Stabilization in a Two-Species Chemostat With Monod Growth Functions
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Abstract—We design feedback controllers for two species chemostats so that an equilibrium with arbitrary prescribed positive species concentrations becomes globally asymptotically stable. We use a new global explicit strict Lyapunov function construction, which allows us to quantify the effects of disturbances using the input-to-state stability paradigm. We assume that only a linear combination of the species concentrations is known. We illustrate our approach using a numerical example.

Index Terms—Bioreactors, integral input-to-state stability (iISS), uncertain states.

I. INTRODUCTION

This work continues our search, begun in [10], [11], for ways to stabilize prescribed equilibrium behaviors in chemostats. See [2], [4], [12] for the fundamental role of chemostats in bioengineering. The basic model for a well mixed chemostat with two competing species is

\[
\begin{align*}
\dot{s} &= D(s_m - s) - \frac{\mu_1(s)}{Y_1} X_1 - \frac{\mu_2(s)}{Y_2} X_2 \\
\dot{X}_i &= [\mu_i(s) - D] X_i, \quad i = 1, 2
\end{align*}
\]

(1)
evolving on \(X := (0, \infty)^3\) [12, Chapter 1]. Here \(s(t)\) is the concentration of the substrate; \(X_1(t)\) and \(X_2(t)\) are the concentrations of the two species of organism; the dilution rate \(D(\cdot)\) and the input nutrient concentration \(s_m(\cdot)\) are controllers that we will specify; \(\mu_1\) and \(\mu_2\) are given uptake functions; and \(Y_1\) and \(Y_2\) are positive constants called yield coefficients. We assume throughout that the \(\mu_i\)’s have the Monod form

\[
\mu_i(s) = \frac{K_i s}{L_i + s}
\]

(2)
where \(K_i\) and \(L_i\) are positive constants we indicate below. Since \(D\) is the ratio of the volumetric flow rate, (with units of volume over time), to the constant reactor volume, it is proportional to the speed of the pump that supplies the reactor with fresh medium containing the nutrient.

The competitive exclusion principle implies that in classical chemostats with one limiting substrate and constant \(D\) and \(s_m\), at most one species can survive generically [12]. This is at odds with the observation that in real ecological systems, it is common for many species to coexist in equilibrium on one limiting nutrient. This paradox has motivated a great deal of research [4], [6]. Also, the species concentrations may not be available for measurement, and there may be actuator errors caused e.g. by variability in the speed of the pump supplying the fresh nutrient. Hence, it is important to quantify the robustness of any stabilizer to uncertainty.

This work addresses all of these issues by designing \(s_m\) and \(D(\cdot)\), depending only on a linear combination \(Y = X_1 + AX_2\) of the species concentrations, (where \(A\) is a given positive constant), that globally stabilize an equilibrium with arbitrary positive prescribed species concentrations. The design of stabilizing feedbacks depending only on a (weighted) sum of the species concentrations is motivated by applications where photometric methods preclude the possibility of separately measuring the individual species levels [4]. Another important feature is that our new construction for an explicit multi-species chemostat strict Lyapunov function makes it possible to quantify the effects of disturbances using integral input-to-state stability (iISS) and input-to-state stability (ISS) [13]; see [3], [14] for the essential role of (i)ISS in nonlinear control and applications. Our explicit Lyapunov function is a significant and original theoretical development.

II. ASSUMPTIONS AND STABILITY THEOREM

Set \(\chi(s) = \mu_2(s) - \mu_1(s)\). We always assume:

Assumption 1: There is a constant \(s_0 > 0\) such that 1) \(\chi(s) = 0\); 2) \(\chi(s) < 0\) when \(0 < s < s_0\); and 3) \(\chi(s) > 0\) when \(s > s_0\). In particular, \(\mu_1(s) = \mu_2(s)\).

Simple calculations and our Assumption 1 readily yield

\[
\mu_i(s) - \mu_i(r) = \frac{L_i}{L_i + r}\mu_i(s - r) \quad \forall s, r > 0
\]

(3)
for \(i = 1, 2\), \(s_0 = (L_1 K_2 - L_2 K_1)/(K_1 - K_2), \mu_1(s) = (L_1 K_2 - L_2 K_1)/(L_1 - L_2), K_2 > K_1, K_1 L_2 > K_2 L_1\), and \(L_2 > L_1\). We transform (1) with the output \(Y = X_1 + AX_2\), using \(x_1 = X_1/Y_1\) and \(x_2 = X_2/Y_2\) to obtain

\[
\begin{align*}
\dot{s} &= D(s_m - s) - \mu_1(s)x_1 - \mu_2(s)x_2 \\
\dot{x}_i &= [\mu_i(s) - D] x_i, \quad i = 1, 2
\end{align*}
\]

(4)
evolving on \(X := (0, \infty)^3\), where \(s_m\) and \(D\) are to be chosen, with the output \(y = Y/Y_1\) satisfying

\[
y = x_1 + ax_2, \quad \text{where } a = AY_2/Y_1.
\]

(5)

