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STRICT LYAPUNOV FUNCTIONS FOR SEMILINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. For families of partial differential equations (PDEs) with particular boundary conditions, strict Lyapunov functions are constructed. The PDEs under consideration are parabolic and, in addition to the diffusion term, may contain a nonlinear source term plus a convection term. The boundary conditions may be either the classical Dirichlet conditions, or the Neumann boundary conditions or a periodic one. The constructions rely on the knowledge of weak Lyapunov functions for the nonlinear source term. The strict Lyapunov functions are used to prove asymptotic stability in the framework of an appropriate topology. Moreover, when an uncertainty is considered, our construction of a strict Lyapunov function makes it possible to establish some robustness properties of Input-to-State Stability (ISS) type.

1. **Introduction.** Lyapunov function based techniques are central in the study of partial differential equations (PDEs). The techniques are useful for the stability analysis of systems of many different families (although other approaches can be used too, especially when parabolic PDEs are studied; see in particular the contributions [9, 10, 16]).

Amongst the remarkable results for PDEs which extensively use Lyapunov functions, it is worth mentioning the following. In [2] a Lyapunov function is used to establish the existence of a global solution to the celebrated heat equation. In [13], Lyapunov functions are designed for the heat equation with unknown destabilizing parameters (see also [26, 27] for further results on the design of output stabilizers). Lyapunov functions have been also used to establish controllability results for semilinear heat equations. For example in [7] the computation of a Lyapunov function, in combination with the quasi-static deformation method, is a key ingredient of the proof of the global controllability of this equation. In all the preceding papers, parabolic PDEs are considered, but Lyapunov functions can also be useful for other kinds of dynamics. For instance, in [5] the stabilization of a linear dynamic equation modeling a rotating beam is achieved through a control design relying on weak

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Lyapunov function, i.e. a Lyapunov function whose derivative along the trajectories of the system which is considered is non-positive (but not necessarily negative definite), and in [8] the controllability of the wave equation is demonstrated via a Lyapunov function. Besides, the knowledge of Lyapunov functions can be useful for the stability analysis of nonlinear hyperbolic systems (see the recent work [4]) or even for designing boundary controls which stabilize a system of conservation laws (see [6]).

To demonstrate asymptotic stability through the knowledge of a weak Lyapunov function, the celebrated LaSalle invariance principle has to be invoked (see e.g. [2, 14, 25]). It requires to demonstrate a precompactness property for the solutions, which may be difficult to prove (and is not even always satisfied, as illustrated by the hyperbolic systems considered in [6]). This technical step is not needed when is available a strict Lyapunov function i.e. a Lyapunov function whose derivative along the trajectories of a system which is considered is negative definite. Thus designing such a Lyapunov function is a way to overcome this technical difficulty. This is not the unique motivation for designing strict Lyapunov functions. From the knowledge of the explicit expression of a strict Lyapunov function, one can estimate the robustness of the stability of a system with respect to the presence of uncertainties and one can analyze the sensitivity of the solutions with respect to external disturbances.

The present paper is devoted to new techniques of constructions of strict Lyapunov functions for parabolic PDEs. For particular families of PDEs with diffusion and convection terms and specific boundary conditions, we modify weak Lyapunov functions, which are readily available, to obtain strict Lyapunov functions given by explicit formulas. The resulting functions have rather simple explicit expressions. The underlying concept of strictification used in our paper is the same as the one exposed in [15], (see also [17, 19]). However, due to the specificity of PDEs, the techniques of construction that we shall present are by no means a direct application of any constructions available for ordinary differential equations.

In the second part of our work, we design strict Lyapunov functions to establish robustness properties of Input-to-State (ISS) type for a family of PDEs with disturbances which are globally asymptotically stable in the absence of disturbances. Although the ISS notion is very popular in the area of the dynamical systems of finite dimension (see e.g. the recent survey [28]) and, for a few years, begins to be used in the domain of the systems with delay (see for instance [11, 18, 22, 23]), the present work is, to the best of our knowledge, the first one which uses it to characterize a robustness property of a PDE.

Our paper is organized as follows. Basic definitions and notations are introduced in Section 2. Constructions of Lyapunov functions under various sets of assumptions are performed in Section 3. In Section 4 the analysis of the robustness of a family of PDEs with uncertainties is carried out by means of the design of a so called ISS Lyapunov function. Two examples in Section 5 illustrate respectively the design of a strict Lyapunov function of an ISS Lyapunov function. Section 6 contains an example of linear system which is globally asymptotically stable and for which there does not exist any ISS Lyapunov function. A technical result is proven in Section 7. Concluding remarks in Section 8 end the work.

Notation. Throughout the paper, the argument of the functions will be omitted or simplified when no confusion can arise from the context. A function $\alpha : [0, +\infty) \rightarrow$

$[0,+\infty)$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, increasing and unbounded. Given a continuously differentiable function $A:\mathbb{R}^n\to\mathbb{R}$, $\frac{\partial A}{\partial \Xi}(\Xi)$ stands for the vector $(\frac{\partial A}{\partial \xi_1}(\Xi).....\frac{\partial A}{\partial \xi_n}(\Xi))\in\mathbb{R}^n$, whereas $\frac{\partial^2 A}{\partial \Xi^2}(\Xi)$ is the square matrix function $(a_{i,j}(\Xi))\in\mathbb{R}^{n\times n}$ with $a_{i,j}(\Xi)=\frac{\partial^2 A}{\partial\xi_i\partial\xi_j}(\Xi)$. The Euclidean inner product of two vectors x and y will be denoted by $x.y$, the induced norm will be denoted by $|\cdot|$. Given a matrix A, its induced matrix norm will be denoted simply by $|A|$, and $\mathtt{Sym}(A)=\frac{1}{2}(A+A^\top)$ stands for the symmetric part of A. The norm $|\cdot|_{L^2(0,L)}$ is defined by: $|\phi|_{L^2(0,L)}=\sqrt{\int_0^L|\phi(z)|^2dz}$. Finally, we denote $C_{\mathrm{L}}=C^2([0,L],\mathbb{R}^n)$, the set of all twice-differentiable $\mathbb{R}^n$-valued functions defined on a given interval $[0,L]$.

2. **Basic definitions and notions.** Throughout our work, we will consider partial differential equations of the form

$$\begin{array}{rcl}\frac{\partial X}{\partial t}(z,t) & = & \frac{\partial^2 X}{\partial z^2}(z,t)+\Delta(z,t)\frac{\partial X}{\partial z}(z,t)\\ & & +f(X(z,t))+u(z,t)\ ,\end{array}\tag{1}$$

with $z\in[0,L]$ and $X(.,t)\in C_{\mathrm{L}}$ for all $t\geq 0$, where $\Delta:[0,L]\times[0,+\infty)\to\mathbb{R}$ is continuous and bounded in norm, where $f:\mathbb{R}^n\to\mathbb{R}^n$ is a continuously differentiable function, and where $u:[0,L]\times[0,+\infty)\to\mathbb{R}^n$ is a continuous function (which typically is unknown and represents disturbances).

Let us introduce the notions of weak and strict Lyapunov functions that we will consider in this paper (see e.g. [14, Def. 3.62]).

