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Interval Observers For Discrete–time Systems

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SUMMARY

Interval observers are constructed for discrete-time systems. First, time-invariant interval observers are proposed for a family of nonlinear systems. Second, it is shown that, for any time-invariant exponentially stable discrete-time linear system with additive disturbances, time-varying exponentially stable discrete-time interval observers can be constructed. The latter result relies on the design of time-varying changes of coordinates which transform a linear system into a nonnegative one.

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1. INTRODUCTION

Twelve years ago, a new state estimation approach has been introduced in [9]. It is based on the notion of interval observers, which are dynamic extensions giving estimates of the solutions of a system in the presence of various types of disturbances through two outputs giving an upper and a lower bound for the solutions. In contrast with classical observers, state estimators of this type supply certain information at any instant: if the initial condition of a solution of the system is unknown but can be bounded between two known values, then the trajectories of the interval observers starting from these bounds will enclose the solution under study. More precisely, an upper and a lower bound is provided *for each component* of the state and the norm of the difference between the bounds converges asymptotically to zero when no disturbance is acting. This type of information cannot be deduced from the knowledge of a classical observer. There are two other reasons why interval observers become more and more popular. First, they can be applied for systems with large uncertainties, which is important when, for instance, is considered an estimation problem of unmeasured coordinates of biological models. Second, it has been successfully applied to many applied problems, as illustrated by the papers [4], [2], [8], [18].

The technique has been developed in several contexts: in particular some works are devoted to families of linear systems [6], [13], [14], [16], [15] and others are devoted to nonlinear systems [2], [3], [20], [21], [18]. A common feature of these results is that they apply only to continuous-time systems. On the other hand, discrete-time systems are very important from a theoretical as well as an applied point of view and the problem of constructing observers or dynamic output feedbacks for them has been extensively studied (see for instance [12], [5], [22], [11, Chapt. 6]). The interest of the discrete-time systems partially stems from the fact that discretization techniques transform continuous-time systems into discrete-time systems. Besides, systems with sampled data often lead to discrete-time systems, as explained for instance in [1]. These systems are frequently affected by

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disturbances, which motivates the development of robust state estimation techniques, like the one based on interval observers.

This motivates the present work and the recent contribution [7], which, to the best of our knowledge, is the only existing work that is devoted to the design of interval observers for discrete-time systems.

Before comparing the results of [7], with those of the present paper, let us give a sketch of our contributions.

First, we will consider a nonlinear system of the form

$$x_{k+1} = \mathcal{F}(y_k, x_k, w_k) \ , \ k \in \mathbb{N}, \tag{1}$$

with $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$ is the output, where $w_k \in \mathbb{R}^\ell$ represents a disturbance and construct time-invariant interval observers, under a stability condition on the function $\mathcal{F}$ of a new type. Next, we will focus our attention on the fundamental family of the linear time-invariant discrete-time systems

$$x_{k+1} = \mathcal{A}x_k \ , \ k \in \mathbb{N}, \tag{2}$$

with $x_k \in \mathbb{R}^n$ when the spectral radius of $\mathcal{A}$ is smaller than 1 i.e. when $\mathcal{A}$ is Schur stable. We will show that in some cases, the technique of construction of time-invariant interval observers developed for (1) does not lead to interval observers for (2). To overcome this limitation, we will show how *time-varying interval observers* can be constructed for a family of linear systems with outputs and disturbances, which encompasses the family of systems (2). The construction we will propose relies on time-varying changes of coordinates that transform linear discrete-time systems into nonnegative discrete-time systems. We will design the change of coordinates by using the fact that any real matrix can be transformed into a matrix of the Jordan canonical form (see [19, Section 1.8]) and next by finding suitable changes of coordinates for elementary Jordan blocks. Surprisingly, although the changes of coordinates we will apply are time-varying, the transformed systems are autonomous. However, the interval observers we will construct are time-varying because they involve a time-varying change of coordinates, and thus they give lower and upper bounds for the state of the studied system that depend on the time.

