphi(t_0 - \tau)) < \mu_0 + \frac{\Delta}{2}$$

and set accordingly

$$u(t_0 - \tau) = \mu_0 \text{ if } \mu_0 - \frac{\Delta}{2} < -g(t_0 - \tau, \varphi(t_0 - \tau)) < \mu_0 + \frac{\Delta}{2}$$

and

$$u(t_0 - \tau) = \mu_0 - \frac{\Delta}{2} \text{ if } \mu_0 - \frac{\Delta}{2} = -g(t_0 - \tau, \varphi(t_0 - \tau))$$

Observe that, as a consequence of the design of  $u(t_0 - \tau)$ ,

$$u(t_0 - \tau) - \frac{\Delta}{2} < -g(t_0 - \tau, \varphi(t_0 - \tau)) < u(t_0 - \tau) + \frac{\Delta}{2}. \quad (12)$$

The control value  $u(t_0 - \tau)$  is applied during the interval  $[t_0, t_1)$ , where the time  $t_1$  is a real number in  $(t_0, \tau]$  that is equal to  $\tau$  if

$$u(t_0 - \tau) - \frac{\Delta}{2} < -g(l - \tau, \varphi(l - \tau)) < u(t_0 - \tau) + \frac{\Delta}{2}$$

for all  $l \in [0, \tau]$  or is such that

$$u(t_0 - \tau) - \frac{\Delta}{2} < -g(l - \tau, \varphi(l - \tau)) < u(t_0 - \tau) + \frac{\Delta}{2}, \forall l \in [0, t_1)$$

and either  $u(t_0 - \tau) + \frac{\Delta}{2} = -g(t_1 - \tau, \varphi(t_1 - \tau))$  or  $u(t_0 - \tau) - \frac{\Delta}{2} = -g(t_1 - \tau, \varphi(t_1 - \tau))$ . Moreover, if the former equality holds, then we set  $u(t_1 - \tau) = u(t_0 - \tau) + \frac{\Delta}{2}$ , otherwise  $u(t_1 - \tau) = u(t_0 - \tau) - \frac{\Delta}{2}$ .

The rest of the sampling times  $t_k$  and the values of  $u$  are defined similarly: given the increasing sequence of times  $t_0, t_1, \dots, t_{k-1}$  in  $[0, \tau]$ , with  $k \geq 2$ , the next time  $t_k$

(i) does not belong to  $[0, \tau]$  if  $t_{k-1} = \tau$ ,

(ii) is  $\tau$  if  $t_{k-1} < \tau$  and for all  $l \in [t_{k-1}, \tau]$ ,

$$u(t_{k-1} - \tau) - \frac{\Delta}{2} < -g(l - \tau, \varphi(l - \tau)) < u(t_{k-1} - \tau) + \frac{\Delta}{2}, \quad (13)$$

(iii) is the smallest value  $r \in (t_{k-1}, \tau]$  such that either  $u(t_{k-1} - \tau) + \frac{\Delta}{2} = -g(r - \tau, \varphi(r - \tau))$  or  $u(t_{k-1} - \tau) - \frac{\Delta}{2} = -g(r - \tau, \varphi(r - \tau))$  if  $t_{k-1} < \tau$  and (13) is not satisfied for all  $l \in [t_{k-1}, \tau]$ .

Then,

- in the case (ii),  $u(t_k - \tau) = u(t_{k-1} - \tau)$ ,
- and in the case (iii),

$$u(t_k - \tau) = u(t_{k-1} - \tau) + \frac{\Delta}{2}$$

if  $-g(t_k - \tau, \varphi(t_k - \tau)) = u(t_{k-1} - \tau) + \frac{\Delta}{2}$  and

$$u(t_k - \tau) = u(t_{k-1} - \tau) - \frac{\Delta}{2}$$

if  $-g(t_k - \tau, \varphi(t_k - \tau)) = u(t_{k-1} - \tau) - \frac{\Delta}{2}$ . Observe for later use that by construction, for all  $t \in [t_{k-1}, t_k)$ ,

$$u(t_{k-1} - \tau) - \frac{\Delta}{2} < -g(t - \tau, \varphi(t - \tau)) < u(t_{k-1} - \tau) + \frac{\Delta}{2}. \quad (14)$$

Now, we give a technical result, which will be instrumental in proving that an accumulation in finite time of the  $t_k$ 's does not occur.

*Lemma 1:* Assume that the system (2) satisfies Assumptions 1 and 2. Then, when the control constructed above is applied, there is a finite number  $\ell_0$  of times  $t_k$  in  $[0, \tau]$ .

*Proof:* The proof is omitted.  $\blacksquare$

Let us establish the following fact, which will allow us to prove by induction the forward-completeness of the closed-loop system we consider.

*Lemma 2:* Assume that the system (2) satisfies Assumptions 1 and 2. Then, when the control constructed above is applied, the solution  $x(t)$  exists for all  $t \in [-\tau, \tau]$  and is globally Lipschitz on  $[-\tau, \tau]$ .

