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Robust output feedback stabilization of the angular velocity of a rigid body

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Abstract

The problem of semi-global stabilization of the angular velocity of an underactuated rigid body in the presence of model errors is addressed and solved using a smooth, time-varying, dynamic, output feedback control law. The proposed scheme improves some of the existing results and provides guidelines for a general stabilization strategy applicable to systems which are not exponentially stabilizable. Simulations results are included. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Output feedback; Euler's equations; Semi-global stabilization; Robustness

1. Introduction

The issue of state feedback stabilization of the angular velocity of a rigid body with two control torques has been studied and solved using various approaches. The design method proposed in [1] relies on the center manifold theory, whereas the solution in [11] makes use of control Lyapunov functions and La Salle invariance principle. Moreover, in [2] a control law which is robust against errors on the principal moments of inertia is presented. Robustness results are also discussed in [9], where the exploitation of the homogeneity properties of the system and the use of homogeneous control laws lead to the design of a state feedback control law yielding robustness with respect to errors in the principal moments of inertia and in the location of the actuators. Finally, a somewhat different approach is pursued in [5], where robustness against external disturbances is studied.

We conclude that a lot of attention has been devoted to the stabilization problem for the angular velocity of a rigid body in the event of actuator failure. On the other hand, the *dual* problem, i.e. the stabilization in the event of sensor malfunction, has not been studied with the same interest. To the best of our knowledge such a problem has been only recently studied in [4], where a hybrid control law yielding exponential convergence has been derived using Lyapunov techniques and exploiting the fact that the system is linear in the unmeasured states. The result in [4] presents two main limitations. First the measured states have to fulfil a strong structural assumption, namely the measured velocities are the ones about the principal axes with the smallest and the largest moments of inertia. Such restriction cannot be removed using the approach in [4]. Secondly, the feedback law requires the precise knowledge of the moments of inertia of the rigid body, and no robustness bound can be derived.

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In the present paper, we improve in some respects the aforementioned results. We design a time-varying, dynamic, output feedback control law for the rigid body with two controls and partial state information. The present construction, which uses completely different arguments from that in [4], yields on one side weaker results, as only semi-global stability can be proved, but on the other hand, stronger result, as the proposed control law is smooth and is able to counteract model errors analogous to those considered in [9].

The leading idea of our design is as follows. We solve an output tracking problem for a bounded trajectory which goes to the origin when the time goes to infinity. It must be noticed that, even locally, such problem is non-trivial as the zero-dynamics of the system with the considered output are stable, but not asymptotically stable, hence the classical results of [7, Section 4.5], relying on partial output linearization, do not apply. Finally, we emphasize that the stabilization tool used in this work does not exploit the homogeneity property of the system to be controlled and does not make use of the fact that the system is linear in the unmeasured states. As a result, both asymptotic stabilization via partial state information and robustness properties are obtained. However, this does not hold globally.

Remark 1. It must be observed that the proposed construction has a general nature, i.e. it might be used to solve stabilization problems for classes of systems. These include nonholonomic chained forms, see [10] for the definition, and nonholonomic canonical systems with maximum growth, see [10] for the definition and [8] where the construction presented in this work is used, together with backstepping and forwarding arguments, to asymptotically stabilize a local description of the *ball and plate* system [6].

Preliminaries

1. Throughout the paper, the symbol c is used to denote generically a strictly positive number. For instance, we may write: $c + 3c^2 + 1 = c$.
2. By $\dot{f}(t)$ we denote the first derivative of the function of time $f(t)$.

2. Problem statement

Consider a rigid body in an inertial reference frame and let Ω_1, Ω_2 and Ω_3 denote the angular velocity components along a body fixed reference frame having the origin at the center of gravity and consisting of the three principal axes. The Euler's equations for the rigid body, subject to two independent controls aligned with two principal axes, are (see [3])

$$
\begin{aligned}
\dot{\Omega}_3 &= a_3\Omega_1\Omega_2,\\
\dot{\Omega}_2 &= a_2\Omega_1\Omega_3 + b_2u_2,\ \ b_2 \neq 0,\\
\dot{\Omega}_1 &= a_1\Omega_2\Omega_3 + b_1u_1,\ \ b_1 \neq 0,
\end{aligned} \tag{1}
$$

where (u_1, u_2) are the controls and the a_i's and the b_i's are constant parameters describing the inertial properties of the rigid body. If $a_3 = 0$, system (1) is not controllable, moreover, if $a_2 = a_1 = 0$, then $a_3 = 0$ as well. Hence, without lack of generality, we assume $a_3 \neq 0$ and $a_2 \neq 0$. If the rigid body possesses a symmetry axis then $a_1 = 0$ and a_3 and a_2 do not have the same sign (see [11]), whereas in the non-symmetric case, all the a_i's are nonzero and the a_i's do not have the same sign. As a result we may assume, without loss of generality, that $a_3a_2 < 0$.