Definition 2.1. *Let $\mu:C_{\mathrm{L}}\to\mathbb{R}$ be a continuously differentiable function. The function μ is said to be a* weak Lyapunov function *for (1), if there are two functions κ_S and κ_M of class $\mathcal{K}_\infty$ such that, for all functions $\phi\in C_{\mathrm{L}}$,*

$$\kappa_S\left(|\phi|_{L^2(0,L)}\right)\leq\mu(\phi)\leq\int_0^L\kappa_M\left(|\phi(z)|\right)dz\tag{2}$$

and, when u is identically equal to zero, for all solutions of (1), for all $t\geq 0$,

$$\frac{d\mu(X(.,t))}{dt}\leq 0\ .$$

The function μ is said to be a strict Lyapunov function *for (1) if, additionally, u is identically equal to zero, there exits $\lambda_1>0$ such that, for all solutions of (1), for all $t\geq 0$,*

$$\frac{d\mu(X(.,t))}{dt}\leq-\lambda_1\mu(X(.,t))\ .$$

The function μ is said to be an ISS Lyapunov function *for (1) if, additionally, there exist $\lambda_1>0$ and a function λ_2 of class $\mathcal{K}$ such that, for all continuous functions u, for all solutions of (1), and for all $t\geq 0$,*

$$\frac{d\mu(X(.,t))}{dt}\leq-\lambda_1\mu(X(.,t))+\int_0^L\lambda_2(|u(z,t)|)dz\ .$$

Remark 1. 1. For conciseness, we will often use the notation $\dot{\mu}$ instead of $\frac{d\mu(X(.,t))}{dt}$.

2. When a strict Lyapunov function exists and when u is identically equal to zero, then the value of a strict Lyapunov function for (1) along the solutions of (1) exponentially decays to zero and therefore each solution $X(z,t)$ satisfies

$\lim_{t\to+\infty} |X(.,t)|_{L^2(0,L)} = 0$. When in addition, there exists a function κ_L of class $\mathcal{K}_\infty$, such that, for all functions $\phi \in C_{\rm L}$,

$$\mu(\phi) \leq \kappa_L\left(|\phi|_{L^2(0,L)}\right) , \tag{3}$$

then the system (1) is globally asymptotically stable for the topology of the L^2 norm. We note that the L^2 convergence of $X(.,t)$ to zero does not imply that for all fixed $z \in (0,L)$, $\lim_{t\to+\infty} X(z,t) = 0$.

3. Let us recall that, when is known a weak Lyapunov function, asymptotic stability can be often established via the celebrated LaSalle invariance principle applies (see [14, Theorem 3.64] among other references).

4. When the system (1) admits an ISS Lyapunov function μ, then, one can check through elementary calculations[1] that, for all solutions of (1) and for all instants $t \geq t_0$, the inequality

$$\begin{aligned}|X(.,t)|_{L^2(0,L)} \leq\ & \kappa_S^{-1}\left(2e^{-\lambda_1(t-t_0)}\int_0^L \kappa_M(|X(z,t_0)|)dz\right)\\ & +\kappa_S^{-1}\left(\tfrac{2}{\lambda_1}\sup_{\ell\in[t_0,t]}\left(\int_0^L \lambda_2(|u(z,\ell)|)dz\right)\right)\end{aligned}$$

holds. This inequality is the analogue for the PDE (1) of the ISS inequalities for ordinary differential equations. It gives an estimate of the influence of the disturbance u on the solutions of the system (1). ◦

3. Constructions of Lyapunov functions.

In this section, we give several constructions of Lyapunov functions for the system

$$\tfrac{\partial X}{\partial t}(z,t) = \tfrac{\partial^2 X}{\partial z^2}(z,t) + f(X(z,t)) \tag{4}$$

with $z \in [0,L]$, $X(z,t) \in \mathbb{R}^n$ and where f is a nonlinear function of class C^1.

3.1. Weak Lyapunov function for the system (4).

To prepare the construction of strict Lyapunov functions of the forthcoming sections, we recall how a weak Lyapunov function can be constructed for the system (4) under the following assumptions:

Assumption 1. *There is a symmetric positive definite matrix Q such that the function W_1 defined by, for all $\Xi \in \mathbb{R}^n$,*

$$W_1(\Xi) := -\tfrac{\partial V}{\partial \Xi}(\Xi)f(\Xi) , \tag{5}$$

with $V(\Xi) = \frac{1}{2}\Xi^\top Q\Xi$, is nonnegative.

Assumption 2. *The boundary conditions are such that, for all $t \geq 0$,*

$$\begin{aligned} \textit{either}\quad & |X(L,t)|\left|\tfrac{\partial X}{\partial z}(L,t)\right| = |X(0,t)|\left|\tfrac{\partial X}{\partial z}(0,t)\right| = 0 ,\\ \textit{or}\quad & X(L,t) = X(0,t)\ \textit{and}\ \tfrac{\partial X}{\partial z}(L,t) = \tfrac{\partial X}{\partial z}(0,t) . \end{aligned} \tag{6}$$

[1]For instance this inequality follows from the fact that we have, for all κ of class $\mathcal{K}_\infty$, and for all positive values a and b,

$$\kappa(a+b) \leq \kappa(2a) + \kappa(2b) ,$$

and from the fact that the function κ_S^{-1} is of class $\mathcal{K}_\infty$.

The problem of the proof of existence of solutions of (4) under Assumptions 1 and 2 is an important issue that has been tackled in the literature depending on the regularity of the function f. Consider e.g. [29, Chap. 15] for local (in time) existence of solution for sufficiently small (with respect to the existence time) and smooth function f. The global (in time) existence of solutions holds as soon as f is globally Lipschitz (see [21, Chap. 6] among other references). When f is superlinear, the finite escape time phenomenon may occur (see for instance [1, Chap. 5] or [20]). In this paper, we do not consider this issue and the results presented here are valid, as long as there exists a solution.

Some comments on Assumptions 1 and 2 follow.

Remark 2. 1. Assumption 1 is equivalent to claiming that V is a weak Lyapunov function for the ordinary differential equation

$$\dot{\Xi} = f(\Xi) \tag{7}$$

with $\Xi \in \mathbb{R}^n$. Therefore it implies that this system is globally stable.

2. Assumption 2 is satisfied in particular if the Dirichlet or Neumann conditions or the periodic conditions, i.e. $X(0,t) = X(L,t)$ and $\frac{\partial X}{\partial z}(0,t) = \frac{\partial X}{\partial z}(L,t)$ for all $t \geq 0$ (see [3]), are satisfied.

3. Since Q is positive definite, there exist two positive real values q_1 and q_2 such that, for all $\Xi \in \mathbb{R}^n$,

$$q_1|\Xi|^2 \leq V(\Xi) \leq q_2|\Xi|^2 \ . \tag{8}$$

The constants q_1 and q_2 will be used in the constructions of strict Lyapunov functions we shall perform later. ○

The construction we perform below is given for instance in [13, 7].

Lemma 3.1. *Under Assumptions 1 and 2, the function*

$$U(\phi) \quad = \quad \int_0^L V(\phi(z))dz \tag{9}$$

is a weak Lyapunov function whose derivative along the solutions of (4) satisfies

$$\dot{U} \quad = \quad -\int_0^L \frac{\partial X^\top}{\partial z}(z,t)Q\frac{\partial X}{\partial z}(z,t)dz - \int_0^L W_1(X(z,t))dz \ . \tag{10}$$

Proof. From (8), it follows that, for all $\phi \in L^2(0,L)$,

$$q_1|\phi|^2_{L^2(0,L)} \leq U(\phi) \leq q_2|\phi|^2_{L^2(0,L)} \ .$$

Since

$$\dot{U} = \int_0^L X^\top(z,t)Q\frac{\partial X}{\partial t}(z,t)dz \ ,$$

it follows that, along a solution of (4),

$$\dot{U} \quad = \quad \int_0^L X^\top(z,t)Q\left[\frac{\partial^2 X}{\partial z^2}(z,t) + f(X(z,t))\right]dz \ .$$

Thus, from (5) in Assumption 1, we have

$$\dot{U} \quad = \quad \int_0^L X^\top(z,t)Q\frac{\partial^2 X}{\partial z^2}(z,t)dz - \int_0^L W_1(X(z,t))dz \ . \tag{11}$$

By integrating by part the first integral in the latter equality, we obtain

$$\begin{array}{rcl}\dot{U} & = & X^\top(L,t)Q\frac{\partial X}{\partial z}(L,t) - X^\top(0,t)Q\frac{\partial X}{\partial z}(0,t) \\ & & -\int_0^L \frac{\partial X^\top}{\partial z}(z,t)Q\frac{\partial X}{\partial z}(z,t)dz - \int_0^L W_1(X(z,t))dz \ .\end{array}$$

From Assumption 2, we deduce that (10) is satisfied. □

3.2. **Strict Lyapunov function for the system (4): first result.** In this section, we show that the function U given in (9) is a strict Lyapunov function for (4) when this system is associated with special families of boundary conditions or when W_1 is larger than a positive definite quadratic function. We state and prove the following result:

Theorem 1. *Assume that the system (4) satisfies Assumptions 1 and 2 and that one of the following property is satisfied:*

(i) there exists a constant $\underline{\alpha} > 0$ such that, for all $\Xi \in \mathbb{R}^n$,

$$W_1(\Xi) \geq \underline{\alpha}|\Xi|^2 \ ,$$

(ii) $X(0,t) = 0$ for all $t \geq 0$,

(iii) $X(L,t) = 0$ for all $t \geq 0$.