*Proof:* To begin with, we prove that  $x(t)$  exists over  $[0, \tau]$ ,  $\mathcal{X}_1 := \max_{t \in [0, \tau]} |x(t)|$  is finite and there exists a constant  $\mathcal{X}_2$  such that, for all  $t_1 \in [0, \tau]$ ,  $t_2 \in [0, \tau]$ ,

$$|x(t_1) - x(t_2)| \leq \mathcal{X}_2 |t_1 - t_2|.$$

From Lemma 1, we deduce that  $[0, \tau] = \cup_{i=0}^{\ell_0-1} [t_i, \min\{\tau, t_{i+1}\}]$ . Let  $t \in [0, \tau]$  be such that  $x(\cdot)$  is defined over  $[-\tau, t)$ . Let  $k \in \{0, \dots, \ell_0 - 1\}$  be such that  $t \in [t_k, t_{k+1})$ . Bearing Assumptions 1 and 2 in mind, and using the Lyapunov function  $S(x) = x^\top x$ , we deduce through elementary calculations that for  $l \in [t_k, t)$ , we have the bound

$$S(x(l)) \leq e^{4f_0(t-t_k)} S(x(t_k)) + \frac{1}{2} f_0^2 (t-t_k) e^{4f_0(t-t_k)} (\mathcal{B} + 1)^2 |u(t_k - \tau)|^2,$$

with  $\mathcal{B}$  defined in (10).

From this property, we deduce that  $x(t)$  is defined for all  $t \in [t_k, \min\{\tau, t_{k+1}\})$  and bounded. Hence, for all  $t \in [0, \tau]$  the solution  $x(t)$  exists, is bounded and satisfies, at each instant  $t$ ,

$$|\dot{x}(t)| \leq \sqrt{2} e^{2f_0 t} |x(t_k)| + f_0 \sqrt{\tau} e^{2f_0 \tau} (\mathcal{B} + 1) |u(t_k - \tau)|$$

for some integer  $k \in \{0, 1, \dots, \ell_0 - 1\}$ ,  $\ell_0 \geq 1$ . Moreover, whenever  $\dot{x}(t)$  exists,

$$|\dot{x}(t)| = |f(t, x(t)) + h(t - \tau, x(t - \tau)) u(t_k - \tau)|,$$

which, in combination with Assumption 2, implies that

$$|\dot{x}(t)| \leq f_0 |x(t)| + f_0 |x(t - \tau)| |u(t_k - \tau)|.$$

Therefore there exists a constant  $\mathcal{X}_3 > 0$  such that

$$|\dot{x}(t)| \leq \mathcal{X}_3,$$

for all  $t \in [0, \tau]$  such that there is no integer  $k$  such that  $t = t_k$ . We deduce that the restriction of  $x(\cdot)$  to the interval  $[0, \tau]$  is a Lipschitz function. We easily deduce that the restriction of  $x(\cdot)$  to the interval  $[-\tau, \tau]$  is a Lipschitz function.  $\blacksquare$

We can now iterate. Assume that, for an integer  $j \geq 1$  the numbers  $t_k$  are defined over  $[0, (j+1)\tau]$ , the feedback  $u$  is defined over  $[0, j\tau]$  and the solution  $x(t)$  is defined over  $[-\tau, (j+1)\tau]$  and Lipschitz over this interval. Then we can define values  $t_k$  over  $[0, (j+1)\tau]$ ,  $u$  over  $[0, (j+1)\tau]$  using the constructions above and that way obtain a corresponding solution  $x(t)$  defined over  $[-\tau, (j+2)\tau]$ , which, according to Lemma 2, is Lipschitz over  $[j\tau, (j+1)\tau]$ . We easily deduce that it is Lipschitz over  $[-\tau, (j+1)\tau]$ .

Observe that, as a consequence of this construction, no accumulation in finite time of the sampling times  $t_k$  is possible and the solution  $x(t)$  of (2) starting from the initial condition  $\varphi \in C_{\text{in}}$  exists for all  $t \geq 0$ . The asymptotic behavior of the solution is studied in the next section.

#### IV. MAIN RESULT

In this section, we give the main result of our work, which requires the following extra assumption:

**Assumption 3.** The delay  $\tau > 0$  satisfies the inequalities

$$\begin{aligned}\tau &\leq \frac{1}{|L|} \ln \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{|L|}{2} \sqrt{\frac{r_p}{f_0}}} \right), \\ \tau &\leq \frac{\sqrt{r_p}}{\sqrt{2f_0} \sqrt{5 + \left(1 + \sqrt{1 + 2|L| \sqrt{\frac{r_p}{f_0}} + \frac{c\sqrt{r_p}}{4f_0}}\right)^2}},\end{aligned}\quad (15)$$

where  $f_0$ ,  $L$  are the constant and matrix introduced in Assumptions 1 and 2,  $P$  is the matrix introduced in Section II and  $r_p$  is the constant defined in (9).

The constant  $\Delta > 0$  satisfies the inequality

$$\Delta \leq \frac{c\sqrt{r_p}}{4f_0}. \quad (16)$$

We state and prove the following result:

*Theorem 1:* Consider the system (2). Assume that it satisfies Assumptions 1 to 3. Then there exists a real number  $w > 0$  (independent of  $\Delta$  and  $\tau$ ) such that all the solutions with initial conditions in  $C_{\text{in}}$  of the system (2) in closed-loop with the control law defined in Section III enter the set  $\mathcal{E} = \{x \in \mathbb{R}^n : |x| \leq w\Delta\}$  and stay in it thereafter.

#### Discussion of Theorem 1.

(i) Discussions on the important Assumption 1 are given in [16].

(ii) If we consider the particular case of the linear time-invariant systems

$$\dot{x}(t) = Ax(t) + Bu(t_k - \tau)$$

where  $A$  and  $B$  are constant matrices, then Assumption 1 is satisfied if and only if there exist matrices  $M$  and  $L$  so that

$$A - L = e^{-\tau L} B M. \quad (17)$$

Moreover Assumption 2 is automatically satisfied and Assumption 3 is satisfied for sufficiently small values of  $\tau$  and  $\Delta$ . Detailed discussions about the solutions of the matrix equation (17) are given in [16]. From them, we deduce that Theorem 1 applies to a rather large family of linear systems.