Therefore, every rigid body with two controls can be described by equations of the form

$$
\begin{aligned}
\dot{\omega}_3 &= -\omega_1\omega_2,\\
\dot{\omega}_2 &= \omega_1\omega_3 + v_2,\\
\dot{\omega}_1 &= s\omega_2\omega_3 + v_1
\end{aligned} \tag{2}
$$

with

$$s \in \{-1, 0, 1\}$$

and

$$
\begin{aligned}
\omega_1 &= \sqrt{-a_2a_3}\,\Omega_1,\\
\omega_2 &= \sqrt{|a_1a_3|}\,\Omega_2,\\
\omega_3 &= -(|a_3|/a_3)\sqrt{|a_1a_2|}\,\Omega_3,\\
v_1 &= \sqrt{-a_2a_3}\,b_1u_1, \quad v_2 = \sqrt{|a_1a_3|}\,b_2u_2.
\end{aligned}
$$

However, when there are errors on the principal moments of inertia and on the location of the actuators, as shown in [9], the angular velocities of the rigid body are described by equations of the form

$$
\begin{aligned}
\dot{\Omega}_3 &= (a_3 + \mu_7)\Omega_1\Omega_2 + \mu_1u_1 + \mu_2u_2,\\
\dot{\Omega}_2 &= (a_2 + \mu_8)\Omega_1\Omega_3 + b_2u_2 + \mu_3u_1 + \mu_4u_2,\\
\dot{\Omega}_1 &= (a_1 + \mu_9)\Omega_2\Omega_3 + b_1u_1 + \mu_5u_1 + \mu_6u_2,
\end{aligned} \tag{3}
$$

where the μ_i's are unknown parameters depending on the *size* of the model errors.

Applying to (3) the change of coordinates and the change of feedback resulting in (2), we deduce that the equations of the rigid body with errors on the principal moments of inertia and on the location of the actuators can be described by the equations

$$\begin{aligned}\dot{\omega}_3 &= -\omega_1\omega_2 + \kappa_3(\omega_1,\omega_2,\omega_3) + \mu_3(v_1,v_2),\\ \dot{\omega}_2 &= \omega_1\omega_3 + v_2 + \kappa_2(\omega_1,\omega_2,\omega_3) + \mu_2(v_1,v_2),\\ \dot{\omega}_1 &= s\omega_2\omega_3 + v_1 + \kappa_1(\omega_1,\omega_2,\omega_3) + \mu_1(v_1,v_2),\end{aligned} \tag{4}$$

where s is unknown, but $s \in [-1,1]$, and the functions $\kappa_i(\omega_1,\omega_2,\omega_3)$ and $\mu_i(v_1,v_2)$, $i=1,2,3$ are unknown but satisfy, for some $\varepsilon > 0$, the following bounds

$$\sum_{i=1}^{3} |\kappa_i(\omega_1,\omega_2,\omega_3)| \leqslant \varepsilon(|\omega_1|\,|\omega_2| + |\omega_1|\,|\omega_3| + |\omega_2|\,|\omega_3|) \tag{5}$$

and

$$\sum_{i=1}^{3} |\mu_i(v_1,v_2)| \leqslant \varepsilon(|v_1| + |v_2|). \tag{6}$$

The objective of the present paper is the asymptotic stabilization of the origin of system (4) by means of an output feedback control law which makes use of the output $y_a = (\omega_3,\omega_1)^\top$ or $y_b = (\omega_2,\omega_1)^\top$. The case $y_c = (\omega_3,\omega_2)^\top$ can be dealt with in an analogous manner and will not be discussed in detail.

3. Main result

Theorem 3.1. *Consider system* (4) *with y_a or y_b as output. Assume that* (5) *and* (6) *are satisfied. Then there exists $\bar{\varepsilon} > 0$ such that if $\varepsilon \in [0,\bar{\varepsilon}]$ system* (4) *is semi-globally stabilizable by dynamic time-varying output feedback.*

Remark 2. It must be noted that an explicit estimate for $\bar{\varepsilon}$ can be determined from the arguments contained in part (ii) of the proof. However, for simplicity we leave the details of the derivation.

Proof. The proof breaks up in two parts. First, we construct an output feedback stabilizing control law for the unperturbed system, i.e. system (4) with $\kappa_i(\cdot) = 0$ and $\mu_i(\cdot) = 0$ for $i = 1,2,3$, then we study the robustness properties of system (4) when this control is applied.

