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BACKSTEPPING WITH BOUNDED FEEDBACKS FOR TIME-VARYING SYSTEMS*

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Abstract. A family of time-varying nonlinear systems is globally uniformly asymptotically stabilized by bounded feedbacks constructed through a new extension of the backstepping approach. Explicit expressions of control laws and Lyapunov functions are given.

Key words. backstepping, bounded feedback, time-varying system

AMS subject classifications. 93D05, 93D15, 93D20

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1. Introduction. One of the most popular nonlinear techniques of design of control laws is the backstepping approach. The multiple advantages offered by it are well known. Observe in particular that this technique yields a wide family of globally asymptotically stabilizing control laws, and it allows one to address robustness issues and to solve adaptive problems. However, for a long time, it was a widely held belief that this technique could not be used to solve the problem of designing feedbacks bounded in norm, which in many practical situations should be addressed: For instance, the possibility of actuator saturation or constraints on actuators imposes bounded input. But it turns out that, as a matter of fact, the backstepping approach can be adapted to the problem of designing bounded feedbacks. In three recent works [22, 2, 10], it is shown that for some time-invariant systems (an n-dimensional chain of integrators, for instance), bounded stabilizing feedbacks can be constructed by applying new versions of this technique: The approach of [22, 2] mainly relies on the nested saturation control laws proposed in [18, 20], and the approach of [10] mainly relies on the determination of a particular family of control Lyapunov functions. However, for families of time-varying systems, no bounded backstepping method has ever been developed, and the main results of [22, 2] and [10] cannot be straightforwardly extended.

In the present work, we address the problem of constructing globally uniformly asymptotically stabilizing differentiable bounded feedbacks and accompanying strict Lyapunov functions, using the backstepping approach for time-varying systems of the following form:

$$\begin{cases} \dot{x} = f(t,x) + g(t,x)z, \\ \dot{z} = p(t)(u + b(t,x,z)) \end{cases} \tag{1}$$

with $x \in R^{n_x}$, $z \in R$, where $u \in R$ is the input, $p(t)$ is a bounded function of t, and $f(t,x)$ and $b(t,x,z)$ satisfy $f(t,0) = 0$, $b(t,0,0) = 0$ for all t.

In the particular case where $p(t)$ is a continuous function larger (resp., smaller) than a strictly positive real number (resp., a strictly negative real number), then a

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Lyapunov design of bounded feedbacks can be carried out, for instance, by combining the results of [23, 24] and [22]. But when $p(t)$ is a time-varying function which is neither strictly positive nor strictly negative, then the construction of globally uniformly stabilizing feedbacks and accompanying strict Lyapunov functions for systems (1) is a challenging open problem: To the best of our knowledge, no technique of construction of this type of Lyapunov functions is available in the literature, even in the case where the systems (1) are stabilized by unbounded control laws. We want to emphasize that in the present paper, we *will not impose* on $p(t)$ to be a function which is never equal to zero: We will only assume that $p(t)$ satisfies a persistency of excitation property and is of class C^1. Observe that the study of nonlinear time-varying systems is motivated in particular by the fact that a tracking problem for a nonlinear system can be reformulated as a stabilization problem for the time-varying error system. Through the family of chained form nonholonomic systems, we will show in section 4 how tracking problems for nonlinear systems may lead to the study of systems of the form (1), where $p(t)$ is a function which takes positive and negative values, and how, by applying the main result of the present work repeatedly, one can solve the open problem of determining explicit expressions of globally uniformly asymptotically and locally exponentially stabilizing bounded feedbacks and of accompanying strict Lyapunov functions for time-varying chains of integrators, which in turn implies that one can solve the problem of constructing globally uniformly asymptotically and locally exponentially stabilizing bounded feedbacks and accompanying strict Lyapunov functions for error equations of systems in chained form.

The approach we propose relies extensively on two results. On the one hand, we exploit the family of changes of coordinates used in [10] to obtain explicit expressions of globally uniformly asymptotically stabilizing bounded feedbacks. On the other hand, we construct explicitly strict Lyapunov functions using the main result of [9].

Observe that the strict Lyapunov functions (which at the same time are control Lyapunov functions) we will construct are not just tools enabling us to establish the asymptotic stability of the closed-loop system: The knowledge of continuously differentiable strict Lyapunov functions can be of great help. The potential benefits they offer are so numerous that they cannot be exhaustively enumerated. However, observe in particular the following:

- Recent advances in stabilization of nonlinear delay systems are based on the knowledge of continuously differentiable Lyapunov functions (see in particular [21, 3, 11]).
- Lyapunov functions are known to be very efficient tools for robustness analysis: For example, many proofs of nonlinear disturbance-to-state L^p stability properties rely on Lyapunov functions (see [19, 8]). Moreover, the control Lyapunov function–based theory has provided control designs with guaranteed robustness to different types of disturbances, including deterministic [1] and stochastic [6], as well as the robustness to unmodeled dynamics [13, 15].
- When a control Lyapunov function satisfying the small control property is available, one can apply universal formulas, in particular the one proposed in [16], and obtain that way the expression of an asymptotically stabilizing feedback which is optimal with respect to the control Lyapunov function as optimal value function.

The expressions of the bounded control law and of the Lyapunov function we propose are far from being the only possible expressions that can be obtained; many different formulas can be determined. Moreover, many extensions of the result can be proved; we have briefly mentioned some of them in the discussion of the main result given in section 3 and in the concluding remarks of section 5. For the sake of clarity, we have chosen to restrict ourselves to the systems (1); the control design can be easily carried out for them. However, it is worth noting that the key ideas of our approach can be used in several contexts beyond the scope of the present work. In particular, they can be utilized to solve the problem of constructing bounded feedbacks for systems of the form

$$\left\{\begin{array}{l} \dot{x} = f(t,x,z) + h(t,x,z,u)u, \\ \dot{z} = p(t)u + b(t,x,z), \end{array}\right. \tag{2}$$

which, due to the term $h(t,x,z,u)u$, are not in feedback form.