The part of the present paper that is devoted to linear systems owes a great deal to [14], which presents constructions of interval observers for continuous-time linear systems by using extensively time-varying changes of coordinates. However, there are fundamental differences between the main results of [14] and those of the second part of the present paper because it turns out that a continuous-time system $\dot{x} = \mathcal{A}x$ is positive if and only if the matrix $\mathcal{A}$ is cooperative whereas a discrete-time system $x_{k+1} = \mathcal{A}x_k$ is positive if and only if no entry of $\mathcal{A}$ is negative. Consequently, the time-varying changes of coordinates we will use to obtain nonnegative linear systems cannot be deduced from the time-varying changes of coordinates used in [14] to transform a Hurwitz matrix into a Hurwitz and cooperative matrix.

Finally, let us observe that the paper [7] differs from the present paper for the following reasons: (i) The first main result of [7] is devoted to time-invariant linear systems with outputs and relies on an observability condition which guarantees the existence of time-invariant changes of coordinates leading to equations for which interval observers can be readily constructed. (ii) The second main result of [7] is devoted to time-varying linear systems.

The paper is organized as follows. Basic definitions and results are presented in Section 2. In Section 3, we state and prove results of construction of interval observers for nonlinear systems. In Section 4, we state and prove that any time-invariant exponentially stable linear discrete-time system can be transformed into a block diagonal system with nonnegative and exponentially stable subsystems. A construction of time-varying interval observers for linear systems with outputs is proposed in Section 5. Section 6 provides some illustrative examples. Concluding remarks are drawn in Section 7.

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2. CLASSICAL DEFINITIONS AND RESULTS

2.1. *Notation, definitions, basic result*

– The notation will be simplified whenever no confusion can arise from the context.
– Any $k \times n$ matrix, whose entries are all 0 is simply denoted 0.
– All the inequalities must be understood componentwise (partial order of $\mathbb{R}^r$) *i.e.* $v_a = (v_{a1}, ..., v_{ar})^\top \in \mathbb{R}^r$ and $v_b = (v_{b1}, ..., v_{br})^\top \in \mathbb{R}^r$ are such that $v_a \leq v_b$ if and only if, for all $i \in \{1, ..., r\}$, $v_{ai} \leq v_{bi}$.
– $\max(A, B)$ for two matrices $A = (a_{ij}) \in \mathbb{R}^{r\times s}$ and $B = (b_{ij}) \in \mathbb{R}^{r\times s}$ of same dimension is the matrix where each entry is $m_{ij} = \max(a_{ij}, b_{ij})$.
– For a matrix $A \in \mathbb{R}^{r\times s}$, $A^+ = \max(A, 0)$, $A^- = \max(-A, 0)$. Thus, $A = A^+ - A^-$.
– For a real number m, we define $\varpi(m) = \max\{m, 0\}$, $\mho(m) = \min\{m, 0\}$.
– A matrix $A \in \mathbb{R}^{r\times s}$ is said to be *nonnegative* if every entry of A is nonnegative.
– A sequence (u_i) is *nonnegative* if for all integer k, u_k is nonnegative.
– The discrete-time dynamical system (1) is *nonnegative* if for every nonnegative initial condition x_0 and nonnegative sequence (w_k), the corresponding solution x_k is nonnegative for all $k \in \mathbb{N}$.
– diag$\{B_1, ..., B_j\}$ denotes the block diagonal matrix $\begin{bmatrix} B_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_j \end{bmatrix}$.

2.2. *Definition of interval observer*

We introduce a general definition of interval observer for discrete-time nonlinear time-varying systems.

Definition 1
Consider a time-varying system

$$x_{k+1} = f_1(k, x_k, w_k)\,,\ k \in \mathbb{N}, \tag{3}$$

with an output $y_k = m(x_k, w_k) \in \mathbb{R}^p$, with $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^\ell$, and where f_1 and m are two functions. The uncertainties w_k are such that there exists a sequence $\overline{w}_k = (w_k^+, w_k^-) \in \mathbb{R}^{2\ell}$ such that, for all $k \in \mathbb{N}$,

$$w_k^- \leq w_k \leq w_k^+. \tag{4}$$

Moreover, the initial condition at the instant k_0, $x_{k_0} \in \mathbb{R}^n$ is assumed to be bounded by two known bounds:

$$x_{k_0}^- \leq x_{k_0} \leq x_{k_0}^+. \tag{5}$$

Then, the dynamical system

$$z_{k+1} = f_2(k, z_k, y_k, \overline{w}_k)\,,\ k \in \mathbb{N}, \tag{6}$$

associated with the initial condition $z_{k_0} = g(k_0, x_{k_0}^+, x_{k_0}^-) \in \mathbb{R}^{n_z}$ and bounds for the solution x_k for all $k > k_0$:

$$x_k^+ = h^+(k, z_k)\ ,\ x_k^- = h^-(k, z_k), \tag{7}$$

where f_2, g, h^+ and h^- are functions, is called an interval observer for (3) if