Then the function U given in (9) is a strict Lyapunov function for (4).

Proof. Let us assume that the property (i) holds. Then it follows straightforwardly from (8) and (10) that

$$\dot{U} \leq -\frac{\underline{\alpha}}{q_2}\int_0^L V(X(z,t))dz$$

and U is a strict Lyapunov function (see Definition 2.1).

We consider now the cases (ii) and (iii). Let us recall the Poincaré inequality (see [12, Lemma 2.1]).

Lemma 3.2. *For any function w, continuously differentiable on $[0,1]$, and for $c = 0$ or $c = 1$,*

$$\int_0^1 |w(z)|^2 dz \quad \leq \quad 2w^2(c) + 4\int_0^1 \left|\frac{\partial w}{\partial z}(z)\right|^2 dz \ . \tag{12}$$

From this lemma, we deduce that for all $L \geq 0$ and for $c = 0$ or L, and for any function w, continuously differentiable on $[0, L]$, the inequality

$$\int_0^L |w(z)|^2 dz \quad \leq \quad 2Lw^2(c) + 4L^2\int_0^L \left|\frac{\partial w}{\partial z}(z)\right|^2 \ dz \ , \tag{13}$$

is valid. We deduce that when $X(0,t) = 0$ for all $t \geq 0$ or $X(L,t) = 0$ for all $t \geq 0$ then, for all $t \geq 0$, the inequality

$$\int_0^L |X(z,t)|^2 dz \quad \leq \quad \frac{2L^2}{q_1}\int_0^L \frac{\partial X^\top}{\partial z}(z,t)Q\frac{\partial X}{\partial z}(z,t)dz$$

where q_1 is the constant in (8), is satisfied. Combining this inequality with (10) yields

$$\dot{U} \quad \leq \quad -\frac{q_1}{2L^2}\int_0^L |X(z,t)|^2 dz \ .$$

Using (8) again, we can conclude that U is a strict Lyapunov function for the system (4). □

3.3. **Strict Lyapunov function for the system (4): second result.** One can check easily that Assumptions 1 and 2 alone do not ensure that the system (4) admits the zero solution as an asymptotically stable solution.[2] Therefore an extra assumption must be introduced to guarantee that a strict Lyapunov function exists. In Section 3.2 we have exhibited simple conditions which ensure that U is a strict Lyapunov function. In this section, we introduce a new assumption, less restrictive than the condition (i) of Theorem 1, which ensures that a strict Lyapunov function different from U can be constructed.

Assumption 3. *There exist a nonnegative function $M : \mathbb{R}^n \to \mathbb{R}$ of class C^2, and a continuous function $W_2 : \mathbb{R}^n \to \mathbb{R}$ such that*

$$M(0) = 0 \ , \ \frac{\partial M}{\partial \Xi}(0) = 0 \ , \tag{14}$$

$$\frac{\partial M}{\partial \Xi}(\Xi) f(\Xi) \leq -W_2(\Xi) \ , \ \forall \Xi \in \mathbb{R}^n \ , \tag{15}$$

$$\left| \frac{\partial^2 M}{\partial \Xi^2}(\Xi) \right| \leq \frac{q_1}{2} \ , \ \forall \Xi \in \mathbb{R}^n \ , \tag{16}$$

and there exists a constant $q_3 > 0$ such that $W_1 + W_2$ is positive definite and

$$W_1(\Xi) + W_2(\Xi) \geq q_3 |\Xi|^2 \ , \ \forall \Xi \in \mathbb{R}^n : |\Xi| \leq 1 \ , \tag{17}$$

where W_1 is the function defined in (5).

We are ready to state and prove the following result:

Theorem 2. *Under Assumptions 1 to 3, there exists a function k of class $\mathcal{K}_\infty$, of class C^2 such that k' is positive, k'' is nonnegative and the function*

$$\overline{U}(\phi) = \int_0^L k(V(\phi(z)) + M(\phi(z)))dz \tag{18}$$

is a strict Lyapunov function for (4).

Remark 3. Assumption 3 seems to be restrictive. In fact, it can be significantly relaxed. Indeed, if the system

$$\dot{\Xi} = f(\Xi) \tag{19}$$

is locally exponentially stable and satisfies one of Matrosov's conditions which ensure that a strict Lyapunov function can be constructed then one can construct a function M which satisfies Assumption 3. For constructions of strict Lyapunov functions under Matrosov's conditions, the reader is referred to [15]. ◦

Proof. Let us consider the functional $\overline{U}$ defined in (18). Since we impose a priori on k to be of class $\mathcal{K}_\infty$ and (16) holds, we deduce easily that inequalities of the type (2) are satisfied.

[2] More precisely, we can construct examples of systems (4) which are not asymptotically stable when Assumption 1 is satisfied and Assumption 2 holds with the Neumann boundary conditions. For example the system $\frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial z^2}$ where $X \in \mathbb{R}$ with Neumann boundary conditions at $z = 0$ and $z = L$ admits all constant functions as solutions and thus it is not asymptotically stable in L^2 norm.

Next, let us evaluate the time derivative of $\overline{U}$ along the solutions of (4). With the notation $S = V + M$, we have

$$\begin{array}{rcl} \dot{\overline{U}} & = & \int_0^L k'(S(X(z,t)))\frac{\partial S}{\partial \Xi}(X(z,t))\frac{\partial X}{\partial t}(z,t)dz \\ & = & \mathcal{T}_1(X(.,t)) + \mathcal{T}_2(X(.,t)) \end{array} \tag{20}$$

with

$$\begin{array}{rcl} \mathcal{T}_1(\phi) & = & \int_0^L k'(S(\phi(z)))\frac{\partial S}{\partial \Xi}(\phi(z))f(\phi(z))dz \ , \\ \mathcal{T}_2(\phi) & = & \int_0^L k'(S(\phi(z)))\frac{\partial S}{\partial \Xi}(\phi(z))\frac{\partial^2 \phi}{\partial z^2}(z)dz \ . \end{array} \tag{21}$$

Since

$$\frac{\partial S}{\partial \Xi}(\Xi)f(\Xi) = \frac{\partial V}{\partial \Xi}(\Xi)f(\Xi) + \frac{\partial M}{\partial \Xi}(\Xi)f(\Xi) \ ,$$

we deduce from Assumptions 1 and 3 that

$$\mathcal{T}_1(\phi) \leq -\int_0^L k'(S(\phi(z)))[W_1(\phi(z)) + W_2(\phi(z))]dz \ . \tag{22}$$