The paper is organized as follows. In section 2, a technical lemma is given. In section 3 the main result is stated and proved. The technique is applied to an illustrative example in section 4. Concluding remarks in section 5 end the work.

Preliminaries.

1. The argument of the functions will be omitted whenever no confusion can arise from the context.

2. We assume throughout the paper that the functions encountered are sufficiently smooth.

3. For a real-valued C^1 function $k(\cdot)$, we denote by $k'(\cdot)$ its first derivative.

4. $|x| = \sqrt{x^\top x}$ stands for the Euclidean norm of vector $x \in R^{n_x}$.

5. A function $k(\cdot) : R_{\geq 0} \to R_{\geq 0}$ is of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing, and unbounded.

6. By S, we denote the set of the functions $\sigma : R \to R$ such that

(a) $\sigma(s)$ is a bounded function,

(b) $s\sigma(s)$ is positive definite,

(c) $s\sigma(s) \leq s^2$,

(d) $\sigma'(s)$ is nonnegative and bounded and $\sigma'(0) = 1$.

7. A function $V(\cdot)$ is a strict Lyapunov function for the time-varying system

$$\dot{\chi} = \varphi(t,\chi)$$

if there exists a positive definite function $W(\chi)$ such that, for all t and χ,

$$\frac{\partial V}{\partial \chi}(t,\chi)\varphi(t,\chi) + \frac{\partial V}{\partial t}(t,\chi) \leq -W(\chi)$$

and there exist two functions $\Gamma_1(\cdot), \Gamma_2(\cdot)$ of class $\mathcal{K}_\infty$ such that, for all t and χ,

$$\Gamma_1(|\chi|) \leq V(t,\chi) \leq \Gamma_2(|\chi|).$$

2. Technical result. In this section, we give a technical result which will be used in the next section to prove the main result of the work. We construct a strict Lyapunov function for the one-dimensional time-varying system

$$\dot{\xi} = -q(t)\sigma(\xi), \tag{3}$$

where $q(t)$ is a nonnegative function of class C^1 such that

$$0 \leq q(t) \leq \delta_1 \qquad \forall t, \tag{4}$$

$$\int_t^{t+T} q(s)ds \geq \delta_2 > 0 \qquad \forall t, \tag{5}$$

where δ_1, δ_2, and T are positive real numbers and where $\sigma(\cdot)$ belongs to the set S defined in the preliminaries. We carry out the construction by adapting the approach of [9] to the case where $q(t)$ is not necessarily a periodic function of t but satisfies the persistency of excitation condition (5). First, observe that the property $0 \leq s\sigma(s) \leq s^2$ implies that the function $\frac{|\sigma(s)|}{|s|}$ is bounded. Moreover, $\sigma'(\cdot)$ is bounded and (4) is satisfied. It follows that

$$M := \sup_{t\in R}\left[T + \left|\int_t^{t+T}(s-t-T)q(s)ds\right| \sup_{s\in R, s\neq 0}\left(\frac{|\sigma(s)| + |s\sigma'(s)|}{|s|}\right)\right] \tag{6}$$

is finite and positive. We are now in position to give a technical lemma.

LEMMA 2.1. *The function*

$$\nu(t,\xi) := (M+1)\xi^2 + \left(\int_t^{t+T}(s-t-T)q(s)ds\right)\xi\sigma(\xi) \tag{7}$$

is a strict Lyapunov function for the system (3).

REMARK 1. *When $q(t)$ is a periodic function, then $\nu(t,\xi)$ is a periodic function of t as well, and these functions have the same period.*

Proof. The derivatives of the functions

$$R_1(t,\xi) := \left(\int_t^{t+T}(s-t-T)q(s)ds\right)\xi\sigma(\xi), \qquad R_2(\xi) := \frac{1}{2}\xi^2 \tag{8}$$

along the trajectories of (3) satisfy

$$\begin{aligned} \dot{R}_1 &= \left[-\int_t^{t+T} q(s)ds + Tq(t)\right]\xi\sigma(\xi) \\ &\quad - \left[\int_t^{t+T}(s-t-T)q(s)ds\right][\sigma(\xi) + \xi\sigma'(\xi)]\, q(t)\sigma(\xi), \\ \dot{R}_2 &= -q(t)\xi\sigma(\xi). \end{aligned} \tag{9}$$

Further, since

$$\nu(t,\xi) = 2(M+1)R_2(\xi) + R_1(t,\xi), \tag{10}$$

it follows from (9), (6), and (5) that

$$\begin{aligned} \dot{\nu}(t,\xi) &\leq -\left(\int_t^{t+T} q(s)ds\right)\xi\sigma(\xi) + Mq(t)\xi\sigma(\xi) - 2(M+1)q(t)\xi\sigma(\xi) \\ &\leq -\delta_2\xi\sigma(\xi) < 0 \qquad \forall \xi \neq 0. \end{aligned} \tag{11}$$

Moreover, (7) and (6) imply that

$$(M+1)\xi^2 - M\xi^2 \leq \nu(t,\xi) \leq (M+1)\xi^2 + M\xi^2, \tag{12}$$

which results in

$$\xi^2 \leq \nu(t,\xi) \leq (2M+1)\xi^2. \tag{13}$$

According to (11) and (13), the function $\nu(t,\xi)$ is a strict Lyapunov function for the system (3). This concludes the proof. $\square$

3. Main result. In this section, we state and prove the main result of the paper. Consider the nonlinear time-varying system (1). We introduce a set of assumptions.

ASSUMPTION A1. *The functions $p(t)$ and $\dot{p}(t)$ are bounded in norm by a positive real number P and two positive numbers T and γ such that, for all t,*

$$\int_t^{t+T} p(s)^2 ds \geq \gamma > 0 \tag{14}$$

are known.