We always assume that \(a \neq 1\). Given arbitrary prescribed constraints \(x_{i<} > 0\) and \(x_{i>} > 0\), we set

\[
\begin{align*}
s_m &= s_0 + x_{i<} + x_{i>} \quad \text{and} \quad \hat{s} = s - s_0; \\
\bar{\Sigma} &= \ln\left(\hat{s} + s_0\right) - \ln\left(s_0\right); \quad \text{and} \\
\bar{\xi}_i &= \ln\left(\hat{x}_i + x_{i<}\right) - \ln\left(x_{i<}\right)
\end{align*}
\]

(6)
for \(i = 1, 2\). For each trajectory \((s, x_1, x_2)\) of (4), we call the vector \((\bar{\Sigma}, \bar{\xi}_1, \bar{\xi}_2)\) as defined by (7) the (corresponding transformed) error vector. This transformation is essential for what follows because it produces a system evolving on \(R^3\). Also, \(\bar{\xi}_i(t) \to +\infty \quad \text{when } x_i(t) \to 0^\pm\) for each \(i\), so the \(\bar{\xi}_i\)’s have the additional benefit of warning us when either species approaches extinction. We use the standard classes of comparison functions \(K_\infty\) and \(K_L\) and the constants

\[
\begin{align*}
D_1 &= \mu_1(s_0), \quad \omega_1 = \frac{D_*}{2|a - 1|} \\
\omega_2 &= \frac{D_*}{4s_0\left(\frac{1}{s_0} - \frac{1}{r_2}\right) + 1}
\end{align*}
\]

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\[
\omega_n = \frac{D_s s_*}{20(a-1)s_n + s_n^2 \left( \frac{n}{a-1} - \frac{1}{a-2} \right)} + 1
\]

and
\[
\omega_1 = \frac{0.72D_s s_n^2 \left( \frac{a}{a-1} - \frac{1}{a-2} \right)^2}{(a-1)s_n + s_n^2 \left( \frac{n}{a-1} - \frac{1}{a-2} \right)^2} + 1.
\] (8)

The following theorem implies that \((s(t), x_1(t), x_2(t)) \rightarrow (s_*, x_1, x_2)\) as \(t \to +\infty\) for all trajectories of (4), i.e., for all initial values in \(\mathcal{X}\), hence globally asymptotically stable coexistence. It uses the saturation \(\sigma(r) = r/\sqrt{1+r^2}\).

**Theorem 1:** Consider the system (4) with output (5) with \(a \neq 1\).

Let \(x_1, x_2 > 0\) be given, let \(s_0 > 0\) satisfy Assumption 1, let \(s_n\) satisfy (6), and let \(\varepsilon \in (0, \min \{\omega_1, \omega_2, \omega_3, \omega_4\}]\). Then \((s_0, x_1, x_2, s_n)\) is a GAS equilibrium of (4) when \(D(y) = D_s - \varepsilon(a - 1)\sigma(y - x_1, x_2)\). More precisely, there exists \(\beta \in \mathcal{K}_{\mathcal{L}}\) such that for all trajectories \((s, x_1, x_2)\) of (4) in closed loop with \(s_n\) and \(D_s\), the corresponding transformed error vector \((\tilde{\Sigma}, \tilde{\xi}_1, \tilde{\xi}_2)\) satisfies (7) satisfies
\[
\| \tilde{\Sigma}, \tilde{\xi}_1, \tilde{\xi}_2(t) \| \leq \beta \| \tilde{\Sigma}, \tilde{\xi}_1, \tilde{\xi}_2(0) \| \| t \|
\]

for all \(t \geq 0\). Notice that ISS guarantees persistence, since if, e.g., \(x_2(t) \to 0\) as \(t \to t^*\), then \(\tilde{\xi}_2(t) = \ln(x_2(t)) - \ln(x_2) \to -\infty\) as \(t \to t^*\), which would violate ISS. Similarly, iISS implies persistence when \(\mathcal{D}\) is integrable.

To specify the bounds on \(\mathcal{D}\), we introduce these constants
\[
C_1 = \frac{s_n}{10} \left( \frac{1}{L_1} - \frac{1}{L_2} \right),
\]

\[
C_2 = \frac{20}{19D_s} \left[ \frac{\varepsilon}{4C_1} \left( \frac{L_1 + s_n - L_2 + s_n}{L_1} \right)^2 + 1 \right],
\]

\[
\kappa_1 = \min \left\{ 1, \frac{D_s}{3} \right\} + x_1 + x_2 + D_s + \varepsilon \| a - 1 \|
\]

\[
\kappa_2 = \frac{\kappa_1}{2s_n} + 2C_2 + \frac{1}{s_n} \left( \kappa_1 C_2 + \max \left\{ \frac{\varepsilon}{105}, \frac{3}{2} \right\} \right)
\]

(11)

See Section VII for an example where we explicitly compute the bounds on \(\mathcal{D}\). Our iISS result is as follows:

**Theorem 2:** Under the preceding assumptions with
\[
\tilde{\Delta} = \min \left\{ 1, \frac{2D_s s_n}{3(3\kappa_1 + 2s_n)} \right\} \frac{1}{2\kappa_2}
\]

(12)

(9) in closed loop with the controls provided by Theorem 1 is iISS in \((\tilde{\Sigma}, \tilde{\xi}_1, \tilde{\xi}_2)\) for disturbances \(d \in \mathcal{D}_2(\tilde{\Delta})\).