(iii) For the sake of simplicity, we did not address the case where the system (2) is subject to additive disturbances. However, Theorem 1 can be adapted to this case by combining the proof we give below and the one of Theorem 2 in [16]. As explained in the paper, this result can be used for instance to apply the technique when only an approximate solution of the matrix equation (17) can be found.

(iv) The proof of Theorem 1 provides an explicit expression for the constant  $w$ .

(v) In contrast to [16], in the present paper we do not address the problem of determining positive solutions for the closed-loop system. From the forthcoming proof, it appears that this problem does not admit a solution similar to the one presented in [16]. However, it is possible that constructions of sampled feedbacks different from the one of Section III may lead to closed-loop systems for which positive solutions may be exhibited. This may be the subject of future studies.

(vi) For reasons explained in [16] and [15], the growth conditions imposed in Assumption 2 cannot be removed

without being replaced by another assumption that prevents the finite escape time phenomenon from happening.

(vii) The requirement (15) can be relaxed by using the fact that, for any invertible matrix  $R$ , the matrix  $L$  satisfies the equality  $L = RGR^{-1}$  with  $G = R^{-1}LR$  and then arguing as in [16]. For the sake of simplicity, we do consider the requirement (15) only.

(viii) Increasing  $\Delta$  leads to a slower sampling rate (see for instance rule (iii) before Lemma 1 and, for a more explicit statement, Remark 1 after the proof of Theorem 1) and reduces the accuracy of the stability induced by the controller.

*Proof:* To begin with, we observe that the arguments of Section III ensure that, under the standing assumptions, the solutions of (2) are defined over  $[-\tau, +\infty)$  i.e. the system (2) is forward-complete.

Since we select the increasing sequence of times  $t_k$  and the feedback defined in Section III and Assumption 1 is satisfied, the system (2) can be rewritten as:

$$\begin{aligned}\dot{x}(t) &= Lx(t) + h(t - \tau, x(t - \tau))u(t_k - \tau) \\ &\quad + e^{-L\tau}h(t, x(t))g(t, x(t)), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}\end{aligned}\quad (18)$$

and as

$$\begin{aligned}\dot{x}(t) &= f(t, x(t)) - e^{L\tau}\bar{f}(t - \tau, x(t - \tau)) \\ &\quad + h(t - \tau, x(t - \tau))[u(t_k - \tau) + g(t - \tau, x(t - \tau))],\end{aligned}\quad (19)$$

$$t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}.$$

Both the representations (18) and (19) will be useful. Next, to ease our analysis, we introduce the operator  $\zeta : \mathbb{R} \times C_{\text{in}} \mapsto \mathbb{R}^n$ ,

$$\zeta(t, \phi) = \phi(0) - \int_{-\tau}^0 e^{L(-\ell-\tau)}h(\ell+t, \phi(\ell))g(\ell+t, \phi(\ell))d\ell \quad (20)$$

and the simplifying notation

$$s(t) = \zeta(t, x_t) \quad (21)$$

where  $x$  is any solution of (18). Then the equality

$$x(t) = s(t) + \int_{t-\tau}^t e^{L(t-\ell-\tau)}h(\ell, x(\ell))g(\ell, x(\ell))d\ell \quad (22)$$

is satisfied for all  $t \geq 0$  and, for all  $t$  such that it does not exist an integer  $m$  such that  $t = t_m$ ,

$$\begin{aligned}\dot{s}(t) &= Ls(t) \\ &\quad + h(t - \tau, x(t - \tau))[u(t_k - \tau) + g(t - \tau, x(t - \tau))].\end{aligned}\quad (23)$$

If the term  $h(t - \tau, x(t - \tau))[u(t_k - \tau) + g(t - \tau, x(t - \tau))]$  was not present in (23) or was satisfying Assumption H4 in [16], then the equations (22), (23), with the help of the main result of [16], would lead straightforwardly to a strict Lyapunov-Krasovskii functional for the closed-loop system. However, the presence of the term  $h(t - \tau, x(t - \tau))[u(t_k - \tau) + g(t - \tau, x(t - \tau))]$  forces us to conduct another construction of Lyapunov-Krasovskii functional because this term does not satisfy Assumption H4 in [16].

In what follows, whenever we take the derivatives of functions depending explicitly on  $s(t)$  and/or  $x(t)$ , these derivatives must be intended to hold everywhere except at the times  $t_k$ , where the functions are only continuous. This is enough to infer our conclusions.

The first part of our construction consists in determining the derivative along the trajectories of the closed-loop system of several functionals. First we notice that the derivative of the nonnegative functional  $Q_0 : \mathbb{R} \times C_{\text{in}} \mapsto \mathbb{R}$ ,

$$Q_0(t, \phi) = \zeta(t, \phi)^\top P \zeta(t, \phi), \quad (24)$$

where  $P \in \mathbb{R}^{n \times n}$  is the matrix in (8), along the trajectories of (18) satisfies, for a.e.  $t \geq 0$ ,

$$\begin{aligned} \dot{Q}_0(t) &\leq -cQ_0(t, x_t) \\ &+ 2s(t)^\top P h(t - \tau, x(t - \tau)) [u(t_k - \tau) - g(t - \tau, x(t - \tau))] \end{aligned} \quad (25)$$