Now, we consider $\mathcal{T}_2$. By integrating by part, we obtain

$$\mathcal{T}_2(\phi) \quad = \quad \mathcal{T}_3(\phi) - \int_0^L \frac{\partial H(\phi(z))}{\partial z}\frac{\partial \phi}{\partial z}(z)dz \tag{23}$$

with

$$\mathcal{T}_3(\phi) \quad = \quad k'(S(\phi(L)))\tfrac{\partial S}{\partial \Xi}(\phi(L))\tfrac{\partial \phi}{\partial z}(L) - k'(S(\phi(0)))\tfrac{\partial S}{\partial \Xi}(\phi(0))\tfrac{\partial \phi}{\partial z}(0) \tag{24}$$

and

$$H(\Xi) = k'(S(\Xi))\frac{\partial S}{\partial \Xi}(\Xi) \ . \tag{25}$$

Since

$$\begin{array}{rcl} \frac{\partial H(\phi(z))}{\partial z} & = & k''(S(\phi(z)))\frac{\partial S}{\partial \Xi}(\phi(z))\frac{\partial \phi}{\partial z}(z)\frac{\partial S}{\partial \Xi}(\phi(z)) \\ & & +k'(S(\phi(z)))\frac{\partial \phi}{\partial z}^{\top}(z)\frac{\partial^2 S}{\partial \Xi^2}(\phi(z)) \ , \end{array} \tag{26}$$

we deduce from (23) that

$$\mathcal{T}_2(\phi) \quad = \quad \mathcal{T}_3(\phi) - \mathcal{T}_4(\phi) - \mathcal{T}_5(\phi) \tag{27}$$

with

$$\begin{array}{rcl} \mathcal{T}_4(\phi) & = & \int_0^L k''(S(\phi(z)))\left(\frac{\partial S}{\partial \Xi}(\phi(z))\frac{\partial \phi}{\partial z}(z)\right)^2 dz \ , \\ \mathcal{T}_5(\phi) & = & \int_0^L k'(S(\phi(z)))\frac{\partial \phi}{\partial z}^{\top}(z)\frac{\partial^2 S}{\partial \Xi^2}(\phi(z))\frac{\partial \phi}{\partial z}(z)dz \ . \end{array} \tag{28}$$

Since we impose on k to be such that k'' is nonnegative, we immediately deduce that

$$\mathcal{T}_2(\phi) \leq \mathcal{T}_3(\phi) - \mathcal{T}_5(\phi) \ . \tag{29}$$

Now, observe that

$$\tfrac{\partial^2 S}{\partial \Xi^2}(\phi(z)) \quad = \quad Q + \tfrac{\partial^2 M}{\partial \Xi^2}(\phi(z)) \ . \tag{30}$$

This equality, inequalities (8), Assumption 3 and the fact that we impose on k to be such that $k' > 0$ ensure that

$$\begin{array}{rcl} \mathcal{T}_5(\phi) & \geq & 2q_1\int_0^L k'(S(\phi(z)))\left|\frac{\partial \phi}{\partial z}(z)\right|^2 dz - \frac{q_1}{2}\int_0^L k'(S(\phi(z)))\left|\frac{\partial \phi}{\partial z}(z)\right|^2 dz \\ & = & \frac{3q_1}{2}\int_0^L k'(S(\phi(z)))\left|\frac{\partial \phi}{\partial z}(z)\right|^2 dz \ . \end{array}$$

It follows that

$$\mathcal{T}_2(\phi) \quad \leq \quad \mathcal{T}_3(\phi) - \tfrac{3q_1}{2} \int_0^L k'(S(\phi(z))) \left| \frac{\partial \phi}{\partial z}(z) \right|^2 dz \ . \tag{31}$$

Hence (20), (22), and (31) give

$$\begin{aligned} \dot{\overline{U}} \quad \leq \quad & - \int_0^L k'(S(X(z,t)))W_3(X(z,t))dz \\ & + \mathcal{T}_3(X(.,t)) - \tfrac{3q_1}{2} \int_0^L k'(S(X(z,t))) \left| \frac{\partial X}{\partial z}(z,t) \right|^2 dz \end{aligned}$$

with $W_3 = W_1 + W_2$. Assumption 2 ensures that, for all $t \geq 0$, $\mathcal{T}_3(X(.,t)) = 0$. We deduce that

$$\dot{\overline{U}} \quad \leq \quad - \int_0^L k'(S(X(z,t)))W_3(X(z,t))dz \ .$$

By (17) and the inequality $S(\Xi) \geq q_1|\Xi|^2$, we can prove the following

Lemma 3.3. *There exist a constant $C > 0$, a twice continuously differentiable function k of class $\mathcal{K}_\infty$ such that k' is positive and k'' is nonnegative and*

$$k'(S(\Xi))W_3(\Xi) \geq Ck(S(\Xi)) \tag{32}$$

for all $\Xi \in \mathbb{R}^n$.

The proof of this lemma is postponed to Appendix 7. Selecting the function k provided by Lemma 3.3, we obtain

$$\dot{\overline{U}} \leq -C\overline{U}(X(.,t)) \ .$$

It follows that $\overline{U}$ is a strict Lyapunov function for (4). □

4. **ISS property for a family of PDEs.** In the previous section, we have constructed Lyapunov functions for PDEs without uncertainties and without convection term. In this section, we show how our technique of construction can be used to estimate the impact of uncertainties on the solutions of PDEs with a convection term and uncertainties of the form

$$\begin{aligned} \frac{\partial X}{\partial t}(z,t) \quad = \quad & \frac{\partial^2 X}{\partial z^2}(z,t) + [D_1 + v(z,t)]\frac{\partial X}{\partial z}(z,t) \\ & + f(X(z,t)) + u(z,t) \ , \end{aligned} \tag{33}$$

where D_1 is a constant matrix, v is an unknown continuous matrix function and u is an unknown continuous function.

Remark 4. For a linear finite dimensional system

$$\dot{x} = Ax + Bu \ , \tag{34}$$

where A and B are constant matrices respectively in $\mathbb{R}^{n\times n}$ and in $\mathbb{R}^{n\times 1}$, it is well-known that if $\dot{x} = Ax$, is asymptotically stable then bounded inputs result in bounded solutions. However, for nonlinear finite dimensional systems, global asymptotic stability does not imply input/state stability of any sort. See [28, Section 2.6] for a simple scalar example, which is globally asymptotically stable but which has solutions with a finite time explosion for a suitable constant input.

Also for linear infinite dimensional system, asymptotic stability does not imply input-to-state stability. More precisely, we exhibit in Section 6 an example of linear system which is globally asymptotically stable without any input, but which can have unbounded solutions in the presence of a bounded input (and thus for this system there does not exist any ISS Lyapunov function). ∘

To cope with the presence of a convection term and the uncertainty v, we introduce the following assumption:

Assumption 4. *There exists a nonnegative real number δ such that*

$$|v(z,t)| \leq \frac{\delta}{|Q|}\ ,\ \forall z \in [0,L]\ , t \geq 0\ , \tag{35}$$

where Q is the the symmetric positive definite matrix in Assumption 1. Moreover, the matrix QD_1 is symmetric.

Moreover, we replace Assumption 3 by a more restrictive assumption:

Assumption 5. *There exists a nonnegative function $M : \mathbb{R}^n \to \mathbb{R}$ such that, for all $\Xi \in \mathbb{R}^n$,*

$$M(0) = 0\ , \quad \tfrac{\partial M}{\partial \Xi}(\Xi) f(\Xi) \quad = \quad -W_2(\Xi)\ , \tag{36}$$

where W_2 is a nonnegative function and there exist $c_a > 0$, $c_b > 0$ and $c_c > 0$ such that, for all $\Xi \in \mathbb{R}^n$, the inequalities

$$\left|\frac{\partial M}{\partial \Xi}(\Xi)\right| \leq c_a |\Xi|\ ,\ \left|\frac{\partial^2 M}{\partial \Xi^2}(\Xi)\right| \leq c_b\ , \tag{37}$$

$$|\Xi|^2 \leq c_c[W_1(\Xi) + W_2(\Xi)]\ , \tag{38}$$

where W_1 is the function defined in (5), are satisfied.