ASSUMPTION A2. *Let ε be a positive real number and n a nonnegative integer. A Lyapunov function $V(t,x)$ such that*

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|), \qquad \left|\frac{\partial V}{\partial x}(t,x)\right| \leq \alpha_3(|x|), \tag{15}$$

where the $\alpha_i(\cdot)$'s are functions of class $\mathcal{K}_\infty$, a positive definite function $W(x)$, and a feedback $z_s(t,x) := p(t)^{n+2}\mu_s(t,x)$, bounded in norm by ε such that $\mu_s(t,0) = 0$ and

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)[f(t,x) + g(t,x)z_s(t,x)] \leq -W(x), \tag{16}$$

are known. Moreover, the functions

$$|\mu_s(t,x)|, \quad \left|\frac{\partial \mu_s}{\partial t}(t,x)\right|, \quad \left|\frac{\partial \mu_s}{\partial x}(t,x)f(t,x)\right|, \quad \left|\frac{\partial \mu_s}{\partial x}(t,x)g(t,x)\right|, \quad |b(t,x,z)| \tag{17}$$

are bounded.

ASSUMPTION A3. *A real-valued function $\zeta(\cdot)$ such that $\zeta(s) > 0$ for all $s \neq 0$ and $\int_0^r \zeta(s)ds$ is of class $\mathcal{K}_\infty$, a function $\alpha_4(\cdot)$ of class $\mathcal{K}_\infty$, and a nonnegative function $\beta(\cdot)$ such that the inequalities*

$$\zeta(V(t,x))\left|\frac{\partial V}{\partial x}(t,x)g(t,x)\right|^2 \leq \frac{1}{2}W(x), \tag{18}$$

$$|f(t,x)| \leq \alpha_4(|x|)\ ,\ |g(t,x)| \leq \beta(|x|) \tag{19}$$

are satisfied for all t, x are known.

ASSUMPTION A3$'$. *The function $W(x)$ is such that, for a real number $c_1 > 0$,*

$$W(x) \geq c_1|x|^2 \qquad \forall x : |x| \leq 1. \tag{20}$$

THEOREM 3.1. *Assume that the system* (1) *satisfies Assumptions* A1, A2, *and* A3. *Then the system* (1) *is globally uniformly asymptotically stabilizable by a bounded feedback $u_s(t,x,z)$ such that, for all t, $u_s(t,0,0) = 0$. For the corresponding closed-loop system, a strict Lyapunov function can be constructed. This Lyapunov function belongs to the family of functions of the form*

$$U(t,x,z) = l(V(t,x)) + k(\nu(t,\Omega(z) - z_s(t,x))) \tag{21}$$

with

$$\nu(t,Z) = (M+1)Z^2 + \left(\int_t^{t+T}(s-t-T)p(s)^{2m}ds\right)Z\sigma(Z), \tag{22}$$

where m is a positive integer, $l(\cdot), k(\cdot)$ are functions of class $\mathcal{K}_\infty$, and $\Omega(\cdot)$ is a real-valued function zero at zero such that $\Omega'(z) \geq 1$ for all z. If in addition Assumption A3$'$ *is satisfied, the system* (1) *is globally uniformly asymptotically and locally exponentially stabilizable by a bounded feedback $u_s(t, x, z)$ such that, for all t, $u_s(t, 0, 0) = 0$, and a strict Lyapunov function for the corresponding closed-loop system with a derivative along the trajectories upper bounded on a neighborhood of the origin by a negative definite quadratic function of (x, z) can be constructed. This Lyapunov function belongs to the family of functions* (21).

Discussion of Theorem 3.1.

- All the real-valued periodic functions of class C^1 which are not identically equal to zero satisfy Assumption A1. In the particular case where, for all t, $p(t) > 0$ or $p(t) < 0$, a simpler proof than the one we shall give can be carried out by taking advantage of the change of feedback $v = p(t)(u + b(t, x, z))$. But assuming that, for all t, $p(t) > 0$ or $p(t) < 0$ is very restrictive.
- The boundedness property of the functions in (17) in Assumption A2 and the growth property in Assumption A3 are not surprising assumptions: In the time-invariant case, similar assumptions have been imposed (see [2, 10]). Due to the finite escape time phenomenon, they cannot be removed without being replaced by other assumptions.
- Assumption A3$'$ ensures that the feedback $z_s(t, x)$ not only globally uniformly asymptotically stabilizes the origin of x-subsystem of (1) but also locally exponentially stabilizes it.
- In the formula of the stabilizing feedback we shall construct (see (27)), the function $V(t, x)$ is not involved: So it turns out that the control design strategy we propose can be applied even when the function $V(t, x)$ is not accurately known.
- An important issue is whether or not Theorem 3.1 can be applied recursively. In general, it appears that the assumptions will not be satisfied repeatedly because the presence of $b(t, x, z)$ in the expression of the control law we will construct (see (27)) typically prevents $u_s(t, x, z)$ and its derivatives along the trajectories from vanishing with $p(t)$. However, in particular cases, Theorem 3.1 can be applied recursively. Basically, this can be done for systems of the form

$$\begin{cases} \dot{x} = f(t, x, z_1), \\ \dot{z}_1 = p_1(t) z_2 + b_1(t, x, z), \\ \quad \vdots \\ \dot{z}_n = p_n(t) u + b_n(t, x, z), \end{cases} \tag{23}$$

when the $b_i(t, x, z)$'s are identically equal to zero or when, roughly speaking, they are sufficiently "small": Indeed, since one can construct explicitly a globally uniformly asymptotically stabilizing feedback with an accompanying strict Lyapunov function for a system (23) in absence of the $b_i(t, x, z)$'s, one can take advantage of these tools to determine in a second step how "small" the terms $b_i(t, x, z)$, regarded as disturbances, should be for not destroying the stability properties of the system stabilized by the control law constructed in their absence. It is quite clear that the choice of the feedback at each step plays an important role in this approach. In particular at each step the control law must anticipate the $p_i(t)$'s that follow: One understands from the mechanism of the control design used to prove Theorem 3.1 that a possible strategy of design for the system (23) consists in repeatedly constructing feedbacks, which for convenience we denote $z_{i,f}(t, x, z_1, \dots, z_{i-1})$ for $i = 2$ to $n+1$, such that $z_{i,f}(t, x, z_1, \dots, z_{i-1}) = p_i(t)^2 \dots p_n(t)^{n-i+2}\psi_i(t, x, z_1, \dots, z_{i-1})$, where $\psi_i(t, x, z_1, \dots, z_{i-1})$'s are sufficiently smooth functions. We will not present a rigorous and complete study of this problem; it would require pages of simple but lengthy calculations which can be inferred from the ideas of the proof of Theorem 3.1. For the sake of simplicity, we restrict ourselves to illustrating the possibility of applying the approach repeatedly by solving in section 4 the problem of stabilizing a three-dimensional chain of integrators with time-varying coefficients.