(i) any solution (x_k, z_k) of (3)-(6) with $\overline{w}_k = 0$ for all $k \in \mathbb{N}$ is such that $\lim_{k\to+\infty} |h^+(k, z_k) - h^-(k, z_k)| = 0$.
(ii) for any vectors x_{k_0}, $x_{k_0}^-$ and $x_{k_0}^+$ in $\mathbb{R}^n$ satisfying (5), the solutions of (3), (6) with respectively x_{k_0}, z_{k_0} as initial condition at $k = k_0$, denoted respectively x_k and z_k satisfy, for all $k > k_0$, the inequalities

$$x_k^- = h^-(k, z_k) \leq x_k \leq h^+(k, z_k) = x_k^+. \tag{8}$$

2.3. Basic result

The following result, which is a direct consequence of [10, Chapt. 5, Proposition 5.6], is instrumental in establishing our main results.

Lemma 1
The system (2) is nonnegative if and only if the matrix $\mathcal{A}$ is nonnegative and the sequence (w_k) is nonnegative.

3. TIME-INVARIANT INTERVAL OBSERVERS

3.1. Interval observers for nonlinear systems

In this section we construct interval observers for nonlinear systems of the form

$$\begin{array}{rcl} x_{k+1} & = & \mathcal{F}(y_k, x_k, w_k) \ , k \in \mathbb{N}, \\ y_k & = & \mathcal{H}(x_k), \end{array} \tag{9}$$

with $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^\ell$ and where $y_k \in \mathbb{R}^p$ is the output. We introduce an assumption:
Assumption 1. *There exists a function $\mathcal{F}_c : \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}^n$ such that*

$$\mathcal{F}(y, x, w) = \mathcal{F}_c(y, x, x, w, w) \ , \ \forall (y, x, w) \in \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^\ell \tag{10}$$

and, for all fixed y, b, d, the function $(a, c) \to \mathcal{F}_c(y, a, b, c, d)$ is nondecreasing with respect to each of its variables and, for all fixed y, a, c, the function $(b, d) \to \mathcal{F}_c(y, a, b, c, d)$ is nonincreasing with respect to each of its variables. Moreover, for any sequence y_k which converges in norm to zero, all the solutions of the system

$$\left\{ \begin{array}{rcl} a_{k+1} & = & \mathcal{F}_c(y_k, a_k, b_k, 0, 0), \\ b_{k+1} & = & \mathcal{F}_c(y_k, b_k, a_k, 0, 0), \end{array} \right. \tag{11}$$

converge to the origin.

Remark 1. From Lemma 6 in Appendix, one deduces easily that if the function $\mathcal{F}$ in (9) is of class C^1, then there exists a function $\mathcal{F}_c : \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}^n$ such that $\mathcal{F}(y, x, w) = \mathcal{F}_c(y, x, x, w, w)$ which possesses the monotonous properties required in Assumption 1. So the restrictive aspect of Assumption 1 is the stability property of the system (11). In [17], the result of Lemma 6 is proved for the family of the global Lipschitz functions.

We state and prove the following result.

Theorem 1
Assume that the system (9) satisfies Assumption 1. Let the sequence (w_k) be bounded by two known sequences (w_k^+), (w_k^-): for all integer $k \geq 0$,

$$w_k^- \leq w_k \leq w_k^+. \tag{12}$$

Then the system

$$\left\{ \begin{array}{rcl} z_{k+1}^+ & = & \mathcal{F}_c(y_k, z_k^+, z_k^-, w_k^+, w_k^-), \\ z_{k+1}^- & = & \mathcal{F}_c(y_k, z_k^-, z_k^+, w_k^-, w_k^+), \end{array} \right. \tag{13}$$

associated with the initial conditions

$$z_{k_0}^+ = x_{k_0}^+ \ , \ z_{k_0}^- = x_{k_0}^-, \tag{14}$$

and the bounds for the solutions x_k

$$x_k^+ = z_k^+ \ , \ x_k^- = z_k^-, \tag{15}$$

is an interval observer for system (9).

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Remark 2. An extension of Theorem 1 to the case where the system (9) time-varying can be obtained. For the sake of brevity, we omit it.