Remark 5. If f is linear and $\dot{\Xi} = f(\Xi)$ is exponentially stable, then Assumption 5 is satisfied with a positive definite quadratic function as function M. ∘

We are ready to state and prove the main result of the section

Theorem 3. *Assume that the system (33) satisfies Assumptions 1, 4 and 5 and is associated with boundary conditions satisfying*

$$X(L,t) = X(0,t) \text{ and } \frac{\partial X}{\partial z}(L,t) = \frac{\partial X}{\partial z}(0,t)\ ,\ \forall t \geq 0\ . \tag{39}$$

Then the function

$$\overline{U}(\phi) = \int_0^L [KV(\phi(z)) + M(\phi(z))]dz \tag{40}$$

with

$$K = \max\left\{1, \frac{2c_b}{q_1}, \frac{8c_c c_a^2(|D_1|+1)^2}{q_1}\right\} \tag{41}$$

satisfies, along the trajectories of (33),

$$\dot{\overline{U}} \leq -\lambda_1 \overline{U}(X(z,t)) + \lambda_2 \int_0^L |u(z,t)|^2 dz \tag{42}$$

for some positive constants λ_1, λ_2, provided that δ in Assumption 4 satisfies

$$\delta \leq \min\left\{|Q|, \frac{\sqrt{q_1}}{2\sqrt{2c_c K}}\right\}\ . \tag{43}$$

Remark 6. Using Assumption 5, one can prove easily that the ISS Lyapunov function $\overline{U}$ defined in (40) is upper and lower bounded by a positive definite quadratic function. We deduce easily that (42) leads to an ISS inequality of the type

$$\begin{array}{rcl} |X(.,t)|_{L^2(0,L)} & \leq & \Lambda_1 e^{-\frac{\lambda_1}{2}(t-t_0)}|X(.,t_0)|_{L^2(0,L)} \\ & & +\Lambda_2 \sup_{m\in[t_0,t]} \sqrt{\int_0^L |u(z,m)|^2 dz} \ , \end{array} \tag{44}$$

where Λ_1, Λ_2 are positive real numbers. ◦

Proof. Since $K \geq 1$ and M is nonnegative, we deduce easily that inequalities of the type (2) are satisfied by $\overline{U}$ defined in (40).

Considering again the function U defined by (9), we obtain that, along the solutions of (33), the equality

$$\begin{array}{rcl} \dot{U} & = & \int_0^L X^\top(z,t)Q\left[\frac{\partial^2 X}{\partial z^2}(z,t) + f(X(z,t))\right]dz \\ & & +\int_0^L X^\top(z,t)QD_1\frac{\partial X}{\partial z}(z,t)dz \\ & & +\int_0^L X^\top(z,t)Qv(z,t)\frac{\partial X}{\partial z}(z,t)dz + \int_0^L X^\top(z,t)Qu(z,t)dz \end{array} \tag{45}$$

is satisfied. Since Assumption 4 guarantees that $QD_1 = \mathtt{Sym}(QD_1)$, where $\mathtt{Sym}$ is the function defined in the introduction, and since the equality

$$\int_0^L \phi^\top(z)\mathtt{Sym}(QD_1)\frac{\partial\phi}{\partial z}(z)dz = \frac{1}{2}[\phi^\top(L)\mathtt{Sym}(QD_1)\phi(L) - \phi^\top(0)\mathtt{Sym}(QD_1)\phi(0)]$$

is satisfied for all $\phi \in C_{\mathrm{L}}$, Assumptions 1, 4 and the condition (39) yield

$$\begin{array}{rcl} \dot{U} & \leq & \int_0^L X^\top(z,t)Q\frac{\partial^2 X}{\partial z^2}(z,t)dz - \int_0^L W_1(X(z,t))dz \\ & & +\delta\int_0^L |X(z,t)|\left|\frac{\partial X}{\partial z}(z,t)\right|dz + |Q|\int_0^L |X(z,t)||u(z,t)|dz \ . \end{array}$$

Next, arguing as we did from the inequality (11) to the end of the proof of Lemma 3.1, we obtain:

$$\begin{array}{rcl} \dot{U} & \leq & -2q_1\int_0^L \left|\frac{\partial X}{\partial z}(z,t)\right|^2 dz - \int_0^L W_1(X(z,t))dz \\ & & +\delta\int_0^L |X(z,t)|\left|\frac{\partial X}{\partial z}(z,t)\right|dz + |Q|\int_0^L |X(z,t)||u(z,t)|dz \ . \end{array} \tag{46}$$

Next, we evaluate the derivative of M along the trajectories of (33)

$$\begin{array}{rcl} \dot{M} & = & \int_0^L \frac{\partial M}{\partial \Xi}(X(z,t))\left[\frac{\partial^2 X}{\partial z^2}(z,t) + f(X(z,t))\right]dz \\ & & +\int_0^L \frac{\partial M}{\partial \Xi}(X(z,t))D_1\frac{\partial X}{\partial z}(z,t)dz + \int_0^L \frac{\partial M}{\partial \Xi}(X(z,t))v(z,t)\frac{\partial X}{\partial z}(z,t)dz \\ & & +\int_0^L \frac{\partial M}{\partial \Xi}(X(z,t))u(z,t)dz \ . \end{array}$$

Using Assumption 5, we deduce that

$$\begin{aligned}\dot{M} \quad &\leq \quad -\int_0^L W_2(X(z,t))dz + \mathcal{S}_1(X(.,t)) \\ &\quad +c_a(|D_1| + \frac{\delta}{|Q|})\int_0^L |X(z,t)| \left|\frac{\partial X}{\partial z}(z,t)\right| dz \\ &\quad +c_a\int_0^L |X(z,t)||u(z,t)|dz\end{aligned}$$

with

$$\mathcal{S}_1(\phi) = \int_0^L \frac{\partial M}{\partial \Xi}(\phi(z))\frac{\partial^2\phi}{\partial z^2}(z)dz \ .$$

By integrating by part, we obtain

$$\begin{aligned}\mathcal{S}_1(\phi) \quad &= \quad \tfrac{\partial M}{\partial \Xi}(\phi(L))\tfrac{\partial \phi}{\partial z}(L) - \tfrac{\partial M}{\partial \Xi}(\phi(0))\tfrac{\partial \phi}{\partial z}(0) \\ &\quad -\int_0^L \frac{\partial \phi}{\partial z}^\top(z)\frac{\partial^2 M}{\partial \Xi^2}(\phi(z))\frac{\partial \phi}{\partial z}(z)dz \ .\end{aligned}$$

From (39) and Assumption 5, we deduce that

$$\mathcal{S}_1(X(.,t)) \quad \leq \quad c_b\int_0^L \left|\frac{\partial X}{\partial z}(z,t)\right|^2 dz \ .$$

It follows that

$$\begin{aligned}\dot{M} \quad &\leq \quad -\int_0^L W_2(X(z,t))dz + c_b\int_0^L \left|\frac{\partial X}{\partial z}(z,t)\right|^2 dz \\ &\quad +c_a(|D_1| + \frac{\delta}{|Q|})\int_0^L |X(z,t)| \left|\frac{\partial X}{\partial z}(z,t)\right| dz \\ &\quad +c_a\int_0^L |X(z,t)||u(z,t)|dz \ .\end{aligned} \tag{47}$$

From (46) and (47), we deduce that

$$\begin{aligned}\dot{U} \quad &\leq \quad (-2q_1K + c_b)\int_0^L \left|\frac{\partial X}{\partial z}(z,t)\right|^2 dz \\ &\quad -\mathcal{S}_2(X(.,t)) + \mathcal{S}_3(X(.,t)) + c_d\int_0^L |X(z,t)||u(z,t)|dz\end{aligned} \tag{48}$$

with $c_d = K|Q| + c_a$,

$$\mathcal{S}_2(\phi) = \int_0^L [KW_1(\phi(z)) + W_2(\phi(z))]dz \ ,$$

$$\mathcal{S}_3(\phi) = [K\delta + c_a(|D_1| + \delta)]\int_0^L |\phi(z)|\left|\frac{\partial \phi}{\partial z}(z)\right| dz \ .$$