Proof of Theorem 3.1.

Step 1: *New variable.* We introduce the variable

$$Z := \Omega(z) - z_s(t, x), \tag{24}$$

where $\Omega(\cdot)$ is the function present in (21). In addition to the properties $\Omega(0) = 0$ and $\Omega'(z) \geq 1$ for all z, we require that this function be such that

(a) $\Omega'(z) = 1$ when $|z| \leq 2\varepsilon$,

(b) $\Omega'(z) \geq |z|$ when $|z| \geq 2\varepsilon + 1$.

Its time derivative satisfies

$$\begin{aligned} \dot{Z} &= \Omega'(z)p(t)(u + b(t, x, z)) - \frac{\partial z_s}{\partial t}(t, x) - \frac{\partial z_s}{\partial x}(t, x)[f(t, x) + g(t, x)z] \\ &= \Omega'(z)p(t)(u + b(t, x, z)) + p(t)\lambda(t, x, z) \end{aligned} \tag{25}$$

with

$$\begin{aligned} \lambda(t, x, z) &= -(n+2)\dot{p}(t)p(t)^n\mu_s(t, x) - p(t)^{n+1}\frac{\partial \mu_s}{\partial t}(t, x) \\ &\quad - p(t)^{n+1}\frac{\partial \mu_s}{\partial x}(t, x)[f(t, x) + g(t, x)z]. \end{aligned} \tag{26}$$

We choose for u

$$u = u_s(t, x, z) := -b(t, x, z) - \frac{p(t)^{2m-1}\sigma(Z) + \lambda(t, x, z)}{\Omega'(z)}, \tag{27}$$

where m is a positive integer and where $\sigma(\cdot)$ is a function belonging to the set S defined in the preliminaries. Such a choice of feedback yields

$$\dot{Z} = -p(t)^{2m}\sigma(Z). \tag{28}$$

One can check readily that Assumption A1 and the properties of $\sigma(\cdot)$ imply that this system is globally uniformly asymptotically and locally exponentially stable. Our objective is now to construct a strict Lyapunov function for the system (1) in closed-loop with (27) by exploiting the stability properties of (28).

Step 2: *Strict Lyapunov function for the system* (28). Using Young's inequality, one can check readily that Assumption A1 implies that for all positive integer m one can find a positive real number γ_m such that, for all t,

$$\int_t^{t+T} p(s)^{2m} ds \geq \gamma_m > 0. \tag{29}$$

Moreover, $p(t)$ is bounded in norm. It follows that Lemma 2.1 applies to the system (28): The function defined in (22), where

$$M = \sup_{t \in R} \left[T + \left| \int_t^{t+T} (s-t-T) p(s)^{2m} ds \right| \sup_{s \in R, s \neq 0} \left(\frac{|\sigma(s)| + |s\sigma'(s)|}{|s|} \right) \right], \tag{30}$$

is a strict Lyapunov function for the system (28). Its time derivative along the trajectories of (28) satisfies

$$\dot{\nu}(t,Z) \leq -\left(\int_t^{t+T} p(s)^{2m} ds \right) Z\sigma(Z) \leq -\gamma_m Z\sigma(Z) < 0 \qquad \forall Z \neq 0. \tag{31}$$

Step 3: *Strict Lyapunov function for the system* (1). We construct a strict Lyapunov function for the system (1) in closed-loop with the feedback (27) by using a combination of the Lyapunov functions $V(t,x)$ and $\nu(t,Z)$. This construction is reminiscent of the constructions of Lyapunov functions presented in [17, 12]. Consider a function $U(t,x,z)$, belonging to the family of functions (21), and require that the function $k(\cdot)$ be such that $k'(s) \geq 1$ for all $s \geq 0$. Thanks to Assumption A2, the properties satisfied by the functions $\Omega(\cdot), k(\cdot), l(\cdot)$, and Lemma 2.1, one can prove that there exist two functions $\gamma_1(\cdot)$, $\gamma_2(\cdot)$ of class $\mathcal{K}_\infty$ such that

$$\gamma_1(|(x,z)|) \leq U(t,x,z) \leq \gamma_2(|(x,z)|). \tag{32}$$

The derivative of $U(t,x,z)$ along the trajectories of the closed-loop system satisfies

$$\begin{aligned} \dot{U} &= l'(V(t,x))\dot{V} + k'(\nu(t,Z))\dot{\nu} \\ &\leq -l'(V(t,x))W(x) + l'(V(t,x))\frac{\partial V}{\partial x}(t,x)g(t,x)(z - z_s(t,x)) \\ &\quad - k'(\nu(t,Z))\gamma_m Z\sigma(Z). \end{aligned} \tag{33}$$