Recall that the control value  $u(t_k - \tau)$  and the sampling time  $t_k$  satisfy (14). It follows that

$$-\frac{\Delta}{2} < u(t_k - \tau) + g(t - \tau, \phi(t - \tau)) < \frac{\Delta}{2} \quad (26)$$

for all  $t \in [t_k, t_{k+1})$ . Hence, for  $t \in [t_k, t_{k+1})$ ,

$$|u(t_k - \tau) + g(t - \tau, x(t - \tau))| \leq \frac{\Delta}{2}. \quad (27)$$

Since there is a symmetric positive definite matrix  $P^{\frac{1}{2}}$  such that  $P = P^{\frac{1}{2}} P^{\frac{1}{2}}$ , from the triangle inequality and (27), it follows that

$$\begin{aligned} \dot{Q}_0(t) &\leq -\frac{1}{2}cQ_0(t, x_t) \\ &+ \frac{1}{c}|P| |h(t - \tau, x(t - \tau))|^2 \frac{\Delta^2}{4}. \end{aligned} \quad (28)$$

From (7) in Assumption 2, we deduce that

$$\begin{aligned} \dot{Q}_0(t) &\leq -\frac{c}{5}Q_0(t, x_t) \\ &+ \frac{|P|}{c} f_0^2 (|x(t - \tau)| + 1)^2 \frac{\Delta^2}{4}. \end{aligned} \quad (29)$$

From the inequalities  $(|x(t - \tau)| + 1)^2 \leq 2(|x(t - \tau)|^2 + 1) \leq 2\left(\frac{1}{q_p}x(t - \tau)^\top Qx(t - \tau) + 1\right)$ , it follows that

$$\begin{aligned} \dot{Q}_0(t) &\leq -\frac{c}{5}Q_0(t, x_t) + \frac{\Delta^2}{2c r_p} f_0^2 Q_1(x(t - \tau)) \\ &+ \frac{|P|}{2c} f_0^2 \Delta^2, \end{aligned} \quad (30)$$

where  $r_p$  is the constant defined in (9) and  $Q_1$  is the quadratic function

$$Q_1(x) = x^\top P x, \quad (31)$$

which is positive definite because  $P$  is symmetric and positive definite.

Next, using (19), we deduce that, for all  $t \in [t_k, t_{k+1})$ ,

$$\begin{aligned} \dot{Q}_1(t) &= 2x(t)^\top P [f(t, x(t)) - e^{L\tau} \bar{f}(t - \tau, x(t - \tau))] \\ &+ 2x(t)^\top P [h(t - \tau, x(t - \tau)) [u(t_k - \tau) + g(t - \tau, x(t - \tau))]] \end{aligned} \quad (32)$$

From Assumption 2 and (27), we deduce that

$$\begin{aligned} \dot{Q}_1(t) &\leq 2|x(t)^\top P| [f_0|x(t)| + f_0 e^{L|\tau|}|x(t - \tau)|] \\ &+ 2|x(t)^\top P| f_0 (|x(t - \tau)| + 1) \frac{\Delta}{2}. \end{aligned} \quad (33)$$

By reorganizing the terms and using the inequality  $P \geq q_p Id$ , we obtain

$$\begin{aligned} \frac{r_p}{2f_0} \dot{Q}_1(t) &\leq Q_1(x(t)) \\ &+ \left(e^{L|\tau|} + \frac{\Delta}{2}\right) \sqrt{Q_1(x(t))} \sqrt{Q_1(x(t - \tau))} \\ &+ \sqrt{Q_1(x(t))} \frac{\sqrt{q_p} \Delta}{2}. \end{aligned} \quad (34)$$

Using the triangle inequality, we obtain

$$\begin{aligned} \frac{r_p}{2f_0} \dot{Q}_1(t) &\leq 2Q_1(x(t)) \\ &+ \left(e^{L|\tau|} + \frac{\Delta}{2}\right)^2 Q_1(x(t - \tau)) + \frac{q_p \Delta^2}{4}. \end{aligned} \quad (35)$$

Next, we consider the functional  $Q_2 : C_{\text{in}} \mapsto \mathbb{R}$ ,

$$\begin{aligned} Q_2(\phi) &= \frac{1}{\tau} \int_{-\tau}^0 \left( \int_m^0 Q_1(\phi(\ell)) d\ell \right) dm \\ &+ \int_{-\tau}^0 Q_1(\phi(\ell)) d\ell. \end{aligned} \quad (36)$$

It is nonnegative and, for all  $t \geq 0$ ,  $Q_2(x_t) = \frac{1}{\tau} \int_{t-\tau}^t \left( \int_m^t Q_1(x(\ell)) d\ell \right) dm + \int_{t-\tau}^t Q_1(x(\ell)) d\ell$ . It follows that

$$\begin{aligned} \dot{Q}_2(t) &= 2Q_1(x(t)) - \frac{1}{\tau} \int_{t-\tau}^t Q_1(x(\ell)) d\ell \\ &- Q_1(x(t - \tau)). \end{aligned} \quad (37)$$

Next, we observe that, it follows from (22) and Assumption 1 that, for all  $t \geq 0$ ,

$$\begin{aligned} Q_1(x(t)) &= \left( s(t) + \int_{t-\tau}^t e^{L(t-\ell)} \bar{f}(\ell, x(\ell)) d\ell \right)^\top \\ &\times P \left( s(t) + \int_{t-\tau}^t e^{L(t-\ell)} \bar{f}(\ell, x(\ell)) d\ell \right) \\ &= s(t)^\top P s(t) + 2s(t)^\top P^{\frac{1}{2}} P^{\frac{1}{2}} \int_{t-\tau}^t e^{L(t-\ell)} \bar{f}(\ell, x(\ell)) d\ell \\ &+ \left( \int_{t-\tau}^t e^{L(t-\ell)} \bar{f}(\ell, x(\ell)) d\ell \right)^\top P \int_{t-\tau}^t e^{L(t-\ell)} \bar{f}(\ell, x(\ell)) d\ell. \end{aligned} \quad (38)$$