Since $K \geq 1$, we deduce from Assumption 5 that

$$\mathcal{S}_2(\phi) \geq \frac{1}{c_c}\int_0^L |\phi(z)|^2 dz \ . \tag{49}$$

From Young's inequality, we deduce that

$$\mathcal{S}_3(\phi) \quad \leq \quad \tfrac{1}{2}q_1K\int_0^L \left|\frac{\partial \phi}{\partial z}(z)\right|^2 dz + \frac{[K\delta + c_a(|D_1| + \frac{\delta}{|Q|})]^2}{2q_1K}\int_0^L |\phi(z)|^2 dz \ , \tag{50}$$

$$\int_0^L |X(z,t)||u(z,t)|dz \quad \leq \quad \tfrac{1}{2c_cc_d}\int_0^L |X(z,t)|^2 dz + \frac{c_cc_d}{2}\int_0^L |u(z,t)|^2 dz \ . \tag{51}$$

Combining (48), (49), (50), (51) and the inequality $[K\delta+c_a(|D_1|+\frac{\delta}{|Q|})]^2 \leq 2K^2\delta^2+2c_a^2(|D_1|+\frac{\delta}{|Q|})^2$, we obtain

$$\begin{array}{rcl}\dot{U} & \leq & (-\frac{3}{2}q_1K+c_b)\displaystyle\int_0^L \left|\frac{\partial X}{\partial z}(z,t)\right|^2 dz \\ & & +\left(\dfrac{K^2\delta^2+c_a^2(|D_1|+\frac{\delta}{|Q|})^2}{q_1K}-\dfrac{1}{2c_c}\right)\displaystyle\int_0^L |X(z,t)|^2dz \\ & & +\dfrac{c_cc_d^2}{2}\displaystyle\int_0^L |u(z,t)|^2dz\ .\end{array}$$

We deduce easily that if (43) is satisfied, then $K^2\delta^2 \leq \frac{q_1}{8c_c}K$ and $c_a^2(|D_1|+\frac{\delta}{|Q|})^2 \leq c_a^2(|D_1|+1)^2$. Therefore we have $K^2\delta^2+c_a^2(|D_1|+\frac{\delta}{|Q|})^2 \leq \frac{q_1}{8c_c}K+c_a^2(|D_1|+1)^2$. This inequality and (41) imply that $K^2\delta^2+c_a^2(|D_1|+\frac{\delta}{|Q|})^2 \leq \frac{q_1}{8c_c}K+\frac{q_1}{8c_c}K$. Next, from this inequality and (41), we deduce easily that

$$\begin{array}{rcl}\dot{U} & \leq & -q_1K\displaystyle\int_0^L \left|\frac{\partial X}{\partial z}(z,t)\right|^2 dz-\dfrac{1}{4c_c}\displaystyle\int_0^L |X(z,t)|^2dz \\ & & +\frac{c_cc_d^2}{2}\displaystyle\int_0^L |u(z,t)|^2dz\ .\end{array}$$

From Assumption 5, we deduce that, for all $\Xi\in\mathbb{R}^n$, $M(\Xi)\leq c_a|\Xi|^2$. It follows that (42) is satisfied with $\lambda_1=\frac{1}{4c_c(Kq_2+c_a)}$, and $\lambda_2=\frac{c_cc_d^2}{2}$. This concludes the proof. ☐

5. **Examples.**

5.1. **First example.**

In this section, we illustrate Theorem 1 through the following linear parabolic system, for all $z\in(0,L)$ and for all $t\geq 0$,

$$\left\{\begin{array}{rcl}\frac{\partial x_1}{\partial t}(z,t) & = & \frac{\partial^2 x_1}{\partial z^2}(z,t)+x_2(z,t)\\ \frac{\partial x_2}{\partial t}(z,t) & = & \frac{\partial^2 x_2}{\partial z^2}(z,t)-x_1(z,t)\end{array}\right. \tag{52}$$

with $X=(x_1,\ x_2)^\top\in\mathbb{R}^2$ and with the Dirichlet boundary conditions: $x_1(0,t)=x_1(L,t)=x_2(0,t)=x_2(L,t)$ for all $t\geq 0$. Equation (52) is a system of two heat equations with a coupling term which does not introduce any stability. Indeed the coupling between both dynamics is given by $f(X)=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}X$ and the matrix $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ is stable but not asymptotically stable. Hence the stability that will be observed is a consequence of the presence of the diffusion terms only. Assumptions 1 and 2 hold with the positive definite quadratic function

$$V(\Xi)=\frac{1}{2}[\xi_1^2+\xi_2^2]$$

and the nonnegative function $W_1(\Xi)=0$, for all $\Xi\in\mathbb{R}^2$. Moreover the condition (i) of Theorem 1 does not hold but conditions (ii) and (iii) are satisfied. Therefore Theorem 1 guarantees that U given in (9) is a strict Lyapunov function for (52).

To numerically check the stability, let us discretize the linear parabolic equation (52) using an explicit Euler discretization[3]. We select the parameters of the numerical scheme so that the CFL condition for the stability holds. More precisely,

[3]The simulation codes for both examples can be downloaded from www.gipsa-lab.fr/~christophe.prieur/Codes/2011-Mazenc-Prieur.zip

with $L = 2\pi$, we divide the space domain $[0, L]$ into 40 intervals of identical length, and choosing 10 as final time, we set a time discretization of 10^{-2}. For the initial condition, we select the functions $x_1(z, 0) = \sin(z)$ and $x_2(z, 0) = z(z - L)$, for all $z \in [0, L]$. The time evolutions of the components x_1 and x_2 of the solution are given in Figures 1 and 2 respectively. We can observe that the solution converges as expected to the equilibrium.

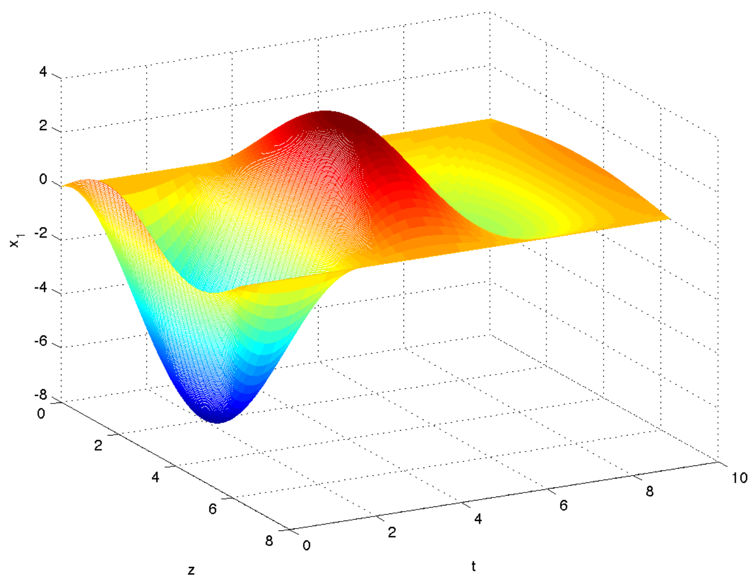

FIGURE 1. Component x_1 of the solution of (52) for $t \in [0, 10]$

5.2. **Second example.** In this section, we illustrate Theorem 3 through the semilinear parabolic system with a convection term, for all $z \in (0, 1)$, and for all $t \geq 0$,

$$\left\{ \begin{array}{rcl} \frac{\partial x_1}{\partial t}(z,t) & = & \frac{\partial^2 x_1}{\partial z^2}(z,t) - \frac{\partial x_1}{\partial z}(z,t) \\ & & +x_2(z,t)[1 + x_1(z,t)^2] + u_1(z,t) \\ \frac{\partial x_2}{\partial t}(z,t) & = & \frac{\partial^2 x_2}{\partial z^2}(z,t) - x_1(z,t)[1 + x_1(z,t)^2] \\ & & -x_2(z,t)[2 + x_1(z,t)^2] + u_2(z,t) \end{array} \right. \tag{53}$$

with $X = (x_1,\ x_2)^\top \in \mathbb{R}^2$ and where u_1 and u_2 are continuous real-valued functions. We associate to (53) the boundary conditions (39) particularized to the two-dimensional case. Equation (53) is a system of two heat equations with a convection term in the first. The interest of this example is that its nonlinear terms are not globally Lipschitz, thus, to the best of our knowledge, no global stability result available in the literature applies to this system.