(i) *Design of a stabilizing control law for the nominal system.*

Consider the new time dependent coordinate

$$z = \omega_1 - \gamma(t), \tag{7}$$

where $\gamma(t)$ is a smooth function of time to be specified. As a result, system (4) with $\kappa_i(\cdot)=0$ and $\mu_i(\cdot)=0$ for $i=1,2,3$ is described by the equations

$$\begin{aligned}\dot{\omega}_3 &= -\gamma(t)\omega_2 - z\omega_2,\\ \dot{\omega}_2 &= \gamma(t)\omega_3 + z\omega_3 + v_2,\\ \dot{z} &= -\dot{\gamma}(t) + s\omega_2\omega_3 + v_1.\end{aligned} \tag{8}$$

Consider the reduced-order observer

$$\begin{aligned}\dot{\hat{\omega}}_3 &= -\gamma(t)\hat{\omega}_2 - z\hat{\omega}_2 + \theta_3,\\ \dot{\hat{\omega}}_2 &= \gamma(t)\hat{\omega}_3 + z\hat{\omega}_3 + v_2 + \theta_2,\end{aligned} \tag{9}$$

where the functions θ_i, $i=2,3$ are to be specified, and the error coordinates $\tilde{\omega}_3 = \omega_3 - \hat{\omega}_3$ and $\tilde{\omega}_2 = \omega_2 - \hat{\omega}_2$. Simple calculations yield

$$\begin{aligned}\dot{\tilde{\omega}}_3 &= -\gamma(t)\tilde{\omega}_2 - z\omega_2 + z\hat{\omega}_2 - \theta_3,\\ \dot{\tilde{\omega}}_2 &= \gamma(t)\tilde{\omega}_3 + z\omega_3 - z\hat{\omega}_3 - \theta_2.\end{aligned} \tag{10}$$

Let

$$\begin{aligned}v_1 &= -k(z)z + \dot{\gamma}(t),\\ v_2 &= -z\hat{\omega}_3 - \gamma(t)\hat{\omega}_2 - \theta_2,\end{aligned} \tag{11}$$

where $k(z)$ is a strictly positive function to be specified. As a result the closed loop system is described by the equations

$$\begin{aligned}\dot{\hat{\omega}}_3 &= -\gamma(t)\hat{\omega}_2 - z\hat{\omega}_2 + \theta_3,\\ \dot{\hat{\omega}}_2 &= \gamma(t)\hat{\omega}_3 - \gamma(t)\hat{\omega}_2,\\ \dot{\tilde{\omega}}_3 &= -\gamma(t)\tilde{\omega}_2 - z\tilde{\omega}_2 - \theta_3,\\ \dot{\tilde{\omega}}_2 &= \gamma(t)\tilde{\omega}_3 + z\tilde{\omega}_3 - \theta_2,\\ \dot{z} &= -k(z)z + s(\hat{\omega}_2 + \tilde{\omega}_2)(\hat{\omega}_3 + \tilde{\omega}_3).\end{aligned} \tag{12}$$

We now consider separately the case with output y_a and the case with output y_b.

Case 1: The output is $y_a = (\omega_3,\omega_1)^\top$. Setting $\theta_3 = \gamma(t)\tilde{\omega}_3$ and $\theta_2 = z\tilde{\omega}_3$, the control laws become

$$\begin{aligned}v_1 &= -k(z)z + \dot{\gamma}(t) + \hat{\omega}_2\hat{\omega}_3,\\ v_2 &= -\gamma(t)\hat{\omega}_2 - z\hat{\omega}_3 + z\tilde{\omega}_3\end{aligned} \tag{13}$$

and system (12) simplifies as follows:

$$\begin{aligned}\dot{\hat{\omega}}_3 &= -\gamma(t)\hat{\omega}_2 - z\hat{\omega}_2 + \gamma(t)\tilde{\omega}_3,\\ \dot{\hat{\omega}}_2 &= \gamma(t)\hat{\omega}_3 - \gamma(t)\hat{\omega}_2,\\ \dot{\tilde{\omega}}_3 &= -\gamma(t)\tilde{\omega}_2 - \gamma(t)\tilde{\omega}_3 - z\tilde{\omega}_2,\\ \dot{\tilde{\omega}}_2 &= \gamma(t)\tilde{\omega}_3,\\ \dot{z} &= -k(z)z + s(\hat{\omega}_2 + \tilde{\omega}_2)(\hat{\omega}_3 + \tilde{\omega}_3).\end{aligned} \tag{14}$$