From the triangular inequality, the inequality

$$\begin{aligned} \dot{U} &\leq -l'(V(t,x))W(x) + l'(V(t,x))^2 \left| \frac{\partial V}{\partial x}(t,x)g(t,x) \right|^2 \\ &\quad + \frac{1}{4}(z - z_s(t,x))^2 - k'(\nu(t,Z))\gamma_m Z\sigma(Z) \end{aligned} \tag{34}$$

can be deduced. According to Assumption A3, a possible choice for $l(\cdot)$ is $l(r) := \int_0^r \zeta(s)ds$, since this function is of class $\mathcal{K}_\infty$. Moreover, for such a choice, the inequality

$$\dot{U} \leq -\frac{1}{2}\zeta(V(t,x))W(x) + \frac{1}{4}(z - z_s(t,x))^2 - \gamma_m k'(\nu(t,Z))Z\sigma(Z) \tag{35}$$

is satisfied. Next, observe that

$$Z^2 = (\Omega(z) - z_s(t,x))^2 = (\Omega(z) - \Omega(z_s(t,x)))^2 \tag{36}$$

because $|z_s(t,x)| \leq \varepsilon$ and $\Omega(s) = s$ when $|s| \leq 2\varepsilon$. It follows that

$$Z^2 = \left(\int_{z_s(t,x)}^{z} \Omega'(s)ds \right)^2. \tag{37}$$

Since $\Omega'(s) \geq 1$ for all s, the inequality

$$Z^2 \geq (z - z_s(t,x))^2 \tag{38}$$

holds. Combining (38) and (35) yields

$$\dot{U} \leq -\frac{1}{2}\zeta(V(t,x))W(x) + \frac{1}{4}Z^2 - \gamma_m k'(\nu(t,Z))Z\sigma(Z). \tag{39}$$

Thanks to the inequalities (13) and the properties of $\sigma(\cdot)$, one can easily determine a function $k(\cdot)$ such that

$$\gamma_m k'(\nu(t,Z))Z\sigma(Z) \geq \frac{1}{2}Z^2. \tag{40}$$

This inequality leads to

$$\dot{U} \ \leq \ -\frac{1}{2}\zeta(V(t,x))W(x) - \frac{\gamma_m}{2}k'(\nu(t,Z))Z\sigma(Z) \ \leq \ -N(x,z), \tag{41}$$

where $N(x,z)$ is the positive definite function

$$N(x,z) := \inf_{t\in R}\left(\frac{1}{2}\zeta(V(t,x))W(x) + \frac{\gamma_m}{2}k'(\nu(t,Z))Z\sigma(Z)\right). \tag{42}$$

It follows that $U(t,x,z)$ is a strict Lyapunov function for the system (1) in closed-loop with the feedback (27). This implies that the system (1) in closed-loop with the feedback (27) is globally uniformly asymptotically stable.

Step 4: *Boundedness of the feedback* (27). Since $\Omega'(z) \geq 1$, $|p(t)| \leq P$, and $|\dot{p}(t)| \leq P$, the inequality

$$\begin{aligned} |u| \leq\ & |b(t,x,z)| + P^{2m-1}|\sigma(Z)| + (n+2)P^{n+1}|\mu_s(t,x)| + P^{n+1}\left|\frac{\partial\mu_s}{\partial t}(t,x)\right| \\ & + P^{n+1}\left|\frac{\partial\mu_s}{\partial x}(t,x)f(t,x)\right| + P^{n+1}\left|\frac{\partial\mu_s}{\partial x}(t,x)g(t,x)\right|\left|\frac{z}{\Omega'(z)}\right| \end{aligned} \tag{43}$$

is satisfied. On the one hand, Assumption A2 ensures that the functions $|b(t,x,z)|$, $|\mu_s(t,x)|$, $|\frac{\partial\mu_s}{\partial t}(t,x)|$, $|\frac{\partial\mu_s}{\partial x}(t,x)f(t,x)|$, and $|\frac{\partial\mu_s}{\partial x}(t,x)g(t,x)|$ are bounded. On the other hand, the functions $\sigma(\cdot)$ and $\Omega(\cdot)$ have been chosen such that $|\sigma(Z)|$ and $|\frac{z}{\Omega'(z)}|$ are bounded. It follows that the feedback (27) is bounded in norm.

Step 5: *The particular case where Assumption* A3$'$ *is satisfied.* When (20) holds, then, according to (40), the function $N(x,z)$ defined in (42) satisfies

$$N(x,z) \geq \inf_{t\in R}\left(\frac{c_1}{2}\zeta(V(t,x))|x|^2 + \frac{1}{4}Z^2\right) \qquad \forall(x,z) : |x| \leq 1. \tag{44}$$

Assumptions A2, A3, and A3$'$ ensure that there exists a positive real number c_2 such that

$$c_2\left|\frac{\partial V}{\partial x}(t,x)g(t,x)\right|^2 \leq \frac{c_1}{2}|x|^2 \leq \frac{1}{2}W(x) \qquad \forall(x,z) : |x| \leq 1. \tag{45}$$

It follows that, if necessary, $\zeta(\cdot)$ can be modified in such a way that $\zeta(0) > 0$ and (18) is satisfied. In that case, $c_3 = \inf_{t\in R, |x|\leq 1}(\frac{1}{2}\zeta(V(t,x)))$ is a positive real number and the property

$$N(x,z) \geq c_1c_3|x|^2 + \frac{1}{4}\inf_{t\in R}\left(Z^2\right) \qquad \forall(x,z) : |x| \leq 1 \tag{46}$$

is satisfied. Through lengthy but simple calculations, one can deduce from (46) that there exists a positive real number c_4 such that

$$N(x,z) \geq c_4(|x|^2 + |z|^2) \qquad \forall (x,z) : |(x,z)| \leq 1. \tag{47}$$

This implies that the system (1) in closed-loop with the feedback (27) is globally uniformly asymptotically stable and locally exponentially stable. This concludes the proof. ∎

4. Illustration: Time-varying chain of integrators. We will illustrate Theorem 3.1 by using it to construct for the particular three-dimensional chain of integrators with time-varying coefficients

$$\begin{cases} \dot{x}_1 = \cos(t)x_2, \\ \dot{x}_2 = \cos(t)x_3, \\ \dot{x}_3 = \cos(t)u \end{cases} \tag{48}$$

a globally uniformly asymptotically and locally exponentially stabilizing bounded state feedback and a strict Lyapunov function for the corresponding closed-loop system. Before doing that, we give a motivation for it.