Let us check that Theorem 3 applies. One can check readily that Assumptions 1, 4 and 5 are satisfied with the positive definite quadratic functions

$$V(\Xi) = \tfrac{1}{2}[\xi_1^2 + \xi_2^2] \ ,$$
$$M(\Xi) = \xi_1^2 + \xi_2^2 + \xi_1\xi_2 \ ,$$

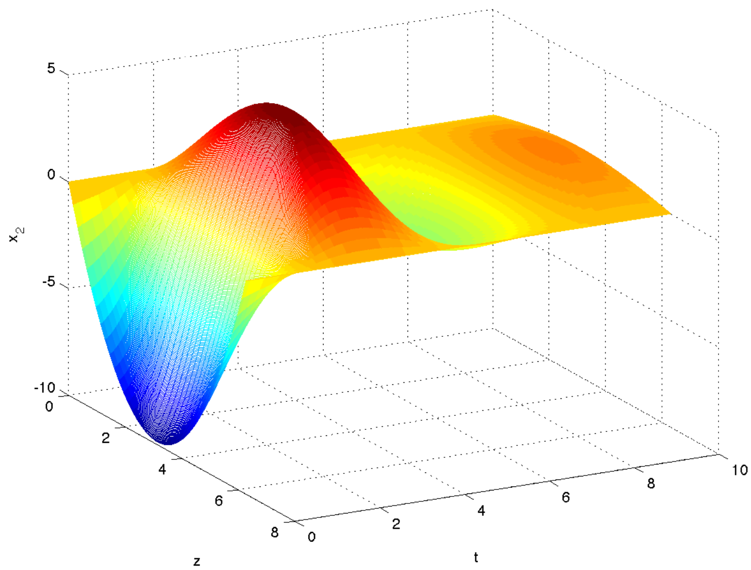

FIGURE 2. Component x_2 of the solution of (52) for $t \in [0, 10]$

and the matrix $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Indeed, with $f(\Xi) = (\xi_2[1+\xi_1^2]\ ,\ -\xi_1[1+\xi_1^2] - \xi_2[2+\xi_1^2])^\top$, we have, for all $\Xi \in \mathbb{R}^2$,

$$\begin{array}{rcl} \frac{\partial V}{\partial \Xi}(\Xi) f(\Xi) & = & -\xi_2^2[2+\xi_1^2]\ , \\ \frac{\partial M}{\partial \Xi}(\Xi) f(\Xi) & = & -(3+\xi_1^2)\xi_2^2 - (1+\xi_1^2)\xi_1^2 - \xi_1\xi_2(2+\xi_1^2) \\ & \leq & -\xi_2^2 - \left(\frac{1}{2} + \frac{3}{4}\xi_1^2\right)\xi_1^2\ . \end{array}$$

Moreover, through elementary calculations, we obtain the following values: $q_1 = \frac{1}{2}$, $|D_1| = 1$, $c_a = c_b = 3$, $c_c = 2$ for the constants in (41). Therefore Theorem 3 guarantees that, if (39) is satisfied, then the function

$$\overline{U}(\phi) = 1153 \int_0^L \left[\phi_1(z)^2 + \phi_2(z)^2\right] dz + \int_0^L \phi_1(z)\phi_2(z) dz \tag{54}$$

is an ISS Lyapunov function for the system (53).

To numerically check the fact that $\overline{U}$ is a Lyapunov function, let us discretize the semilinear parabolic partial differential equation (53) using an explicit Euler discretization We select the parameters of the numerical scheme so that the CFL condition for the stability holds. More precisely, with $L = 2\pi$, we divide the space domain $[0, L]$ into 40 intervals of identical length, and, choosing 10 as final time, we set a time discretization of 10^{-2}. For the initial condition, we select the functions $x_1(z, 0) = \cos(z)$ and $x_2(z, 0) = 1$, for all $z \in [0, L]$. The numerical simulations are performed with $u_1(z,t) = \sin^2(t)$, $u_2(z,t) = 0$ for all $z \in [0, L]$ and $t \in [0, 5]$ and $u_1(z,t) = u_2(z,t) = 0$ for all $z \in [0, L]$ and for all $t \in (5, 10]$. The time evolutions of the components x_1 and x_2 of the solution are given in Figures 3 and 4 respectively. We can observe that the solution converges as expected to the equilibrium after the perturbations vanish (i.e. after $t = 5$).

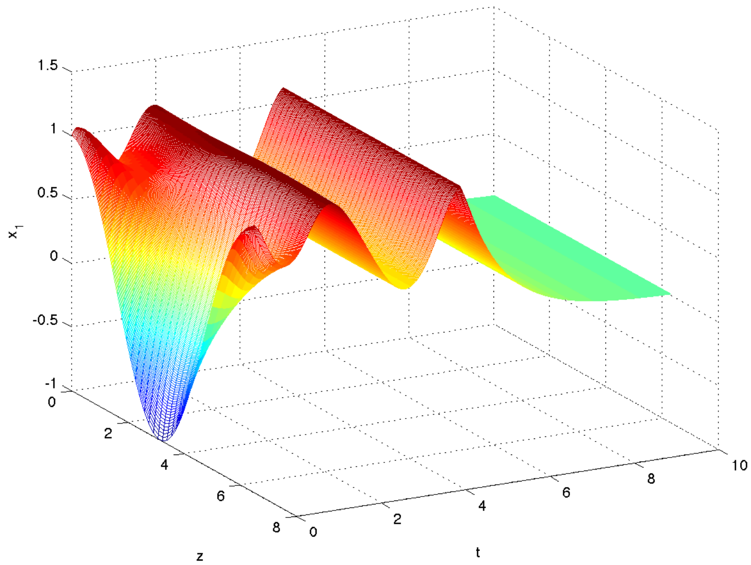

FIGURE 3. Component x_1 of the solution of (53) for $t \in [0, 10]$

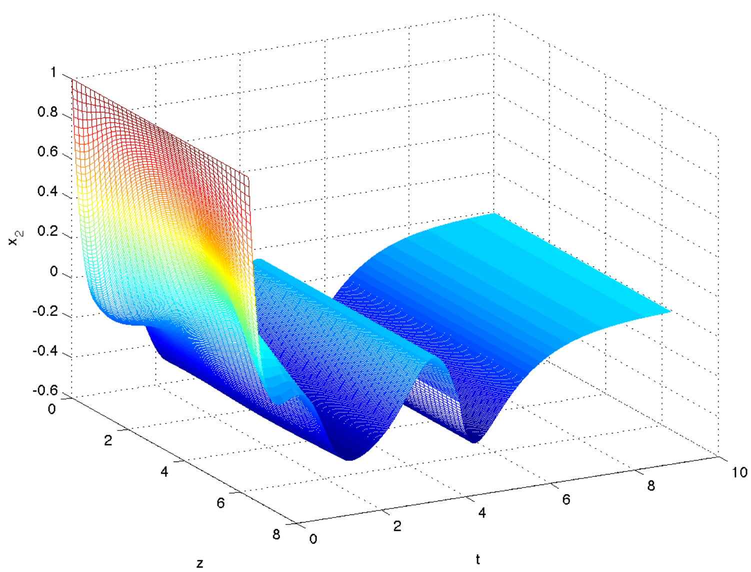

FIGURE 4. Component x_2 of the solution of (53) for $t \in [0, 10]$

Moreover we may check on Figure 5 that the Lyapunov function $\overline{U}$ defined in (54) converges to zero after the perturbations vanish.