Using the triangle inequality, we obtain

$$\begin{aligned} Q_1(x(t)) &\leq 2s(t)^\top P s(t) \\ &+ 2 \left( \int_{t-\tau}^t e^{L(t-\ell)} \bar{f}(\ell, x(\ell)) d\ell \right)^\top P \int_{t-\tau}^t e^{L(t-\ell)} \bar{f}(\ell, x(\ell)) d\ell. \end{aligned} \quad (39)$$

From (6) in Assumption 2, it follows that, for all  $t \geq 0$ ,

$$Q_1(x(t)) \leq 2Q_0(t, x_t) + 2|P| e^{2L|\tau|} f_0^2 \left( \int_{t-\tau}^t |x(\ell)| d\ell \right)^2. \quad (40)$$

Using successively the Cauchy-Schwartz inequality and  $q_p Id \leq P$ , we deduce that, for all  $t \geq 0$ ,

$$\begin{aligned} Q_1(x(t)) &\leq 2Q_0(t, x_t) \\ &+ 2|P| f_0^2 e^{2L|\tau|} \tau \int_{t-\tau}^t |x(\ell)|^2 d\ell \\ &\leq 2Q_0(t, x_t) \\ &+ \frac{2}{r_p} f_0^2 e^{2L|\tau|} \tau \int_{t-\tau}^t Q_1(x(\ell)) d\ell. \end{aligned} \quad (41)$$

So, to summarize, we have, for a.e.  $t \geq 0$ ,

$$\begin{aligned} \dot{Q}_0(t) &\leq -\frac{c}{5}Q_0(t, x_t) + \frac{\Delta^2}{2c r_p} f_0^2 Q_1(x(t - \tau)) \\ &+ \frac{|P|}{2c} f_0^2 \Delta^2, \\ \dot{Q}_1(t) &\leq \frac{4f_0}{r_p} Q_1(x(t)) \\ &+ \frac{2f_0}{r_p} \left( e^{L|\tau|} + \frac{\Delta}{2} \right)^2 Q_1(x(t - \tau)) \\ &+ |P| f_0 \frac{\Delta}{2}, \end{aligned} \quad (42)$$

$$\begin{aligned} \dot{Q}_2(t) &= 2Q_1(x(t)) - \frac{1}{\tau} \int_{t-\tau}^t Q_1(x(\ell)) d\ell \\ &- Q_1(x(t - \tau)), \end{aligned}$$

$$\begin{aligned} Q_1(x(t)) &\leq 2Q_0(t, x_t) \\ &+ \frac{2f_0^2}{r_p} e^{2L|\tau|} \tau \int_{t-\tau}^t Q_1(x(\ell)) d\ell. \end{aligned}$$

These inequalities lead us to consider the functional  $U : \mathbb{R} \times C_{\text{in}} \mapsto \mathbb{R}$ ,

$$U(t, \phi) = Q_0(t, \phi) + \frac{a_1}{|P|f_0} Q_1(\phi(0)) + a_2 Q_2(\phi), \quad (43)$$

where  $a_1$  and  $a_2$  are positive real numbers to be selected later. According to the inequalities in (42), for a.e.  $t \geq 0$ ,

$$\begin{aligned} \dot{U}(t) \leq & -\frac{c}{2} Q_0(t, x_t) + \frac{f_0^2 \Delta^2}{2c r_p} Q_1(x(t-\tau)) \\ & + \frac{|P|f_0^2}{2c} \Delta^2 + \frac{4a_1}{q_p} Q_1(x(t)) \\ & + \frac{2a_1}{q_p} b_1 Q_1(x(t-\tau)) + a_1 \frac{\Delta^2}{2} \\ & + 2a_2 Q_1(x(t)) - \frac{a_2}{\tau} \int_{t-\tau}^t Q_1(x(\ell)) d\ell \\ & - a_2 Q_1(x(t-\tau)), \end{aligned} \quad (44)$$

with  $b_1 = (e^{L|\tau} + \frac{\Delta}{2})^2$ . Due to space limitation, we omit the choices and the calculations that lead to the inequality

$$\dot{U}(t) \leq -b_8 U(t, x_t) + b_2 \Delta^2, \quad (45)$$

with

$$b_8 = \min \left\{ \frac{c}{8}, \frac{f_0}{r_p}, \frac{f_0 |P|}{b_7} \right\}.$$

It follows that there exists  $T \geq 0$  such that, for all  $t \geq T$ ,

$$U(t, x_t) \leq 2 \frac{b_2}{b_8} \Delta^2. \quad (46)$$

From the definition of  $U$ , we deduce that, for all  $t \geq T$ ,

$$q_p |x(t)|^2 \leq Q_1(x(t)) \leq 2 \frac{|P|f_0 b_2}{a_1 b_8} \Delta^2. \quad (47)$$

It follows that

$$|x(t)| \leq w \Delta, \quad (48)$$

for all  $t \geq T$ , with  $w = \sqrt{\frac{2f_0 b_2}{r_p a_1 b_8}}$ . Observing that the constants present in the formula of  $w$  are independent from  $\tau$  and  $\Delta$ , we deduce that  $w$  is independent from  $\tau$  and  $\Delta$ . This allows us to conclude.