See also Remarks 2–4 for iISS and related results with less stringent disturbance bounds.

**IV. COMPARISON WITH KNOWN RESULTS**

Lyapunov functions have not frequently been used to study stability in multi-species chemostats, and where they are used, they are often nonstrict Lyapunov functions which do not lend themselves to robustness analysis. An exception is the one from [8]; see also [12, Theorem 4.1] and [7]. Strict Lyapunov functions for two species chemostats were constructed in [11], but no robustness to disturbances was established there. However, [11] provides simple linear feedback stabilizers that yield local asymptotic stability of a prescribed periodic trajectory. See [9], [15] where weak Lyapunov functions are used with variants of the LaSalle Invariance Principle.

This raises the important question of whether strict Lyapunov functions can be explicitly constructed for multi-species chemostats that are globally stabilized through static output feedbacks, and whether Lyapunov functions can be used to quantify the effects of uncertainty. For chemostats with one species that are made oscillating through a suitable \(D(\cdot)\), this problem was solved in [10]. Our work owes a great deal to [4], [5], which stabilize chemostats in which only the sum of the species concentrations can be measured, using an appropriate \(D\). However, [4] and [5] do not include our work since [4] does not rely on a Lyapunov approach and the Lyapunov functions from [5] are nonstrict and so do not lend themselves to ISS. See [6] for results where only the substrate level (multiplied by an a priori bounded error) is available for measurement.

**V. PROOF OF THEOREM 1**

Since \(D_s = \mu_1(s_0) = \mu_2(s_0)\), the change of feedback \(D_s \rightarrow D_s + \nu\), (3) with the choice \(r = s_n\), and (7) transform (4) into
\[
\dot{s} = (D_s + \nu) s_n - s_0 - \mu_1(s)x_1 - \mu_2(s)x_2
\]

\[
\dot{\xi}_i = \frac{L_i}{L_i + s_n} \mu_i(s), \quad i = 1, 2.
\]

(13)
A. Step 1: Construction of a Weak Lyapunov Function

We first show that in terms of (10)
\[
V(\hat{s}, \hat{\xi}_1, \hat{\xi}_2) = \frac{1}{2} A_0(\hat{s}) + \frac{1}{x_{1*}} s^2 - \frac{1}{x_{1*}} s \cdot \nu - \left[ \frac{\xi_{\hat{s}}}{x_{1*}} + \frac{\xi_{\hat{\xi}_2}}{x_{2*}} \right] v
\]
(14)
is a weak Lyapunov function for (13) in the sense that
\[
\dot{V} = -H(\hat{s}) \dot{s}^2 + \frac{1}{x_{1*}} \hat{s} (s_m - s) \nu - \left[ \frac{\hat{x}_{\hat{s}}}{x_{1*}} + \frac{\hat{x}_{\hat{\xi}_2}}{x_{2*}} \right] v
\]
holds along all trajectories of (13), where \( H \) is defined by
\[
H(\hat{s}) = \frac{1}{2} \left[ D_1 \{ \mu_1(s) - \mu_1(s_s) \} \{ x_{1*}/s \} + \{ \mu_2(s) - \mu_2(s_s) \} \{ x_{2*}/s \} \right], \quad \hat{s} \neq 0
\]
(16)
is everywhere positive. To this end, first notice that
\[
\frac{d}{dt} A_0 = \langle \hat{s}, \hat{s} \rangle \frac{d}{dt} \hat{s}, (d/dt) \hat{x}_1 = \langle \hat{x}_1, \hat{s} \rangle d/dt \hat{s}, \quad (d/dt) \hat{x}_2 = \langle \hat{x}_2, \hat{s} \rangle d/dt \hat{s}, \quad \text{by (7)}.
\]
Hence, along the trajectories of (13), we easily get
\[
\dot{V} = \frac{1}{x_{1*}} D_1 \{ s_m - s \} - \frac{1}{x_{1*}} \mu_1(s) \frac{s}{\hat{s} v} - \frac{\hat{x}_{\hat{s}}}{x_{1*}} \frac{s}{\hat{s} v} + \frac{\hat{x}_{\hat{\xi}_2}}{x_{2*} s} \frac{s}{\hat{s} v} - \frac{1}{x_{1*} s} \mu_2(s) \frac{s}{\hat{s} v}.
\]
Since \(-x_{1*} + \hat{x}_{\hat{s}} = -x_{1*} \), for \( i = 1, 2 \), our choices of \( \hat{s} \) and \( \hat{\xi}_2 \) from (10) give
\[
\dot{V} = \frac{1}{x_{1*}} D_1 \{ s_m - s \} - \frac{1}{x_{1*}} \mu_1(s) \frac{s}{\hat{s} v} - \frac{1}{x_{1*} s} \mu_2(s) \frac{s}{\hat{s} v} + \frac{\hat{x}_{\hat{s}}}{x_{1*}} \frac{s}{\hat{s} v} + \frac{\hat{x}_{\hat{\xi}_2}}{x_{2*} s} \frac{s}{\hat{s} v}.
\]
(17)
The formulas \( s_m = s_s \), \( x_{1*} \), and \( D_1 = \mu_1(s_s) - \mu_2(s_s) \) now easily give (15).