Case 2: The output is $y_b = (\omega_2, \omega_1)^\top$. Setting $\theta_3 = z\tilde{\omega}_2$ and $\theta_2 = \gamma(t)\tilde{\omega}_2$ the control laws become

$$
\begin{aligned}
v_1 &= -k(z)z + \dot{\gamma}(t), \\
v_2 &= -z\hat{\omega}_3 - \gamma(t)\hat{\omega}_2 - \gamma(t)\tilde{\omega}_2
\end{aligned} \tag{15}
$$

and system (12) simplifies as follows:

$$
\begin{aligned}
\dot{\hat{\omega}}_3 &= -\gamma(t)\hat{\omega}_2 + z\tilde{\omega}_2, \\
\dot{\hat{\omega}}_2 &= \gamma(t)\hat{\omega}_3 - \gamma(t)\hat{\omega}_2, \\
\dot{\tilde{\omega}}_3 &= -\gamma(t)\tilde{\omega}_2, \\
\dot{\tilde{\omega}}_2 &= -\gamma(t)\tilde{\omega}_2 + \gamma(t)\tilde{\omega}_3 + z\tilde{\omega}_3, \\
\dot{z} &= -k(z)z + s(\hat{\omega}_2 + \tilde{\omega}_2)(\hat{\omega}_3 + \tilde{\omega}_3).
\end{aligned} \tag{16}
$$

In the remainder of the proof we consider only the case where the output is y_a. An analogous demonstration can be carried out for the output y_b.

Let $\gamma(t)$ be a bounded, strictly positive and decreasing function. Consider the two positive definite and radially unbounded functions

$$P(l, m) = l^2 + m^2 + lm$$

and

$$Q(l, m) = P(-l, m).$$

One can readily check that

$$
\overbrace{P(\tilde{\omega}_3, \tilde{\omega}_2) + z^2}^{\cdot} \leqslant -\gamma(t)P(\tilde{\omega}_3, \tilde{\omega}_2) + c|z|[P(\tilde{\omega}_3, \tilde{\omega}_2) + Q(\hat{\omega}_2, \hat{\omega}_3)] - 2k(z)z^2 \tag{17}
$$

and

$$
\overbrace{Q(\hat{\omega}_3, \hat{\omega}_2)}^{\cdot} \leqslant -\gamma(t)Q(\hat{\omega}_3, \hat{\omega}_2) + |z|Q(\hat{\omega}_3, \hat{\omega}_2) + \gamma(t)\sqrt{Q(\hat{\omega}_3, \hat{\omega}_2)}\sqrt{P(\tilde{\omega}_3, \tilde{\omega}_2)}. \tag{18}
$$

Consider now the function

$$U(\hat{\omega}_3, \hat{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_2, z) = P(\tilde{\omega}_3, \tilde{\omega}_2) + z^2 + \lambda Q(\hat{\omega}_3, \hat{\omega}_2),$$

where λ is a strictly positive parameter to be specified. It must be noted that this function is positive definite, radially unbounded, zero at zero and differentiable. Using the triangular inequality, one has

$$
\begin{aligned}
\dot{U} \leqslant& -\tfrac{3}{4}\gamma(t)P(\tilde{\omega}_3, \tilde{\omega}_2) - 2k(z)z^2 - \tfrac{3}{4}\lambda\gamma(t)Q(\hat{\omega}_3, \hat{\omega}_2) \\
&+ c|z|(P(\tilde{\omega}_3, \tilde{\omega}_2) + Q(\hat{\omega}_3, \hat{\omega}_2)) \\
&+ \lambda c\gamma(t)P(\tilde{\omega}_3, \tilde{\omega}_2).
\end{aligned}
$$

Choosing

$$\lambda \in \left] 0, \frac{1}{1 + 4c\sup_{t \geqslant 0}\gamma(t)} \right],$$

we obtain

$$
\begin{aligned}
\dot{U} \leqslant& -\tfrac{1}{2}\gamma(t)P(\tilde{\omega}_3, \tilde{\omega}_2) - 2k(z)z^2 \\
&- \tfrac{3}{4}\lambda\gamma(t)Q(\hat{\omega}_3, \hat{\omega}_2) \\
&+ c|z|(P(\tilde{\omega}_3, \tilde{\omega}_2) + Q(\hat{\omega}_3, \hat{\omega}_2)).
\end{aligned}
$$

Finally, using Joung's inequality and setting $k(z) = k_1(|z| + 1)$ with $k_1 \geqslant 4$, yields

$$
\begin{aligned}
\dot{U} \leqslant& -\tfrac{1}{2}\gamma(t)P(\tilde{\omega}_3, \tilde{\omega}_2) - k(z)z^2 - \tfrac{3}{4}\lambda\gamma(t)Q(\hat{\omega}_3, \hat{\omega}_2) \\
&+ \frac{c}{k_1}[P(\tilde{\omega}_3, \tilde{\omega}_2) + Q(\hat{\omega}_3, \hat{\omega}_2)]^{3/2}.
\end{aligned}
$$

As a result

$$\dot{U} \leqslant \left[-\tfrac{1}{2}\gamma(t) + \frac{c}{k_1}\sqrt{U(t)} \right] U(t). \tag{19}$$

This inequality allows to establish the semi-global asymptotic stabilizability of the undisturbed system (4) if $\gamma(t)$ is properly selected. Many choices are possible and this flexibility is important from a qualitative and practical point of view to ensure a good convergence. In what follows we propose one such selection.