4.1. Motivation: Systems in chained form. For the sake of simplicity, we will restrict our attention to the system (48). But it is worth noting that Theorem 3.1 can be successfully applied repeatedly to any time-varying chain of integrators

$$\begin{cases} \dot{x}_n = p_n(t)x_{n-1}, \\ \dot{x}_{n-1} = p_{n-1}(t)x_{n-2}, \\ \quad\vdots \\ \dot{x}_1 = p_1(t)u, \end{cases} \tag{49}$$

where the $p_i(t)$'s are bounded functions with bounded first derivatives such that the product $p_1(t)\cdots p_n(t)$ satisfies Assumption A1.

One of the motivations for solving the problem of globally uniformly asymptotically stabilizing time-varying chains of integrators by bounded feedback arises from the tracking problem for systems in chained form under input saturation. The importance of this family of systems is well known: The kinematic model of several nonholonomic systems can be transformed into a system in chained form, and a lot of interest has been devoted to the stabilization and the tracking of these systems. In particular, in [4, 5, 7] the backstepping approach has been used to achieve for these systems the global tracking of trajectories. Let us briefly recall how. Systems in chained form of order n with two inputs (see, for instance, [14]) are described by the equations

$$\begin{cases} \dot{z}_n = z_{n-1}v_1, \\ \quad\vdots \\ \dot{z}_3 = z_2 v_1, \\ \dot{z}_2 = v_2, \\ \dot{z}_1 = v_1. \end{cases} \tag{50}$$

Assume that the trajectory to be tracked satisfies

$$\left\{\begin{array}{rcl}\dot{z}_{n,r}(t) &=& z_{n-1,r}(t)v_{1,r}(t),\\ &\vdots& \\ \dot{z}_{3,r}(t) &=& z_{2,r}(t)v_{1,r}(t),\\ \dot{z}_{2,r}(t) &=& v_{2,r}(t),\\ \dot{z}_{1,r}(t) &=& v_{1,r}(t)\end{array}\right. \tag{51}$$

and is bounded. Then, after the change of feedbacks $v_1 = v_{1,r}(t)+u_1$, $v_2 = v_{2,r}(t)+u_2$, and denoting $v_{1,r}(t)$ simply by $p(t)$, the error equation is

$$\left\{\begin{array}{rcl}\dot{z}_{n,e} &=& p(t)z_{n-1,e} + z_{n-1}u_1,\\ &\vdots& \\ \dot{z}_{3,e} &=& p(t)z_{2,e} + z_2u_1,\\ \dot{z}_{2,e} &=& u_2,\\ \dot{z}_{1,e} &=& u_1,\end{array}\right. \tag{52}$$

where $z_{i,e} = (z_i - z_{i,r}(t))$ for all $i = 1$ to n. Assume that for the chain of integrators

$$\left\{\begin{array}{rcl}\dot{z}_{n,e} &=& p(t)z_{n-1,e},\\ &\vdots& \\ \dot{z}_{3,e} &=& p(t)z_{2,e},\\ \dot{z}_{2,e} &=& u_2\end{array}\right. \tag{53}$$

there are a bounded control law $u_2(t, z_e)$ with $z_e = (z_{2,e}, \ldots, z_{n,e})$, a strict Lyapunov function $V_e(t, z_e)$, and a positive definite function $W_e(z_e)$ such that the derivative of $V_e(\cdot)$ along the trajectories of (53) in closed-loop with $u_2(t, z_e)$ satisfies

$$\dot{V}_e \leq -W_e(z_e). \tag{54}$$

Then the derivative of the function

$$U_e(t, z_e, z_{1,e}) = V_e(t, z_e) + \frac{1}{2}z_{1,e}^2 \tag{55}$$

along the trajectories of (52) in closed-loop with $u_2(t, z_e)$ and the bounded feedback

$$u_1(t, z_{1,e}, z_e) = -\frac{\frac{\partial V_e}{\partial z_{n,e}}(t, z_e)z_{n-1} + \cdots + \frac{\partial V_e}{\partial z_{3,e}}(t, z_e)z_2 + z_{1,e}}{1 + \left(\frac{\partial V_e}{\partial z_{n,e}}(t, z_e)z_{n-1}\right)^2 + \cdots + \left(\frac{\partial V_e}{\partial z_{3,e}}(t, z_e)z_2\right)^2 + z_{1,e}^2} \tag{56}$$

satisfies

$$\dot{U}_e \leq -W_e(z_e) - \frac{\left(\frac{\partial V_e}{\partial z_{n,e}}(t, z_e)z_{n-1} + \cdots + \frac{\partial V_e}{\partial z_{3,e}}(t, z_e)z_2 + z_{1,e}\right)^2}{1 + \left(\frac{\partial V_e}{\partial z_{n,e}}(t, z_e)z_{n-1}\right)^2 + \cdots + \left(\frac{\partial V_e}{\partial z_{3,e}}(t, z_e)z_2\right)^2 + z_{1,e}^2}. \tag{57}$$

One can check readily that this implies that $U_e(\cdot)$ is a strict Lyapunov function for the system (52) in closed-loop with the bounded feedbacks $u_1(t, z_{1,e}, z_e)$, $u_2(t, z_e)$.

Consequently, we have shown that the problem of determining globally uniformly asymptotically stabilizing bounded feedbacks and strict Lyapunov functions for error equations resulting from the problem of tracking a bounded trajectory of a system in chained form can be reduced to the problem of determining globally uniformly asymptotically stabilizing bounded feedbacks for time-varying chains of integrators (49) and accompanying strict Lyapunov functions.

4.2. Control design for the system (48). We begin the construction with a preliminary result which will be used throughout the remainder of the section.

Preliminary result. By applying Lemma 2.1, one can prove that the derivative of the function

$$V(t,x) = 80x^2 + \left(\int_t^{t+2\pi} (s-t-2\pi)\cos^6(s)ds\right)\frac{x^2}{\sqrt{1+x^2}} \tag{58}$$

along the trajectories of

$$\dot{x} = -\cos^6(t)\frac{x}{\sqrt{1+x^2}} \tag{59}$$

satisfies

$$\dot{V}(t,x) \leq -\frac{5\pi}{8}\frac{x^2}{\sqrt{1+x^2}}. \tag{60}$$

Moreover, $V(t,x)$ is periodic of period 2π and

$$70x^2 \leq V(t,x) \leq 90x^2. \tag{61}$$

We are ready now to carry out the backstepping construction of a bounded feedback for (48) by applying repeatedly the main result of section 3.