B. Step 2: Construction of a Strict Lyapunov Function

Set \( \hat{y} = \hat{x}_1 + \hat{x}_2, \hat{Z} = \hat{s} + \hat{x}_1 + \hat{x}_2, \Psi(r) = (1/2)r^2, \), and
\[
V_2(\hat{s}, \hat{\xi}_1, \hat{\xi}_2) = x_{1*} V(\hat{s}, \hat{\xi}_1, \hat{\xi}_2) + \frac{1}{2} \frac{20}{19 D_1} \left[ 2 s_s + \frac{1}{x_{1*} s} \frac{s}{\hat{s} v} + \frac{\hat{x}_{\hat{s}}}{x_{1*}} \frac{s}{\hat{s} v} + \frac{\hat{x}_{\hat{\xi}_2}}{x_{2*} s} \frac{s}{\hat{s} v} \right] v \cdot \hat{Z} \]
(18)
where \( V \) is from (14). We show that \( V_2 \) is a strict Lyapunov function for (13) in the sense that its time derivative along the trajectories of (13) with \( \nu = -\varepsilon (a - 1) \sigma(\hat{y}) \) satisfies
\[
\dot{V}_2 \leq -W_2, \quad \text{where}
\]
\[
W_2(\hat{s}, \hat{\xi}_1, \hat{\xi}_2) = \frac{3}{4} D_1 s^2 + \frac{1}{x_{1*}} \frac{1}{L_1 - L_2} \varepsilon \sigma(\hat{y}) + \hat{Z}^2
\]
(19)
and \( \hat{s}, \hat{x}_1, \) and \( \hat{\xi}_2 \) are related by (7). The fact that \( W_2 \) is positive definite follows because \( a \neq 1, L_2 > L_1, \) and \( r \mapsto r \sigma(r) \) is positive definite.

Since \( \varepsilon \leq \omega_1 \), our choice of \( D \) gives \( D(y) = D_1 + v \geq D_1 - ||a - 1||^{1/2} 20 \) everywhere. Therefore, (4) gives \( \dot{Z} = D(y)[s_m - s - x_{1*} - x_{2*}] = -D(y) \hat{Z}, \) hence
\[
\frac{d}{dt} \Psi(\hat{Z}) \leq -\frac{10}{19} D_1 \hat{Z}^2.
\]
(20)
Moreover, our assumption that \( a \neq 1 \) gives \( \hat{x}_1 = (a[\hat{Z} - s] - \hat{y})/(a - 1) \) and \( \hat{x}_2 = (\hat{y} - \hat{Z} + s)/(a - 1). \) Substituting the preceding formulas for the \( \hat{x}_i \)'s and (10) into (15) gives
\[
\dot{V} = -H(\hat{s}) \dot{s}^2 + \frac{1}{x_{1*}} \frac{s}{\hat{s} v} - \frac{1}{x_{1*}} \mu_1(s) \frac{s}{\hat{s} v} - \frac{1}{x_{1*} s} \mu_2(s) \frac{s}{\hat{s} v} + \frac{\hat{x}_{\hat{s}}}{x_{1*}} \frac{s}{\hat{s} v} + \frac{\hat{x}_{\hat{\xi}_2}}{x_{2*} s} \frac{s}{\hat{s} v} \cdot \hat{Z} v.
\]
(16)
Taking \( v = -\varepsilon (a - 1) \sigma(\hat{y}) \) and recalling (16) gives
\[
\dot{V}_2 = -\frac{1}{x_{1*}} \left[ D_2 \{ s_m - s \} - x_{1*} \mu_1(s) - x_{2*} \mu_2(s) \right] \frac{s}{\hat{s} v} + \frac{\hat{x}_{\hat{s}}}{x_{1*}} \frac{s}{\hat{s} v} + \frac{\hat{x}_{\hat{\xi}_2}}{x_{2*} s} \frac{s}{\hat{s} v} \cdot \hat{Z} \sigma(\hat{y}).
\]
(21)
Since \( \varepsilon \leq \omega_2 \) and \( |\sigma(\cdot)| \) is bounded by \( 1 \), we get \(-\dot{V}_2 \) \((a s_s/L_1) - (s_s/L_2)\) \( \leq -D_1 \hat{Z}^2/(4 s_s) \). Moreover
\[
\left[ \frac{L_1 + s_s}{L_1} - \frac{L_2 + s_s}{L_2} \right] \hat{Z} \sigma(\hat{y}) \leq s_s \left[ \frac{1}{L_1} - \frac{1}{L_2} \right] \varepsilon \sigma(\hat{y})^2
\]
(18)
Since \( \sigma(\hat{y}) \) \( \leq \hat{y} \sigma(\hat{y}) \), we easily deduce from (21) that
\[
\dot{V}_2 = -\frac{3 D_1 s^2}{4 s_s} + \frac{9 s_s}{10} \left[ \frac{1}{L_1} - \frac{1}{L_2} \right] \varepsilon \sigma(\hat{y})^2
\]
(22)
Notice that
\[
\left\{ \frac{\sqrt{D_1}}{\sqrt{s_s}} \right\} \varepsilon \sigma(\hat{y})^2 \leq \frac{\sqrt{D_1}}{D_1 s_s} \sigma(\hat{y})^2
\]
(23)
for all \( s > 0 \). Using (23) to bound the term in braces in (22)