Proof. Let us consider vectors x_{k_0}, $x^+_{k_0}$, $x^-_{k_0}$, $z^+_{k_0}$, $z^-_{k_0}$ in $\mathbb{R}^n$ such that

$$z^-_{k_0} = x^-_{k_0} \leq x_{k_0} \leq x^+_{k_0} = z^+_{k_0}. \tag{16}$$

Next, let us consider the solutions (x_k), (z^+_k), (z^-_k) of the systems (9), (13) with initial conditions x_{k_0}, $z^+_{k_0}$, $z^-_{k_0}$. Using the equality (10), we obtain, for all integer $k \geq k_0$,

$$\begin{array}{l} z^+_{k+1} - x_{k+1} = \mathcal{F}_c(y_k, z^+_k, z^-_k, w^+_k, w^-_k) - \mathcal{F}_c(y_k, x_k, x_k, w_k, w_k), \\ x_{k+1} - z^-_{k+1} = \mathcal{F}_c(y_k, x_k, x_k, w_k, w_k) - \mathcal{F}_c(y_k, z^-_k, z^+_k, w^-_k, w^+_k). \end{array} \tag{17}$$

Now, we prove by induction that for all $k \geq k_0$, $z^+_k - x_k \geq 0$, $x_k - z^-_k \geq 0$. According to (16), the property is satisfied at the instant k_0. Assume that it is satisfied at the step $k \geq k_0$. Then, the monotonicity properties of $\mathcal{F}_c$ and the fact that for all integer j, $w^+_j - w_j \geq 0$, $w_j - w^-_j \geq 0$, imply that $\mathcal{F}_c(y_k, z^+_k, z^-_k, w^+_k, w^-_k) - \mathcal{F}_c(y_k, x_k, x_k, w_k, w_k) \geq 0$, $\mathcal{F}_c(y_k, x_k, x_k, w_k, w_k) - \mathcal{F}_c(y_k, z^-_k, z^+_k, w^-_k, w^+_k) \geq 0$. It follows that $z^+_{k+1} - x_{k+1} \geq 0$ and $x_{k+1} - z^-_{k+1} \geq 0$. Consequently, the induction assumption is satisfied at the step $k+1$. Finally, we conclude by observing that the global asymptotic stability of the system (11) implies that the solutions of the system (13), when the sequences (w^+_k) and (w^-_k) are identically equal to zero are such that $\lim_{k \to +\infty} |z^+_k - z^-_k| = 0$. ■

We show now that if a system can be transformed through a change of coordinates into a system that satisfies Assumption 1, then again an interval observer can be constructed.

Theorem 2
Consider the system

$$\begin{array}{rcl} s_{k+1} & = & \mathcal{G}(r_k, s_k, w_k)\ , k \in \mathbb{N}, \\ r_k & = & \mathcal{J}(s_k)\ , \end{array} \tag{18}$$

with $s_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^\ell$ and where $r_k \in \mathbb{R}^p$ is the output. Let the sequence (w_k) be bounded by two known sequences (w^+_k), (w^-_k): for all integer $k \geq 0$,

$$w^-_k \leq w_k \leq w^+_k. \tag{19}$$

Assume that there is a diffeomorphism $\theta : \mathbb{R}^n \to \mathbb{R}^n$ such that $\theta(0) = 0$ and the change of coordinates $x_k = \theta(s_k)$ transforms (18) into a system

$$x_{k+1} = \mathcal{F}(y_k, x_k, w_k)\ , k \in \mathbb{N}, \tag{20}$$

with $y_k = \mathcal{J}(\theta^{-1}(x_k))$ and

$$\mathcal{F}(y_k, x_k, w_k) = \theta(\mathcal{G}(y_k, \theta^{-1}(x_k), w_k)),$$

that satisfies Assumption 1. Let $\theta_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\rho_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be two functions such that, for all $x \in \mathbb{R}^n$,

$$\theta(x) = \theta_c(x, x)\ ,\ \theta^{-1}(x) = \rho_c(x, x) \tag{21}$$

and both θ_c and ρ_c are nondecreasing with respect to each of their n first variables and nonincreasing with respect to their n last variables.