The x_1-subsystem. According to the preliminary result, the time derivative of the function $V(t,x_1)$ along the trajectories of (48) satisfies

$$\begin{aligned}\dot{V} &\leq -\frac{5\pi}{8}\frac{x_1^2}{\sqrt{1+x_1^2}} + \frac{\partial V}{\partial x_1}(t,x_1)\cos(t)\left[x_2 + \cos^5(t)\frac{x_1}{\sqrt{1+x_1^2}}\right]\\ &\leq -\frac{5\pi}{8}\frac{x_1^2}{\sqrt{1+x_1^2}} + 180|x_1|\left|x_2+\cos^5(t)\frac{x_1}{\sqrt{1+x_1^2}}\right|.\end{aligned} \tag{62}$$

The x_2-subsystem. Consider the variable

$$X_2 = \Omega_a(x_2) + \cos^5(t)\frac{x_1}{\sqrt{1+x_1^2}}, \tag{63}$$

where $\Omega_a(\cdot)$ is the odd function

$$\Omega_a(r) = \int_0^r \left(1+\max\{0, 9(|s|-2)^3\}\right)ds. \tag{64}$$

An immediate calculation yields

$$\begin{aligned}\dot{X}_2 &= \Omega_a'(x_2)\cos(t)x_3 - 5\sin(t)\cos^4(t)\frac{x_1}{\sqrt{1+x_1^2}} + \cos^6(t)\frac{x_2}{(1+x_1^2)^{\frac{3}{2}}}\\ &= -\cos^6(t)\frac{X_2}{\sqrt{1+X_2^2}} + \Omega_a'(x_2)\cos(t)x_3 + \cos^6(t)\frac{X_2}{\sqrt{1+X_2^2}}\\ &\quad - 5\sin(t)\cos^4(t)\frac{x_1}{\sqrt{1+x_1^2}} + \cos^6(t)\frac{x_2}{(1+x_1^2)^{\frac{3}{2}}}.\end{aligned} \tag{65}$$

Considering x_3 as a fictitious input v and choosing for it

$$v(t,x_1,x_2) = \frac{-\cos^5(t)\frac{X_2}{\sqrt{1+X_2^2}} + 5\sin(t)\cos^3(t)\frac{x_1}{\sqrt{1+x_1^2}} - \cos^5(t)\frac{x_2}{(1+x_1^2)^{\frac{3}{2}}}}{\Omega_a'(x_2)}, \tag{66}$$

the dynamics (65) become

$$\dot{X}_2 = -\cos^6(t)\frac{X_2}{\sqrt{1+X_2^2}}. \tag{67}$$

The derivative of

$$U(t,x_1,X_2) = l(V(t,x_1)) + k(V(t,X_2)) \tag{68}$$

satisfies

$$\begin{aligned}
\dot{U} \leq & -\frac{5\pi}{8}l'(V(t,x_1))\frac{x_1^2}{\sqrt{1+x_1^2}} + 180l'(V(t,x_1))|x_1|\left|x_2 + \cos^5(t)\frac{x_1}{\sqrt{1+x_1^2}}\right| \\
& - \frac{5\pi}{8}k'(V(t,X_2))\frac{X_2^2}{\sqrt{1+X_2^2}} \\
\leq & -\frac{5\pi}{8}l'(V(t,x_1))\frac{x_1^2}{\sqrt{1+x_1^2}} + 180l'(V(t,x_1))|x_1||X_2| \\
& - \frac{5\pi}{8}k'(V(t,X_2))\frac{X_2^2}{\sqrt{1+X_2^2}} \\
\leq & -\frac{\pi}{2}l'(V(t,x_1))\frac{x_1^2}{\sqrt{1+x_1^2}} + \frac{2}{\pi}180^2\sqrt{1+x_1^2}l'(V(t,x_1))X_2^2 \\
& - \frac{5\pi}{8}k'(V(t,X_2))\frac{X_2^2}{\sqrt{1+X_2^2}}.
\end{aligned} \tag{69}$$

Choosing for $l(\cdot)$

$$l(r) = \sqrt{(80-4\pi^2)+r} - \sqrt{80-4\pi^2} \tag{70}$$

leads to

$$\begin{aligned}
\dot{U} \leq & -\frac{\pi}{2}l'(V(t,x_1))\frac{x_1^2}{\sqrt{1+x_1^2}} + \frac{180^2\sqrt{1+x_1^2}}{\pi\sqrt{(80-4\pi^2)+(80-4\pi^2)x_1^2}}X_2^2 \\
& - \frac{5\pi}{8}k'(V(t,X_2))\frac{X_2^2}{\sqrt{1+X_2^2}}.
\end{aligned} \tag{71}$$

Choosing for $k(\cdot)$

$$k(r) \ = \ 480\left[\left((80-4\pi^2)+r\right)^{\frac{3}{2}} - \left(80-4\pi^2\right)^{\frac{3}{2}}\right] \tag{72}$$

leads to

$$\dot{U} \leq -\frac{\pi}{2}l'(V(t,x_1))\frac{x_1^2}{\sqrt{1+x_1^2}} - \frac{\pi}{2}k'(V(t,X_2))\frac{X_2^2}{\sqrt{1+X_2^2}}. \tag{73}$$