\[
V_2 \leq -\frac{D_s}{2s} s^2 - \frac{9s}{10} \left( \frac{1}{\mathcal{L}_1} - \frac{1}{\mathcal{L}_2} \right) \| \hat{y} \| \sigma(\hat{y}) - \hat{z}^2 + \left\{ \frac{a \left( 1 - \frac{1}{1 + \frac{\hat{z}}{\mathcal{L}_1}} \right)}{D_s s} \right\}^2 \sigma^2(\hat{y}) \right\}
\]  

(24)

We consider two cases. Case A: If \( s \geq (9/10)s_0 \), then (24) and the fact that \( \epsilon \leq \omega_1 \) give

\[
V_2 \leq -\frac{D_s}{4s} s^2 - \frac{\epsilon}{10} \left( \frac{1}{\mathcal{L}_1} - \frac{1}{\mathcal{L}_2} \right) \| \hat{y} \| \sigma(\hat{y}) - \hat{z}^2
\] 

(25)

because in this case, \( 1/s \leq 10/(9s_0) \), so the term in braces in (24) is at most \( 10/(9s_0) \left( (1/\mathcal{L}_1) - (1/\mathcal{L}_2) \right) \| y \| \sigma(y) \). Notice our use of 0.72 from (8) and \( \| y \| \leq \| y \| \). Case B: If \( s \leq (9/10)s_0 \), then \( \| y \| \geq (s_0/10) \). Hence, (24), the fact that \( \epsilon \leq \omega_3 \), and \( \| y \| < \| y \| \); \( \epsilon \) give

\[
V_2 \leq -\frac{D_s}{4s} s^2 - \left( \frac{a}{1 + \frac{\hat{z}}{\mathcal{L}_1}} \right) \| \hat{y} \| \sigma(\hat{y}) - \hat{z}^2 + \left\{ \frac{s}{D_s s} \right\}^2 \sigma^2(\hat{y})
\] 

(26)

because \( \epsilon \leq \omega_1 \) guarantees that the term in braces is at most \( (D_s/400s_0)^2 \). In either case, we get (19). However, \( V_2 \) becomes unbounded as \( \hat{s} \to \infty \), so extra care is needed to obtain \( \beta \in \mathcal{K}\mathcal{L}; \) see Appendix A. This proves Theorem 1.

**VI. PROOF OF THEOREM 2**

Fix \( d \in D_2(\hat{A}) \) and let \( \Delta \) denote its sup norm, so \( \Delta \leq \hat{\Delta} \). We use the notation from the proof of Theorem 1, (11) and

\[
C_3 = \frac{\sigma_{max}}{\sigma_{min}} \left( 2 \mathcal{C}_2 \kappa_1 + 2 \max \left( \mathcal{A}, \frac{\dot{x}_r}{x_r} \right) \right) \mathcal{S}_m + 2 \kappa_1 + 8 \mathcal{C}_2 \kappa_2^2, \quad W_i(\hat{s}, \hat{\xi}, \hat{\xi}_2) = \frac{D_s}{20s^2} + C_1 \| y \| \sigma(\hat{y}) + \frac{1}{2} \hat{z}^2
\]  

(26)

where we used the relations (7). Rearranging terms in (9) and again using (7) gives

\[
\hat{s} = D(y) - \mu_1(s)x_1 - \mu_2(s)x_2 + g_1(\hat{s}, \hat{y}, d), \quad \hat{\xi}_i = \mu_i(s) - D(y) - d_2, \quad i = 1, 2
\]  

(27)

with \( g_1(\hat{s}, \hat{y}, d) = d_2(\hat{s} + x_1 + x_2) + [D(y) + d_2]d_1 \). Along the trajectories of (27)

\[
V_2 \leq -W_2 + \frac{\partial V_2}{\partial s} g_1(\hat{s}, \hat{y}, d) - d_2 \left\{ \frac{\partial V_2}{\partial \hat{\xi}_1} + \frac{\partial V_2}{\partial \hat{\xi}_2} \right\}
\]  

(28)

by (19). In terms of (7) and the constants from (11), we have \( g_1(\hat{s}, \hat{y}, d) \leq \mathcal{A}(\kappa_1 + s) \)

\[
W_2(\hat{s}, \hat{\xi}_1, \hat{\xi}_2) = \frac{D_s}{4s^2} + C_1 \| y \| \sigma(\hat{y}) + \hat{z}^2
\]  