*Remark 1:* From the proof of Theorem 1 and Lemmas 1 and 2 we can obtain, through lengthy but simple calculations, an upper bound for the sampling rate over each interval  $[j\tau, (j+1)\tau]$ . In the special case of time-invariant systems, these bounds do not depend explicitly on the time. ■

## V. ILLUSTRATION

To illustrate Theorem 1, we consider the one-dimensional system

$$\dot{x}(t) = \sin(x(t)) + u(t_k - \tau). \quad (49)$$

Then, using the notation of Section II, Assumption 1 is satisfied with  $L = -1$ ,  $P = \frac{1}{2}$  and

$$\begin{aligned} \bar{f}(t, x) &= \sin(x) + x, \\ h(t, x) &= 1, \quad g(t, x) = e^{-\tau} (\sin(x) + x). \end{aligned} \quad (50)$$

Moreover,  $c = 1$  and Assumption 2 is satisfied with  $f_0 = 2$ . Finally, Assumption 3 is satisfied with

$$\Delta = \frac{1}{8\sqrt{2}}, \quad \tau = \frac{1}{\sqrt{22} \sqrt{5 + \left( \sqrt{1 + \sqrt{2}} + \frac{9}{8} \right)^2}} \quad (51)$$

and therefore Theorem 1 applies to (49) when the two equalities in (51) are satisfied.

## VI. CONCLUSION

We have established a result of global practical stabilization for a family of nonlinear time-varying systems with state-dependent sampling and delay in the inputs. In further studies, we shall consider other types of discontinuous systems, which in particular include systems in feedback form and feedforward systems. Moreover, for the sake of simplicity, in this paper we have assumed that the input is scalar, the sampling is driven by a function of the state and the delay is constant. The case of multi-input systems in which the sampling depends on any component of the state (independently of the others) and the delays are time-varying represents an interesting extension.

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**For the convenience of the reviewers of this draft, we include the following material. The sections to follow are not part of the manuscript that the authors are proposing for possible inclusion in the proceedings.**

## VII. PROOF OF LEMMA 1

Without loss of generality, we can restrict our attention to the case  $\ell_0 \geq 4$  (if  $\ell_0 < 4$ ,  $\ell_0$  is finite). Observe for later use that this assumption and Assumption 1 on the function  $g$  imply that the constant

$$\mathcal{G}_M := \max_{m \in [-\tau, 0], |\xi| \leq \mathcal{B}} \left\{ \left| \frac{\partial g}{\partial x}(m, \xi) \right| \right\} \mathcal{C} + \max_{m \in [-\tau, 0], |\xi| \leq \mathcal{B}} \left\{ \left| \frac{\partial g}{\partial t}(m, \xi) \right| \right\}, \quad (52)$$

with  $\mathcal{B}$  defined in (10) and  $\mathcal{C}$  the Lipschitz constant of  $\varphi$ , is a well-defined nonnegative real number.

Next, we determine a lower bound on  $t_m - t_{m-1}$  where  $m \in \{2, \dots, \ell_0 - 2\}$ . Since  $m - 1 > 0$  and  $m < \ell_0 - 1$ , we know that, necessarily, both  $t_{m-1}$  and  $t_m$  are defined by (iii) and therefore

$$\begin{aligned} |u(t_m - \tau) - u(t_{m-1} - \tau)| &= \frac{\Delta}{2}, \\ u(t_m - \tau) &= -g(t_m - \tau, \varphi(t_m - \tau)), \\ u(t_{m-1} - \tau) &= -g(t_{m-1} - \tau, \varphi(t_{m-1} - \tau)). \end{aligned}$$

It follows that

$$|g(t_m - \tau, \varphi(t_m - \tau)) - g(t_{m-1} - \tau, \varphi(t_{m-1} - \tau))| = \frac{\Delta}{2}. \quad (53)$$

On the other hand, since, for all  $a \in [-\tau, 0]$ ,  $b \in [-\tau, 0]$ ,

$$\begin{aligned} |g(a, \varphi(a)) - g(b, \varphi(b))| &\leq |g(a, \varphi(a)) - g(a, \varphi(b))| \\ &\quad + |g(a, \varphi(b)) - g(b, \varphi(b))| \\ &\leq \max_{|\xi| \leq \mathcal{B}} \left\{ \left| \frac{\partial g}{\partial x}(a, \xi) \right| \right\} \mathcal{C} |a - b| \\ &\quad + \max_{m \in [a, b], |\xi| \leq \mathcal{B}} \left\{ \left| \frac{\partial g}{\partial t}(m, \xi) \right| \right\} |a - b| \end{aligned} \quad (54)$$

and  $t_{m-1} \in [t_0, t_0 + \tau]$ ,  $t_m \in [t_0, t_0 + \tau]$ , we deduce that

$$|g(t_m - \tau, \varphi(t_m - \tau)) - g(t_{m-1} - \tau, \varphi(t_{m-1} - \tau))| \leq (t_m - t_{m-1}) \mathcal{G}_M \quad (55)$$

with  $\mathcal{G}_M$  defined in (52). Combining (53) and (55), we obtain

$$(t_m - t_{m-1}) \mathcal{G}_M \geq \frac{\Delta}{2}. \quad (56)$$

We deduce that

$$t_m - t_{m-1} \geq \frac{\Delta}{2(\mathcal{G}_M + 1)}. \quad (57)$$

Since, for all  $l \in \{1, \dots, \ell_0 - 3\}$ ,  $[t_1, t_2] \cup \dots \cup [t_l, t_{l+1}]$ , is included in  $[t_0, t_0 + \tau]$ , we deduce that

$$(t_2 - t_1) + \dots + (t_{l+1} - t_l) \leq (t_0 + \tau) - t_0 = \tau \quad (58)$$

for all  $l \in \{1, \dots, \ell_0 - 3\}$ . From (57), we deduce that

$$l \leq \tau \frac{2(\mathcal{G}_M + 1)}{\Delta} \quad (59)$$

for all  $l \in \{1, \dots, \ell_0 - 3\}$ . It follows that necessarily  $\ell_0$  is a finite integer.