Then the system

$$\left\{ \begin{array}{rcl} z^+_{k+1} & = & \mathcal{F}_c(r_k, z^+_k, z^-_k, w^+_k, w^-_k), \\ z^-_{k+1} & = & \mathcal{F}_c(r_k, z^-_k, z^+_k, w^-_k, w^+_k), \end{array} \right. \tag{22}$$

where $\mathcal{F}_c$ is the function provided by Assumption 1, associated with the initial conditions

$$z^+_{k_0} = \theta_c(s^+_{k_0}, s^-_{k_0})\ ,\ z^-_{k_0} = \theta_c(s^-_{k_0}, s^+_{k_0}) \tag{23}$$

and the bounds for the solutions s_k

$$s_k^+ = \rho_c(z_k^+, z_k^-)\ ,\ s_k^- = \rho_c(z_k^-, z_k^+), \tag{24}$$

is an interval observer for the system (18).

Proof. To begin with, we observe that the existence of the functions θ_c, ρ_c satisfying the conditions of Theorem 2 is a consequence of Lemma 6 in Appendix. Now, let us consider the solutions (s_k), (z_k^+), (z_k^-) of the systems (18), (22) with initial conditions $(s_{k_0}, s_{k_0}^+, s_{k_0}^-) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, $(z_{k_0}^+, z_{k_0}^-) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$s_{k_0}^- \leq s_{k_0} \leq s_{k_0}^+ \tag{25}$$

and the equalities (23) are satisfied.

From (25) and the monotonicity properties of θ_c, it follows that the inequalities

$$\theta_c(s_{k_0}^-, s_{k_0}^+) \leq \theta_c(s_{k_0}, s_{k_0}) \leq \theta_c(s_{k_0}^+, s_{k_0}^-) \tag{26}$$

are satisfied. From (23) and (21), it follows that

$$z_{k_0}^- \leq \theta(s_{k_0}) \leq z_{k_0}^+. \tag{27}$$

On the other hand, we know that, for all $k \in \mathbb{N}$,

$$\begin{array}{rcl} z_{k+1}^+ & = & \mathcal{F}_c(r_k, z_k^+, z_k^-, w_k^+, w_k^-), \\ \theta(s_{k+1}) & = & \mathcal{F}(r_k, \theta(s_k), w_k) = \mathcal{F}_c(r_k, \theta(s_k), \theta(s_k), w_k, w_k), \\ z_{k+1}^- & = & \mathcal{F}_c(r_k, z_k^-, z_k^+, w_k^-, w_k^+). \end{array} \tag{28}$$

Arguing as we did to prove Theorem 1, we deduce that, for all integer $k \geq k_0$, the inequalities

$$z_k^- \leq \theta(s_k) \leq z_k^+ \tag{29}$$

are satisfied. From the monotonous properties of ρ_c and the inequalities (29), we deduce that

$$\rho_c(z_k^-, z_k^+) \leq \rho_c(\theta(s_k), \theta(s_k)) \leq \rho_c(z_k^+, z_k^-). \tag{30}$$

Using (21), we obtain that, for all integer $k \geq k_0$,

$$\rho_c(z_k^-, z_k^+) \leq \theta^{-1}(\theta(s_k)) \leq \rho_c(z_k^+, z_k^-). \tag{31}$$

Thus the inequalities

$$s_k^- \leq s_k \leq s_k^+ \tag{32}$$

are satisfied for all integer $k \geq k_0$. Using $\rho_c(0,0) = \theta^{-1}(0) = 0$, we can conclude the proof. ■

3.2. *Interval observers for linear systems*

In this section we analyze the consequences of the results of Section 3.1 when particularized to the family of the linear time-invariant systems.

As an direct consequence of Theorem 1, we have the following result:

Corollary 1

Consider the system

$$x_{k+1} = \mathcal{A}x_k + w_k\ , k \in \mathbb{N}, \tag{33}$$

with $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^n$, where $\mathcal{A} \in \mathbb{R}^{n\times n}$ is a constant matrix. Assume that the matrix

$$\mathcal{A}^* = \begin{bmatrix} \mathcal{A}^+ & -\mathcal{A}^- \\ -\mathcal{A}^- & \mathcal{A}^+ \end{bmatrix} \tag{34}$$