(29)

holds in both cases. Hence, if we define \( \mathcal{V} \) and the positive definite function \( W \) by \( \mathcal{V}(\hat{s}, \hat{\xi}_1, \hat{\xi}_2) = V_2(\hat{s}, \hat{\xi}_1, \hat{\xi}_2) \) and \( W(\hat{s}, \hat{\xi}_1, \hat{\xi}_2) = \mathcal{V}(\hat{s}, \hat{\xi}_1, \hat{\xi}_2) \), then we get \( \frac{d}{dt} \mathcal{V} \leq -W_2 + C_2 \mathcal{V} \). We can also find \( \alpha_1, \alpha_2 \in \mathcal{K}_{\infty} \) so that \( \alpha_1(\| \hat{s} \|, \| \hat{\xi}_1 \|, \| \hat{\xi}_2 \|) \leq \alpha_2(\| \hat{s} \|, \| \hat{\xi}_1 \|, \| \hat{\xi}_2 \|) \). Hence, a standard argument [1, p.1088] gives (IIS) with \( \alpha = 1/(2C_3) \), proving Theorem 2.

**Remark 2:** The system (9) is ISS in \( (\hat{s}, \hat{\xi}_1, \hat{\xi}_2) \) when \( d \in D_2(\hat{A}) \), where, in terms of (12)

\[
\hat{\Delta} = \min \left\{ \hat{\Delta}_1, \frac{1}{2C_3} \min \left\{ \alpha_1, \alpha_2 \right\} \right\}
\]  

(33)

\( \hat{\alpha}_1 \in \mathcal{K}_{\infty} \) is so that \( \alpha_1(r) \leq \hat{r} \) and \( W_2(\hat{s}, \hat{\xi}_1, \hat{\xi}_2) \geq \alpha_1(\| \hat{s} \|, \| \hat{x}_1 \|, \| \hat{x}_2 \|) \) hold everywhere; \( \hat{s}_i \) and \( \hat{\xi}_i \) are related by (7); and \( C_2 \) is from (26). For the proof, see Appendix E.

**Remark 3:** An important special case of Theorem 2 is where \( d_1 = 0 \) and \( d_2 \neq 0 \). This often occurs when the speed of the pump that supplies
Theorem 2 still holds if (12). For example, a variant of our proof of Theorem 2 shows that
where
we take

In that case, we can take

with

so that Theorem 2 remains true if

is replaced by

and

for large

and (18), one checks that

overshoot determined by the iISS estimate and the magnitude of

15%, and we simulate (35) with the sinusoidal input nu-

; in particular, if

that

is replaced by

get robustness to disturbances

and the convergence

and so validates our theorems.

Remark 4: By reducing the parameter

we can enlarge our bound (12). For example, a variant of our proof of Theorem 2 shows that
Theorem 2 still holds if

is replaced by

where

with

initial values, all trajectories of (35) with our controllers

are bounded by

and obtained Fig. 1 with

plotted against time in hours.

We next consider the important case where

We provide the variant of the standard argument needed to obtain

from Theorem 1. This will give the desired stability estimate

where

We simulated (35) with the sinusoidal input nutrient concentration disturbance

and the initial value

and obtained Fig. 1 with

plotted against time in hours. This illustrates the persistence of

This is obtained by using formulas (10), (14), and (18), one checks that

V2(s, \xi_1, \xi_2) \geq \left\{ \frac{s_x}{x_1}, (e^{r_1} - 1 - \xi_1) + \alpha_1 (e^{r_2} - 1 - \xi_2) \right\} x_1,

\geq x_1, \text{min} \left\{ \frac{s_x}{x_1}, \frac{\alpha_1}{\beta} \right\} \times \left[ E \left( \xi_1 \right) + E \left( \xi_1 \right) + E \left( \xi_2 \right) \right]

\geq \alpha_1 \left( \left( \xi_1, \xi_2 \right) \right)

where

\( E(p + q + r) \leq E(3p) + E(3q) + E(3r) \) for all

and the fact that

\( e^{r} - 1 - r \geq e^{-r} - 1 + r \) for all

Since

V1 and V2 are non-trivial and not bounded by

and

we have persistence of both species.

If instead the disturbance is only added to

then we can use (34) to get robustness to disturbances

that are bounded by

or about 15% of

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\( e^{r} - 1 - r \geq e^{-r} - 1 + r \) for all

Since

V1 and V2 are non-trivial and not bounded by

and

we have persistence of both species.
we get \(a_1 \in \mathcal{K}_\infty\). We can also find \(a_2 \in \mathcal{K}_\infty\) so that \(V_2(s, \hat{x}_1, \hat{x}_2) \leq a_2((\hat{x}_1, \hat{x}_2), \hat{x}_2) \leq a_2((\hat{x}_1, \hat{x}_2), \hat{x}_2)\), everywhere, e.g., \(a_2(r) = k_1 E_1(r) + k_2 e^{-1} r^2\) for large enough constants \(k_1, k_2 > 0\), since \(r \rightarrow E_1(r)\) is also of class \(\mathcal{K}_\infty\). Defining \(\mathcal{V}\) and \(\mathcal{W}\) as in the proof of Theorem 2 except with \(W_2\) instead of \(W_1\) gives \(a_0((\hat{x}_1, \hat{x}_2)) \leq a_0((\hat{x}_1, \hat{x}_2), \hat{x}_2) \leq a_0((\hat{x}_1, \hat{x}_2), \hat{x}_2)\), so \(a_0((\hat{x}_1, \hat{x}_2), \hat{x}_2) \leq a_0((\hat{x}_1, \hat{x}_2), \hat{x}_2)\). Since \(a_0 \neq 1\), \(Q(s, \hat{x}_1, \hat{x}_2) = s^2 \geq \hat{x}_1 + \hat{x}_2 + 2 \hat{x}_1 2 \hat{x}_2 \) is positive definite. Let \(\lambda_{\min} > 0\) be the smallest eigenvalue of the positive definite matrix \(P\) for which \(Q(s, \hat{x}_1, \hat{x}_2) = (\hat{x}_1, \hat{x}_2)^T P (\hat{x}_1, \hat{x}_2)\). Since \(Q(s, \hat{x}_1, \hat{x}_2) \geq N^2 \lambda_{\min}\) everywhere, we can take \(a_1(N) = \min\{1, \lambda_{\min}\} K(N) N^2\).