Lemma 3.2. *Let $R \geqslant 5000$ and E_R be the set*

$$E_R = \{(\hat{\omega}_3, \hat{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_2, z) : U(\hat{\omega}_3, \hat{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_2, z) \leqslant R\}.$$

Consider an initial condition $(\hat{\omega}_3, \hat{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_2, z)$ belonging to E_R. Let

$$\gamma(t) = \frac{4\gamma_0}{1 + t},$$

where $\gamma_0 = \frac{1}{32}\sqrt{R}$ and $k_1 \geqslant 64(c + 1)$, where c is the positive number c in Eq. (19).

Then for all $t \geqslant 0$ the inequality

$$\dot{U} \leqslant -\frac{\gamma_0}{2(1 + t)}U(t) \tag{20}$$

holds.

Proof. See Appendix A. □

Inequality (20) yields

$$\lim_{t \to +\infty} U(t) = 0.$$

It follows that every trajectory converges to the origin. However, since function $\gamma(t)$ depends on R, we do not yet have proved semi-global asymptotic stability. To complete the proof, consider $\mathscr{R} > 1$ and a trajectory with an initial condition $(\omega_1, \omega_2, \omega_3, \hat{\omega}_2, \hat{\omega}_3)$ such that

$$U(\hat{\omega}_3, \hat{\omega}_2, \omega_3, \omega_2, \omega_1) \leqslant \mathscr{R}.$$

We now show that for a suitable choice of feedback the considered trajectory converges to the origin. To begin with observe that the following inequality

$$U(\hat{\omega}_3, \hat{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_2, z) \leqslant c\mathscr{R} + 2\gamma_0^2, \quad c \geqslant 1$$

is satisfied. Let $R = \frac{512}{511} c\mathscr{R}$. According to Lemma 3.2, if $\mathscr{R}$ is sufficiently large then for some feedback and $\gamma_0 = \frac{1}{32}\sqrt{R}$, the set E_R is contained within the basin of attraction of the corresponding closed-loop system. Since $\gamma_0 = \frac{1}{32}\sqrt{R}$, we have $R = c\mathscr{R} + 2\gamma_0^2$. It then follows that system (4) is semi-globally asymptotically stabilized by the family of control laws (13).

(ii) *Robustness analysis.*

In this part, we prove that the family of feedback (13) semi-globally asymptotically stabilizes (1) when disturbances satisfying bounds (5) and (6) with $\varepsilon \in [0, \bar{\varepsilon}]$ are present.

Step 1: Consider the functions $\kappa_i\ (\omega_1, \omega_2, \omega_3)$, $i = 1, 2, 3$. Using inequality (19), we get

$$\dot{U} \leqslant \left[-\tfrac{1}{2}\gamma(t) + \frac{c}{k_1}\sqrt{U(t)} \right] U(t) + c\sqrt{U(t)} \sum_{i=1}^{3} |\kappa_i(\omega_1, \omega_2, \omega_3)|.$$

Hence, as immediate consequence of the bound (5), one has

$$\begin{aligned} \dot{U} &\leqslant \left[-\tfrac{1}{2}\gamma(t) + \frac{c}{k_1}\sqrt{U(t)} \right] U(t) \\ &\quad + c\varepsilon\sqrt{U(t)}[|\omega_1|\,|\omega_2| + |\omega_1|\,|\omega_3| + |\omega_2|\,|\omega_3|] \\ &\quad + c\varepsilon\gamma(t)U(t) \\ &\leqslant \left[-\tfrac{1}{2}\gamma(t) + \frac{c}{k_1}\sqrt{U(t)} \right] U(t) + c\varepsilon U(t)^{3/2} \\ &\quad + c\varepsilon\gamma(t)U(t). \end{aligned}$$

Therefore, if $\varepsilon \in \left[0, \frac{1}{16c}\right]$, we conclude

$$\begin{aligned} \dot{U} &\leqslant \left[-\tfrac{3}{8}\gamma(t) + \left(\frac{c}{k_1} + c\varepsilon \right) \sqrt{U(t)} \right] U(t) \\ &\leqslant \left[-\tfrac{3}{8}\gamma(t) + \tfrac{1}{8}\sqrt{U(t)} \right] U(t). \end{aligned} \tag{21}$$