## VIII. EXTRA PART OF THE PROOF OF THEOREM 1

Choosing

$$a_2 = \frac{\Delta^2}{2cr_p} f_0^2 + \frac{2a_1}{q_p} b_1 \quad (60)$$

and reorganizing the terms, we obtain, for a.e.  $t \geq 0$ ,

$$\begin{aligned} \dot{U}(t) &\leq -\frac{c}{2} Q_0(t, x_t) - \frac{a_1}{q_p} Q_1(x(t)) \\ &\quad + \left[ \frac{5a_1}{q_p} + \frac{f_0^2 \Delta^2}{cr_p} + \frac{4a_1 b_1}{q_p} \right] Q_1(x(t)) \\ &\quad - \frac{a_2}{\tau} \int_{t-\tau}^t Q_1(x(\ell)) d\ell + b_2 \Delta^2, \end{aligned} \quad (61)$$

with

$$b_2 = \frac{|P|f_0^2}{2c} + \frac{a_1}{2}.$$

Using the last inequality in (42), we deduce that, for a.e.  $t \geq 0$ ,

$$\begin{aligned} \dot{U}(t) &\leq \left[ -\frac{c}{2} + 2 \left( \frac{5a_1}{q_p} + \frac{|P|\Delta^2}{cq_p} f_0^2 + \frac{4a_1 b_1}{q_p} \right) \right] Q_0(t, x_t) \\ &\quad - \frac{a_1}{q_p} Q_1(x(t)) - \frac{|P|f_0}{q_p \tau} b_3 \int_{t-\tau}^t Q_1(x(\ell)) d\ell \\ &\quad + b_2 \Delta^2, \end{aligned} \quad (62)$$

with

$$\begin{aligned} b_3 &= -2 \left[ \frac{5a_1}{q_p} + \frac{\Delta^2}{cr_p} f_0^2 + \frac{4a_1 b_1}{q_p} \right] f_0 e^{2|L|\tau} \tau^2 + \frac{\Delta^2}{2c} f_0 \\ &\quad + \frac{2a_1}{|P|f_0} e^{2|L|\tau}. \end{aligned}$$

From (16) it follows that  $2 \frac{|P|\Delta^2}{cq_p} f_0^2 \leq \frac{c}{8}$ . Using this inequality, we obtain, for a.e.  $t \geq 0$ ,

$$\begin{aligned} \dot{U}(t) &\leq \left[ -\frac{3c}{8} + \frac{2a_1}{q_p} (5 + 4b_1) \right] Q_0(t, x_t) \\ &\quad - \frac{a_1}{q_p} Q_1(x(t)) - \frac{f_0 b_3}{r_p \tau} \int_{t-\tau}^t Q_1(x(\ell)) d\ell \\ &\quad + b_2 \Delta^2. \end{aligned} \quad (63)$$

By grouping the terms, we obtain

$$\begin{aligned} b_3 &= 2a_1 e^{2|L|\tau} \left[ -5 \frac{f_0}{q_p} \tau^2 - \frac{4f_0 b_1}{q_p} \tau^2 + \frac{1}{|P|f_0} \right] \\ &\quad + \frac{\Delta^2 f_0}{2c} \left[ -\frac{4}{r_p} f_0 e^{2|L|\tau} \tau^2 + 1 \right]. \end{aligned}$$

The first inequality in (15) implies that  $-\frac{4}{r_p} f_0 e^{2|L|\tau} \tau^2 + 1 \geq 0$ . We deduce that, for a.e.  $t \geq 0$ ,

$$\begin{aligned} \dot{U}(t) &\leq \left[ -\frac{3c}{8} + \frac{2a_1}{q_p} (5 + 4b_1) \right] Q_0(t, x_t) \\ &\quad - \frac{a_1}{q_p} Q_1(x(t)) \\ &\quad - \frac{2a_1}{q_p \tau} e^{2|L|\tau} b_4 \int_{t-\tau}^t Q_1(x(\ell)) d\ell \\ &\quad + b_2 \Delta^2, \end{aligned} \quad (64)$$

with

$$b_4 = 1 - \frac{f_0^2}{r_p} (5 + 4b_1) \tau^2.$$

From the first inequality in (15) and (16), we deduce that

$$\begin{aligned} b_1 &\leq \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{|L|}{2} \sqrt{\frac{r_p}{f_0}} + \frac{c\sqrt{r_p}}{8f_0}} \right)^2 \\ &\leq \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{|L|}{2\sqrt{f_0}} + \frac{c}{8f_0}} \right)^2 \\ &\leq \left( 2\sqrt{\frac{1}{4} + \frac{|L|}{2\sqrt{f_0}} + \frac{c}{8f_0}} \right)^2 \end{aligned}$$