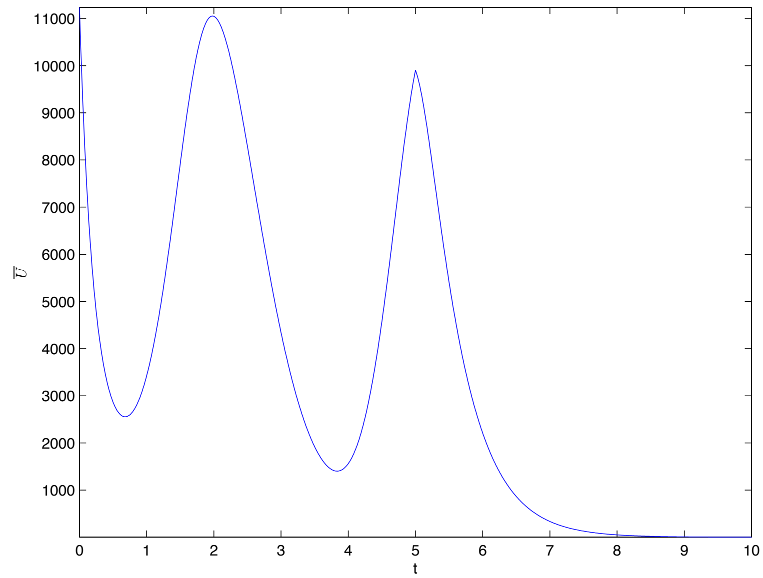

FIGURE 5. Time-evolution of the function $\overline{U}$ given by (54)

6. **Appendix: Illustration of Remark 4.** In this section, we exhibit an example of a linear infinite dimensional system which is globally asymptotically stable in the absence of input, but which may admit unbounded solutions in presence of a bounded input. Consequently, this system is globally asymptotically stable but it doesnot admit any ISS Lyapunov function.

This system is composed by an infinite number of scalar ordinary differential equations written by, for each $n \in \mathbb{N}$ and for all $t \geq 0$,

$$\dot{X}_n(t) = -\frac{1}{n+1} X_n(t) + u_n(t) \ . \tag{55}$$

Given an initial condition in $l^2(\mathbb{N})$ and for $u_n(t) = 0$ for all $t \geq 0$ and for all $n \in \mathbb{N}$, each system (55) admits a solution in $l^2(\mathbb{N})$ defined for all time $t \geq 0$. Moreover the system (55) is globally asymptotic stable when, for all $n \in \mathbb{N}$ and all $t \geq 0$, $u_n(t) = 0$.

Let us consider the input satisfying, for each $n \in \mathbb{N}$, and for all $t \geq 0$,

$$u_n(t) = \frac{1}{(n+1)^2} e^{-\frac{1}{(n+1)^2} t} \ .$$

Note that the function $t \mapsto \|(u_n(t))_{n\in\mathbb{N}}\|_{l^2(\mathbb{N})}$, where $\|\cdot\|_{l^2(\mathbb{N})}$ denotes the usual norm in $l^2(\mathbb{N})$, is bounded. This function is even so that, for all $t \geq 0$, $\|(u_n(t))_{n\in\mathbb{N}}\|_{l^2(\mathbb{N})} \leq \frac{\pi^2}{6} e^{-t}$, which implies that $\|(u_n(t))_{n\in\mathbb{N}}\|_{l^2(\mathbb{N})}$ converges exponentially to zero. On the other hand, the solution of (55) with, for all $n \in \mathbb{N}$, $X_n(0) = 0$ is given by, for all $t \geq 0$,

$$X_n(t) = \frac{1}{n} \left[e^{-\frac{1}{(n+1)^2} t} - e^{-\frac{1}{n+1} t} \right]$$

for all $n \geq 1$. Therefore $(X_n(t))_{n\in\mathbb{N}}$ is not in $l^2(\mathbb{N})$ for $t > 0$. It follows that the system composed of the systems (55) is not ISS with respect to $(u_n(t))_{n\in\mathbb{N}}$.

7. **Appendix: Proof of Lemma 3.3.** Before proving Lemma 3.3, let us state and prove the following technical result:

Lemma 7.1. *Let $P : \mathbb{R} \to \mathbb{R}$ be a positive and increasing function. Then the function*

$$\kappa(m) = me^{\int_1^m P(\ell)d\ell}$$

is such that

$$\kappa'(m)m \geq \min\left\{1, \frac{1}{P(1)}\right\} P(m)\kappa(m) \ ,$$

for all $m \geq 0$.

Proof. An elementary calculation gives

$$\kappa'(m) = [1 + mP(m)]\, e^{\int_1^m P(\ell)d\ell} \ .$$

Therefore

$$\kappa'(m)m = [1 + mP(m)]\, me^{\int_1^m P(\ell)d\ell} = [1 + mP(m)]\, \kappa(m) \ .$$

Therefore, when $m \geq 1$,

$$\kappa'(m)m \geq P(m)\kappa(m)$$

and, when $m \in [0, 1]$, we deduce from the fact that P is positive and increasing that

$$\kappa'(m)m \geq \kappa(m) \geq \frac{P(m)}{P(1)}\kappa(m) \ .$$

This allows us to conclude the proof of Lemma 7.1. □

We are now in position to prove Lemma 3.3. Since $S(\Xi) = V(\Xi) + M(\Xi)$, due to (8), (14) and (16), the inequalities

$$q_1|\Xi|^2 \leq S(\Xi) \leq \left(q_2 + \frac{q_1}{2}\right) |\Xi|^2$$

hold for all $\Xi \in \mathbb{R}^n$. Next, we observe that, due to (17), W_3 is positive definite and $W_3(\Xi) \geq q_3|\Xi|^2$ for all $\Xi \in \mathbb{R}^n$ such that $|\Xi| \leq 1$. We deduce that there exists a positive, increasing and continuously differentiable function Γ such that, for all $\Xi \in \mathbb{R}^n$,

$$\Gamma(q_1|\Xi|^2)W_3(\Xi) \geq \left(q_2 + \frac{q_1}{2}\right) |\Xi|^2 \ .$$

Therefore, for all $\Xi \in \mathbb{R}^n$,

$$W_3(\Xi) \geq \frac{S(\Xi)}{\Gamma\left(S(\Xi)\right)} \ .$$

This implies that, for all continuously differentiable functions k of class $\mathcal{K}_\infty$, the inequality

$$k'(S(\Xi))W_3(\Xi) \geq \frac{k'(S(\Xi))S(\Xi)}{\Gamma\left(S(\Xi)\right)} \ , \tag{56}$$

is satisfied, for all $\Xi \in \mathbb{R}^n$. Next, from Lemma 7.1, we deduce that the function

$$k(m) = me^{\int_1^m \Gamma(\ell)d\ell} \ ,$$

which is of class $\mathcal{K}_\infty$, twice continuously differentiable such that k' is positive, is such that, for all $\Xi \in \mathbb{R}^n$,

$$\frac{k'(S(\Xi))S(\Xi)}{\Gamma\left(S(\Xi)\right)} \geq \min\left\{1, \frac{1}{\Gamma(1)}\right\} k(S(\Xi)) \ . \tag{57}$$

The combination of (56) and (57) implies that (32) is satisfied with $C = \min\left\{1, \frac{1}{\Gamma(1)}\right\}$. Finally observe that since Γ is increasing and of class C^1, k'' is nonnegative. This concludes the proof of Lemma 3.3.

8. **Conclusion.**

For important families of PDEs, we have shown how weak Lyapunov functions can be transformed into strict Lyapunov functions. Robustness properties of ISS type can be inferred from our constructions. In a different context, the stability of hyperbolic systems has been established using the Lyapunov theory. More precisely, based on [4], strictification techniques have been used for a class of time-varying hyperbolic equations in [24]. One natural perspective of the present paper could be to consider time-varying semilinear parabolic equations.

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