Step 2: We now include also the functions $\mu_i\ (v_1, v_2)$, $i = 1, 2, 3$. According to bound (6), and using inequality (21), we obtain

$$\begin{aligned} \dot{U} &\leqslant \left[-\tfrac{3}{8}\gamma(t) + \tfrac{1}{8}\sqrt{U(t)} \right] U(t) \\ &\quad + c\varepsilon\sqrt{U(t)}(|-k(z)z + \dot{\gamma}(t)| \\ &\quad + |-\gamma(t)\hat{\omega}_2 - z\hat{\omega}_3 + z\tilde{\omega}_3|) \\ &\leqslant \left[-\tfrac{3}{8}\gamma(t) + \tfrac{1}{8}\sqrt{U(t)} \right] U(t) \\ &\quad + k(z)c\varepsilon\sqrt{U(t)}|z| + c\varepsilon\sqrt{U(t)}|\dot{\gamma}(t)| \\ &\quad + c\varepsilon\gamma(t)U(t) + c\varepsilon U(t)^{3/2}. \end{aligned} \tag{22}$$

As a result, if $\varepsilon \in [0, \frac{1}{16c}]$,

$$\begin{aligned} \dot{U} &\leqslant \left[-\tfrac{1}{4}\gamma(t) + \tfrac{3}{16}\sqrt{U(t)} \right] U(t) \\ &\quad + k(z)c\varepsilon\sqrt{U(t)}|z| + c\varepsilon\sqrt{U(t)}|\dot{\gamma}(t)|. \end{aligned} \tag{23}$$

Moreover, since $k(z)$ does not depend on R, we obtain, if k_1 is sufficiently large

$$\begin{aligned} \dot{U} + \dot{\overbrace{|z|^{3/2}}} &\leqslant \left[-\tfrac{1}{4}\gamma(t) + \tfrac{3}{16}\sqrt{U(t)} \right] U(t) \\ &\quad + c\varepsilon\sqrt{U(t)}|z| + c\varepsilon\sqrt{U(t)}|\dot{\gamma}(t)| \\ &\quad - \tfrac{3}{2}k(z)|z|^{3/2} + cU(t)\sqrt{|z|} \\ &\leqslant \left[-\tfrac{1}{4}\gamma(t) + \tfrac{1}{4}\sqrt{U(t)} \right] U(t) \\ &\quad - \tfrac{1}{2}k(z)|z|^{3/2} + c\varepsilon\sqrt{U(t)}|\dot{\gamma}(t)|. \end{aligned} \tag{24}$$

It follows that the positive definite and radially unbounded function

$$\begin{aligned} &\mathscr{U}(\hat{\omega}_3, \hat{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_2, z) \\ &\quad = U(\hat{\omega}_3, \hat{\omega}_2, \tilde{\omega}_3, \tilde{\omega}_2, z) + |z|^{3/2}, \end{aligned}$$

satisfies

$$\dot{\mathscr{U}} \leqslant \left[-\tfrac{1}{4}\gamma(t) + \tfrac{1}{4}\sqrt{\mathscr{U}(t)} \right] \mathscr{U}(t) + c\varepsilon\sqrt{\mathscr{U}(t)}|\dot{\gamma}(t)|. \tag{25}$$

Finally, selection $\gamma(t) = \frac{4\gamma_0}{1+t}$, yields

$$\dot{\mathscr{U}} \leqslant \left[-\frac{\gamma_0}{1+t} + \tfrac{1}{2}\sqrt{\mathscr{U}(t)} \right] \mathscr{U}(t) + c\varepsilon \frac{\gamma_0^{3/2}}{(1+t)^3} \tag{26}$$

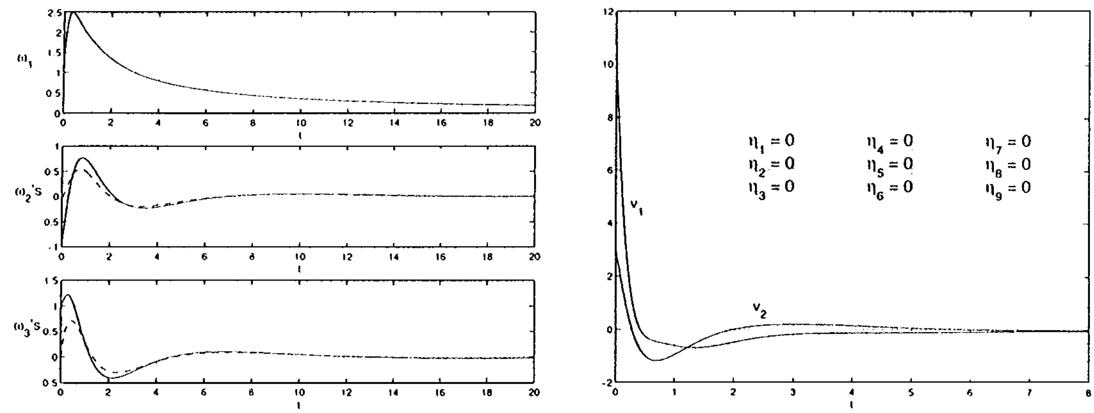

Fig. 1. Nominal simulations. State histories (left) and control actions (right) from the initial state $(\omega_1(0), \omega_2(0), \omega_3(0)) = (1, -1, 1)$. The states ω_2 and ω_3 are displayed together with their estimates (dashed lines) $\hat{\omega}_2$ and $\hat{\omega}_3$. The variables ω's are in rad/s.