**APPENDIX E**

**DISTURBANCE BOUND FOR ISS**

We prove the ISS assertion from Remark 2, using the function \(\hat{a}_1 \in \mathcal{K}_\infty\) from Appendix D. We indicate the changes necessary in the proof of Theorem 2. Arguing as in the proof of Theorem 2 with \(\Delta := |d|_{\infty}\) along all trajectories of the error dynamics. Set

\[
\bar{\sigma} = \min\left\{\frac{s}{s}, \frac{D_{s} s_{t}^{2}}{80 [C_{\Delta} + 2 s_{t}^{2}]}, \frac{s_{t}}{s_{t}}\right\}
\]

(37)

We next transform (32) into an ISS decay estimate in the transformed error vector \((\hat{x}_1, \hat{x}_2)\) from (7). To do so, we view (32) simply as a relation in \(\hat{x}_1, \hat{x}_2,\) and \(\Delta \in [0, \Delta]\), rather than an estimate along trajectories. Here \(\Delta\) is the tighter ISS bound in (33). We consider two cases.

**Case I:** Assume that \(s_{t} \hat{x}_1 + \hat{x}_2 \geq s_{t} \hat{x}_1 + \hat{x}_2\) or \(s_{t} \hat{x}_1 \leq s_{t} \hat{x}_1 + \hat{x}_2\). If \(|\hat{x}| \geq s_{t}\), then \(W_1((\hat{x}_1, \hat{x}_2)) \geq 2 s_{t}^2\), so since \(s_{t} + \hat{x}_1 \leq s_{t} + \hat{x}_1 + \hat{x}_2\), we immediately obtain \(V_2 \leq - W_1 + \Delta M\Delta\) from (32). Hence, \(V_2 \leq - W_1 + \Delta M\Delta\) and \(\Delta M\Delta\), where \(M = (\Delta/2) C_{\hat{a}_1}(D_2 + |r| - 1)\) and the last inequality uses \(\Delta \leq \Delta\) and the relation \(\Delta \leq (1/2)^2 + (1/2) m^2\). Consider two cases. If \(|s| \geq 4 s_{t}/5\), then \(\bar{\sigma} s_{t}/4 s_{t}\) \(\geq 1\) gives

\[
V_2 \leq - W_1 + \Delta M\Delta
\]

where \(V_1, V_2, \) and \(V_3\) are from (26), since our bound (34) implies that the term in brackets is at most \((s/4)(0.16) D_2 s_{t}/s = D_{s} s_{t}/s^2\). On the other hand, if \(|s| \leq 4 s_{t}/5\), then \(s \leq s_{t}/5\). Hence, \(s \leq 4 s_{t}/5\) gives. This and (36) gives \(V_2 \leq - W_1 + \Delta M\Delta\) in either case, where \(M = M + 4(D_2 + |r| - 1)\). The argument concludes exactly as in the last part of the proof of Theorem 2, except with \(C_{\hat{a}_1}\) replaced by \(M_{1}\).

**APPENDIX D**

**LOWER BOUND FOR \(W_1\)**

We construct a function \(a_{\hat{a}_1} \in \mathcal{K}_\infty\) so that \(W_3(\hat{x}_1, \hat{x}_2) \leq a_{\hat{a}_1}((\hat{x}_1, \hat{x}_2), \hat{x}_2)\) and \(a_{\hat{a}_1}(r) \leq r^2\) everywhere, as required by Remark 2. Set \(N = (\hat{x}_1, \hat{x}_2)\). Then \(s = s_{t} + \hat{x}_1 \leq s_{t} + \hat{x}_2 + \hat{x}_2\) and \(\hat{x}_1 + \hat{x}_2 + \hat{x}_2\) \(\leq \hat{x}_1 + \hat{x}_2 + \hat{x}_2\) \(\leq s_{t} + \hat{x}_2 + \hat{x}_2\) \(\leq \hat{x}_1 + \hat{x}_2 + \hat{x}_2\). Hence, it follows from our choice of \(s_{t}\) that \(W_3(\hat{x}_1, \hat{x}_2) \leq K(N) s_{t}^2 + \hat{x}_1 + \hat{x}_2 + \hat{x}_2\) \(\leq \hat{x}_1 + \hat{x}_2 + \hat{x}_2\) \(\leq \hat{x}_1 + \hat{x}_2 + \hat{x}_2\). Hence, \(K(N) = \min\{\Delta(\hat{x}_1, \hat{x}_2), \hat{x}_1 + \hat{x}_2 + \hat{x}_2\}\).
CONSERVATIVENESS OF THE DISTURBANCE BOUND

We prove the assertion from Remark 4. We indicate the changes needed in the proof of Theorem 2. We argue as in the earlier proof up through (31), and we let $M \geq 2$ be any constant. Consider two cases.