and therefore

$$b_1 \leq 2 + \frac{4|L|}{\sqrt{f_0}} + \frac{c^2}{32f_0^2}.$$

It follows that, for a.e.  $t \geq 0$ ,

$$\begin{aligned} \dot{U}(t) &\leq \left[ -\frac{3c}{8} + \frac{2a_1}{q_p} \left( 13 + \frac{16|L|}{\sqrt{f_0}} + \frac{c^2}{8f_0^2} \right) \right] Q_0(t, x_t) \\ &\quad - \frac{a_1}{q_p} Q_1(x(t)) \\ &\quad - \frac{2}{q_p \tau} a_1 e^{2|L|\tau} b_5 \int_{t-\tau}^t Q_1(x(\ell)) d\ell + b_2 \Delta^2, \end{aligned} \quad (65)$$

with

$$b_5 = 1 - \frac{1}{r_p} \left( 13f_0^2 + 16|L|f_0\sqrt{f_0} + \frac{c^2}{8} \right) \tau^2.$$

Choosing

$$a_1 = \frac{cq_p}{104 + \frac{128|L|}{\sqrt{f_0}} + \frac{c^2}{f_0^2}}$$

and noticing that (15) implies that  $b_4 \geq \frac{1}{2}$ , we obtain, for a.e.  $t \geq 0$ ,

$$\begin{aligned} \dot{U}(t) &\leq -\frac{c}{8} Q_0(t, x_t) - \frac{a_1}{q_p} Q_1(x(t)) \\ &\quad - \frac{f_0 a_1}{r_p \tau} \int_{t-\tau}^t Q_1(x(\ell)) d\ell + b_2 \Delta^2. \end{aligned} \quad (66)$$

Now, recall that

$$\begin{aligned} U(t, \phi) &= Q_0(t, \phi) + \frac{a_1}{|p|f_0} Q_1(\phi(0)) \\ &\quad + \left[ \frac{\Delta^2}{2cr_p} f_0^2 + \frac{2a_1}{q_p} \left( e^{|L|\tau} + \frac{\Delta}{2} \right)^2 \right] Q_2(\phi) \end{aligned} \quad (67)$$

and observe that, for all  $\phi \in C_{\text{in}}$ , the inequality

$$Q_2(\phi) \leq 2 \int_{-\tau}^0 Q_1(\phi(\ell)) d\ell$$

is satisfied. It follows that, for all  $t \in \mathbb{R}$  and for all  $\phi \in C_{\text{in}}$ , the inequality

$$\begin{aligned} U(t, \phi) &\leq Q_0(t, \phi) + \frac{a_1}{|p|f_0} Q_1(\phi(0)) \\ &\quad + \frac{b_6}{\tau} \int_{-\tau}^0 Q_1(\phi(\ell)) d\ell, \end{aligned} \quad (68)$$

with  $b_6 = \frac{\Delta^2}{cr_p} f_0^2 + \frac{a_1}{q_p} \left( 8 + \frac{16|L|}{\sqrt{f_0}} + \frac{c^2}{8f_0^2} \right) \tau$ , is satisfied. By using (16) and (15), we obtain

$$\begin{aligned} b_6 &\leq \frac{1}{cr_p} f_0^2 \frac{c^2 r_p}{16f_0^2} \\ &\quad + \frac{a_1}{q_p} \left( 8 + \frac{16|L|}{\sqrt{f_0}} + \frac{c^2}{8f_0^2} \right) \frac{1}{|L|} \ln \left( \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{|L|}{2} \sqrt{\frac{r_p}{f_0}}} \right) \\ &\leq \frac{c}{16} + \frac{a_1}{q_p} \left( 8 + \frac{16|L|}{\sqrt{f_0}} + \frac{c^2}{8f_0^2} \right) \frac{1}{|L|} \ln \left( 1 + \sqrt{\frac{|L|}{2\sqrt{f_0}}} \right) \\ &\leq \frac{c}{16} + \frac{a_1}{q_p} \left( 8 + \frac{16|L|}{\sqrt{f_0}} + \frac{c^2}{8f_0^2} \right) \frac{1}{\sqrt{2}\sqrt{|L|f_0^{\frac{1}{4}}}} \\ &\leq \frac{a_1}{q_p} \left( \frac{13}{3} + \frac{8|L|}{\sqrt{f_0}} + \frac{c^2}{16f_0^2} \right) \\ &\quad + \frac{a_1}{q_p} \left( 8 + \frac{16|L|}{\sqrt{f_0}} + \frac{c^2}{8f_0^2} \right) \frac{1}{\sqrt{2}\sqrt{|L|f_0^{\frac{1}{4}}}} \\ &= \frac{a_1}{q_p} b_7, \end{aligned} \quad (69)$$

with  $b_7 = \left( \frac{26}{3} + \frac{16|L|}{\sqrt{f_0}} + \frac{c^2}{8f_0^2} \right) \left( \frac{1}{2} + \frac{1}{\sqrt{2}\sqrt{|L|f_0^{\frac{1}{4}}}} \right)$ . Therefore, for all  $t \in \mathbb{R}$  and for all  $\phi \in C_{\text{in}}$ , the inequality

$$\begin{aligned} U(t, \phi) &\leq Q_0(t, \phi) + \frac{a_1}{|p|f_0} Q_1(\phi(0)) \\ &\quad + \frac{b_7}{q_p} \frac{a_1}{\tau} \int_{-\tau}^0 Q_1(\phi(\ell)) d\ell \end{aligned} \quad (70)$$

is satisfied. We deduce that, for a.e.  $t \geq 0$ , the inequality (45) is satisfied.