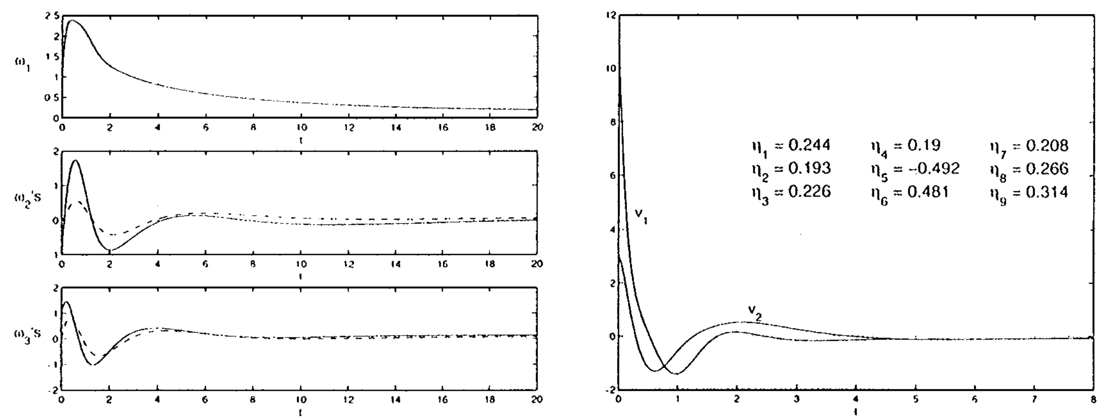

Fig. 2. Simulations with model errors. State histories (left) and control actions (right) from the initial state $(\omega_1(0), \omega_2(0), \omega_3(0))=(1, -1, 1)$. The states ω_2 and ω_3 are displayed together with their estimates (dashed lines) $\hat{\omega}_2$ and $\hat{\omega}_3$. The variables ω's are in rad/s.

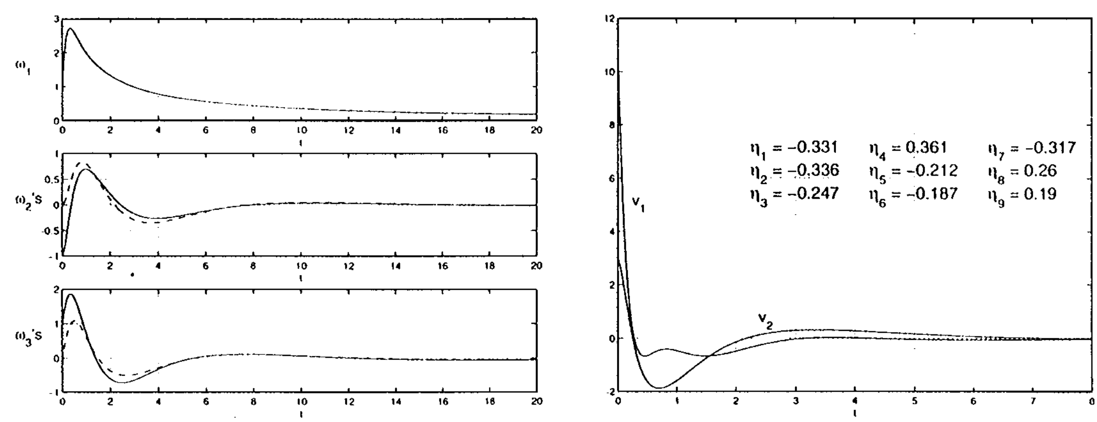

Fig. 3. Simulations with model errors. State histories (left) and control actions (right) from the initial state $(\omega_1(0), \omega_2(0), \omega_3(0))=(1, -1, 1)$. The states ω_2 and ω_3 are displayed together with their estimates (dashed lines) $\hat{\omega}_2$ and $\hat{\omega}_3$. The variables ω's are in rad/s.