The overall system. From the above analysis, it results that the system (48) is equivalent to the system

$$\begin{cases}
\dot{x}_1 \ = \cos(t)x_2, \\
\dot{X}_2 \ = -\cos^6(t)\dfrac{X_2}{\sqrt{1+X_2^2}} + \Omega_a'(x_2)\cos(t)(x_3 - v(t,x_1,x_2)), \\
\dot{x}_3 \ = \cos(t)u
\end{cases} \tag{74}$$

with $x_2 = \Omega_a^{-1}(X_2 - \cos^5(t)\frac{x_1}{\sqrt{1+x_1^2}})$. Using the inequality

$$
\begin{aligned}
&\frac{(k^{-1})'(U(t,x_1,X_2))}{\sqrt{1+k^{-1}(U(t,x_1,X_2))}}\left[\left|\frac{\partial U}{\partial X_2}((t,x_1,X_2))\right| + \left|\frac{\partial U}{\partial x_1}((t,x_1,X_2))\right|\right] \\
&\qquad \leq c\left(l'(V(t,x_1))\frac{x_1^2}{\sqrt{1+x_1^2}} + k'(V(t,X_2))\frac{X_2^2}{\sqrt{1+X_2^2}}\right),
\end{aligned}
\tag{75}
$$

where c is a positive constant,[1] and observing that the function $\int_0^r \frac{(k^{-1})'(s)}{\sqrt{1+k^{-1}(s)}}ds$ is of class $\mathcal{K}_\infty$, one can prove that Assumptions A1, A2, A3, and A3′ of Theorem 3.1 are satisfied by the system (74) with $(x_1, X_2)^\top$ playing the role of x, x_3 playing the role of z, and $U(\cdot)$ playing the role of $V(\cdot)$. It follows that the construction of a globally uniformly asymptotically and locally exponentially stabilizing bounded feedback for the system (74) can be achieved. The last part of the section is devoted to this construction.

The function $v(t, x_1, x_2)$ defined in (66) satisfies

$$
|v(t,x_1,x_2)| \leq \frac{6+|x_2|}{1+\max\{0,(|x_2|-2)^3\}} \leq 10. \tag{76}
$$

These inequalities lead one to consider the change of variable

$$
X_3 = \Omega_b(x_3) - v(t,x_1,x_2), \tag{77}
$$

where $\Omega_b(\cdot)$ is the odd function defined as

$$
\Omega_b(r) = \int_0^r \left(1+\max\{0, 21(|s|-20)^3\}\right) ds. \tag{78}
$$

Its time derivative satisfies

$$
\dot{X}_3 = \Omega_b'(x_3)\cos(t)u - \dot{v}(t,x_1,x_2) \tag{79}
$$

with

$$
\begin{aligned}
\dot{v} &= \cos^2(t)\zeta(t,x_1,x_2,x_3), \\
\zeta &= \frac{5\left[\sin(t)\cos^2(t)\left(\frac{X_2}{\sqrt{1+X_2^2}} + \frac{x_2}{(1+x_1^2)^{\frac{3}{2}}}\right) + (\cos^4(t)+3\sin^2(t))\frac{x_1}{\sqrt{1+x_1^2}}\right]}{1+\max\{0,9(|x_2|-2)^3\}} \\
&\quad + \frac{\cos^2(t)}{1+\max\{0,9(|x_2|-2)^3\}}\left[\frac{5\sin(t)}{(1+x_1^2)^{\frac{3}{2}}}x_2 - \frac{3\cos^2(t)x_2^2}{(1+x_1^2)^{\frac{5}{2}}} - \cos^3(t)\frac{x_3}{(1+x_1^2)^{\frac{3}{2}}}\right] \\
&\quad - \frac{\cos^4(t)\left[(1+\max\{0,9(|x_2|-2)^3\})x_3 - 5\sin(t)\cos^3(t)\frac{x_1}{\sqrt{1+x_1^2}} + \cos^5(t)\frac{x_2}{(1+x_1^2)^{\frac{3}{2}}}\right]}{(1+\max\{0,9(|x_2|-2)^3\})(1+X_2^2)^{\frac{3}{2}}} \\
&\quad + \cos(t)\frac{\cos^2(t)\frac{X_2}{\sqrt{1+X_2^2}} - 5\sin(t)\frac{x_1}{\sqrt{1+x_1^2}} + \cos^2(t)\frac{x_2}{(1+x_1^2)^{\frac{3}{2}}}}{(1+\max\{0,9(|x_2|-2)^3\})^2}\max\{0,27(|x_2|-2)^2\}.
\end{aligned}
\tag{80}
$$

[1]The explicit value of c can be determined through lengthy calculations. But an explicit value is useless; to carry out the proof, it is only required that we know that c exists.

One can check readily that the feedback

$$u = \frac{-\cos^5(t)\frac{X_3}{\sqrt{1+X_3^2}} + \cos(t)\zeta(t,x_1,x_2,x_3)}{\Omega_b'(x_3)} = \frac{-\cos^5(t)\frac{X_3}{\sqrt{1+X_3^2}} + \cos(t)\zeta(t,x_1,x_2,x_3)}{1+\max\{0,21(|x_3|-20)^3\}} \tag{81}$$

yields

$$\dot{X}_3 = -\cos^6(t)\frac{X_3}{\sqrt{1+X_3^2}}. \tag{82}$$

Moreover, the feedback u is bounded:

$$|u| \leq \frac{1+|\zeta(t,x_1,x_2,x_3)|}{1+\max\{0,21(|x_3|-20)^3\}} \leq 25 + \frac{6|x_2|+x_2^2}{1+\max\{0,9(|x_2|-2)^3\}} + \frac{2|x_3|}{1+\max\{0,21(|x_3|-20)^3\}} \leq 94. \tag{83}$$

REMARK 2. *The feedback we have constructed is bounded in norm by* 94*. But for all* $\varepsilon > 0$*, one can modify the design in such a way that the resulting control law is bounded by* ε *instead of* 94*.*

5. Conclusion. We have proposed a new extension of the backstepping technique which applies to time-varying nonlinear systems and thereby can be utilized in particular for solving problems of global tracking. We have proposed families of bounded control laws. We have not explored all the possible extensions of the approach: We want to emphasize that the key ideas of the technique are even more important than the results themselves. Much remains to be done; robustness and disturbance attenuation issues and applications to the control design for systems with delay are some issues that may be pursued.

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