Case 1M: If $\|s + \dot{x}_1 + \dot{x}_2\| \geq M \mathbf{m}_n$, then $\dot{V} = \|s + \dot{x}_1 + \dot{x}_2\| \leq 2\|s + \dot{x}_1 + \dot{x}_2\|$, by Lemma 2 since $M \geq 2$. We deduce that $\Delta \psi_1 \leq \Delta \psi_1 (s + \dot{x}_1 + \dot{x}_2)^2 / (M^2 \mathbf{m}_n^2)$, $\Delta \psi_2 (2 \mathbf{k}_1 \dot{x}_1 + \dot{x}_2^2) \leq \Delta \psi_2 (2 \mathbf{k}_1 (s + \dot{x}_1 + \dot{x}_2)^2 / (M \mathbf{m}_n)) + 2\|s + \dot{x}_1 + \dot{x}_2\|$, $\Delta \psi (\dot{x}_1^2) \leq 2 \Delta \psi (s + \dot{x}_1 + \dot{x}_2)^2 / (M \mathbf{m}_n)$, and $\Delta \psi (\dot{x}_2^2) \leq 2 \Delta \psi (s + \dot{x}_1 + \dot{x}_2)^2 / (M \mathbf{m}_n)$, by multiplying by $\|s + \dot{x}_1 + \dot{x}_2\| / (M \mathbf{m}_n) \geq 1$. Substituting into (31), noting that $\Delta \leq 1 / (2 \mathbf{k}_2)$, and grouping terms gives $V_2 \leq -W_4 (s, \dot{x}_1, \dot{x}_2) + \Delta \psi (s, \dot{x}_1, \dot{x}_2)$, where $\Delta$ is now redefined to be $\Delta = 2 \mathbf{k}_1 + 2 \mathbf{k}_2 (2 + M \mathbf{m}_n^2 + 2 \mathbf{m}_n \mathbf{m}_n^2 + \|s + \dot{x}_1, \dot{x}_2\|/2 + M \mathbf{m}_n)$. The rest of the proof is exactly as before, because we again have (32).

REFERENCES


APPENDIX F

Sufficient Conditions for Finite-Time Stability of Impulsive Dynamical Systems

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Abstract—The finite-time stability problem for state-dependent impulsive dynamical linear systems (SD-ILDS) is addressed in this note. SD-ILDS are a special class of hybrid systems which exhibit jumps when the state trajectory reaches a resetting set. A sufficient condition for finite-time stability of SD-ILDS is provided. $\mathcal{L}_\infty$-procedure arguments are exploited to obtain a formulation of this sufficient condition which is numerically tractable by means of Differential Linear Matrix Inequalities. Since such a formulation may be in general more conservative, a procedure which permits to automate its verification, without introduce conservatism, is given both for second order systems, and when the resetting set is ellipsoidal.

Index Terms—Finite-time stability (FTS), state-dependent impulsive dynamical linear systems (SD-ILDS).

I. INTRODUCTION

The concept of finite-time stability (FTS) dates back to the Sixties, when it was introduced in the control literature [1]. A system is said to be finite-time stable if, given a bound on the initial condition, its state does not exceed a certain threshold during a specified time interval. It is important to recall that FTS and Lyapunov Asymptotic Stability (LAS) are independent concepts. In particular, due to possible elongations of the system trajectories, LAS is not sufficient to guarantee FTS. Moreover, while LAS deals with the behavior of a system within an infinite time interval, FTS studies the behavior of the system within a finite (possibly short) interval. It follows that an unstable system can be FTS if the considered time interval is sufficiently small. It is worth noticing that Lyapunov stability becomes necessary for FTS of linear systems if the considered time interval becomes infinite.

In [2], [3] sufficient conditions for FTS and finite-time stabilization of continuous-time linear systems have been provided; such conditions are based on the solution of a feasibility problem involving either Linear Matrix Inequalities (LMIs [4]) or Differential Linear Matrix Inequalities (DLMIs [5]). The former approach is less demanding from the computational point of view, while the latter is less conservative.

The increasing interest that the researchers have devoted in the last decade to the theory and application of hybrid systems represents a natural stimulus to the extension of the FTS concept to such context, which is the objective of the present work. Indeed, in this note, we will focus on a class of hybrid systems, namely state-dependent impulsive dynamical linear systems (SD-ILDS) [6], where the state jumps occur when the trajectory reaches an assigned subset of the state space, the so-called resetting set. Many results concerning the classical Lyapunov

Manuscript received August 04, 2008; revised October 30, 2008 and October 30, 2008. Current version published April 08, 2009. Recommended by Asso-