and as long as

$$\frac{\gamma_0}{1+t} \geqslant 4\sqrt{\mathscr{U}(t)}$$

$$\overbrace{\mathscr{U}(t) - \frac{\delta}{(1+t)^2}}^{\bullet} \leqslant -\frac{\gamma_0}{2(1+t)}\left[\mathscr{U}(t) - \frac{\delta}{(1+t)^2}\right], \tag{27}$$

where

$$\delta = -\frac{c\varepsilon\gamma_0^{3/2}}{\gamma_0 - 2}.$$

Observe now that for γ_0 sufficiently large $|\delta| \leqslant c\varepsilon\sqrt{\gamma_0}$. Moreover, if ε is smaller than a constant number independent of R, and γ_0 is sufficiently large, then, using arguments similar to those invoked to prove Lemma 3.2, we conclude semi-global stabilizability of system (4) with model errors satisfying (5) and (6) directly from inequality (27). This concludes our proof. □

Remark 3. It is worth mentioning that there are many possible choices for $\gamma(t)$. For instance

$$\gamma(t) = \gamma_0\left[\rho(t) + \frac{1-\rho(t)}{1+t}\right]$$

with $\rho(t)=1$ if $t \leqslant \chi(R)$ and $\rho(t)=\mathrm{e}^{-t+\chi(R)}$ if $t \geqslant \chi(R)$ where $\chi(\cdot)$ is an increasing function suitably chosen is a possible alternative choice.

4. Simulation

In this section some simulations, showing the performance and the robustness properties of the proposed design, are presented. We consider system (4) with $s=-1$ and output $y_a = (\omega_3, \omega_1)^\top$. The parameters η_1 through η_9 are randomly generated variables with zero mean and variance equal to 0.3. The control law has been built according to the result proved in Lemma 3.2 with $\gamma_0 = 1$.

Fig. 1 displays the state variables $\omega_1, \omega_2, \omega_3, \hat{\omega}_2$ and $\hat{\omega}_3$, and the control actions v_1 and v_2, from the initial state [2] $(1,-1,1,0,0)$ in the nominal case, i.e. $\eta_i=0$ for $i=1,\dots,9$; whereas Figs. 2 and 3 show two simulation results for nonzero η's. Observe the convergence of the trajectories of the closed-loop system despite the presence of substantial model errors.

[2] The controller states are always initialized at $(\hat{\omega}_2(0), \hat{\omega}_3(0)) = (0,0)$.

5. Concluding remarks

The problem of robust output feedback stabilization of the angular velocity of a rigid body has been addressed and solved via a dynamic, time-varying control law. The proposed result improves in several respects existing ones and provides a new tool for stabilization of systems with non-stabilizable linearization. Various problems are left open. First, the extension of the proposed semi-global stabilizer to a global stabilizer and secondly the much more challenging problem of output feedback stabilization with one control torque and one measured state.

Appendix A. Proof of Lemma 3.2

Let

$$\Gamma(t) = -\frac{2\gamma_0}{1+t} + \frac{c}{k_1}\sqrt{U(t)}.$$

Since $U(0) \leqslant R$, $k_1 \geqslant 64(c+1)$ and $\gamma_0 = \frac{1}{32}\sqrt{R}$, function $\Gamma(t)$ is such that

$$\Gamma(0) \leqslant -\gamma_0 - \frac{1}{32}\sqrt{R} + \frac{c}{k_1}\sqrt{R} \leqslant -\gamma_0.$$

Assume that there exists $t_* \in\,]0,+\infty[$ such that

$$\Gamma(t_*) = -\frac{\gamma_0}{1+t_*} \tag{A.1}$$

and $\Gamma(t) \leqslant -\gamma_0/(1+t)$ for all $t \in [0, t_*[$. Using inequality (19), we deduce

$$\dot{U} \leqslant -\frac{\gamma_0}{1+t}U(t), \quad \forall t \in [0, t_*[$$

and, after a simple integration,

$$U(t) \leqslant \frac{U(0)}{(1+t)^{\gamma_0}} \leqslant \frac{R}{(1+t)^{\gamma_0}}, \quad \forall t \in [0, t_*]. \tag{A.2}$$

Moreover, Eq. (A.1) implies

$$U(t_*) = \left(\frac{k_1}{c}\frac{\gamma_0}{1+t_*}\right)^2. \tag{A.3}$$

Hence by Eqs. (A.2) and (A.3) we conclude

$$\left(\frac{k_1}{c}\frac{\gamma_0}{1+t_*}\right)^2 \leqslant \frac{R}{(1+t_*)^{\gamma_0}} \tag{A.4}$$

and, since $k_1 \geqslant 64(1+c)$, one has

$$16384\gamma_0^2 \leqslant R(1+t_*)^{2-\gamma_0}. \tag{A.5}$$

Finally, since $\gamma_0 = \frac{1}{32}\sqrt{R}$ and $R \geqslant 5000$, inequality (A.5) is false, hence inequality (20) is satisfied for all $t \geqslant 0$. □

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