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is Schur stable. Let (w_k) be a sequence bounded by two known sequences (w_k^+), (w_k^-): for all integer $k \geq 0$,

$$w_k^- \leq w_k \leq w_k^+. \tag{35}$$

Then the system

$$\left\{ \begin{array}{lcl} z_{k+1}^+ & = & \mathcal{A}^+ z_k^+ - \mathcal{A}^- z_k^- + w_k^+, \\ z_{k+1}^- & = & \mathcal{A}^+ z_k^- - \mathcal{A}^- z_k^+ + w_k^-, \end{array} \right. \tag{36}$$

associated with the initial conditions

$$z_{k_0}^+ = x_{k_0}^+ \ , \ z_{k_0}^- = x_{k_0}^- \tag{37}$$

and the bounds for the solutions x_k

$$x_k^+ = z_k^+ \ , \ x_k^- = z_k^-, \tag{38}$$

is an interval observer for the system (33).

Proof. We observe that, for all $x \in \mathbb{R}^n$, $\mathcal{A}x = \mathcal{L}(x,x)$ with $\mathcal{L}(a,b) = \mathcal{A}^+ a - \mathcal{A}^- b$. Since the function $\mathcal{L}$ is nondecreasing with respect to the variables a_i and nonincreasing with respect to the variables b_i and since the system (36) is exponentially stable when (w_k^+) and (w_k^-) are identically equal to zero, we can conclude by applying Theorem 1. ■

From Corollary 1, a question arises. If the matrix $\mathcal{A}$ is Schur stable, is the corresponding matrix $\mathcal{A}^*$ necessarily Schur stable ? If the answer to the question was positive, then by Corollary 1 it would be possible to construct interval observers for any exponentially stable linear discrete-time system. Unfortunately, the answer is negative. For instance,

$$\mathcal{A} = \left[\begin{array}{cc} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{array} \right] \tag{39}$$

is a Schur matrix, but the spectral radius of the associated matrix $\mathcal{A}^*$ is larger than 1.

Then, from Theorem 2, another question arises. If a linear time-invariant discrete-time system (33) is Schur stable, is it always possible to apply Theorem 2 with a linear change of coordinates θ ? If the answer was positive, then one might always transform an exponentially stable linear discrete-time system into a system for which an interval observer could be designed. But we conjecture that the answer is negative. The reason why we conjecture this is the following lemma, which establishes that some Schur stable systems cannot be transformed through a linear time-invariant change of coordinates into a system which satisfies the conditions of Corollary 1.

Lemma 2
Let

$$\mathcal{A} = \left[\begin{array}{cc} -\omega & \omega \\ -\omega & -\omega \end{array} \right] \in \mathbb{R}^{2\times 2} \tag{40}$$

with $\omega \in \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$. The spectral radius of $\mathcal{A}$ is $\sqrt{2}\omega < 1$. For any invertible matrix

$$L = \left[\begin{array}{cc} l_{11} & l_{12} \\ l_{21} & l_{22} \end{array} \right] \in \mathbb{R}^{2\times 2}, \tag{41}$$

the matrix

$$\mathcal{A}_L = L\mathcal{A}L^{-1} \tag{42}$$

is such that the spectral radius of the matrix

$$\mathcal{A}_L^* = \left[\begin{array}{cc} \mathcal{A}_L^+ & -\mathcal{A}_L^- \\ -\mathcal{A}_L^- & \mathcal{A}_L^+ \end{array} \right] \in \mathbb{R}^{4\times 4} \tag{43}$$

is larger than 1.

Proof. To begin with, we observe that the eigenvalues of $\mathcal{A}$ are $-\omega + i\omega$ and $-\omega - i\omega$. Therefore the spectral radius of $\mathcal{A}$ is $\sqrt{2}\omega$.

Using the fact that, for all vector $V = (v^\top, -v^\top)^\top \in \mathbb{R}^4$, $\mathcal{A}_L^* V = \begin{pmatrix} (\mathcal{A}_L^+ + \mathcal{A}_L^-)v \\ -(\mathcal{A}_L^+ + \mathcal{A}_L^-)v \end{pmatrix}$, we deduce that the spectral radius of $\mathcal{A}_L^*$ is not smaller than the spectral radius of the matrix $\mathcal{A}_L^P = \mathcal{A}_L^+ + \mathcal{A}_L^-$. Therefore if we can prove that the spectral radius of $\mathcal{A}_L^P$ is larger than 1, the proof is completed. To analyze the spectral radius of $\mathcal{A}_L^P$, we observe that simple calculations lead to the equality

$$\mathcal{A}_L = \omega \begin{bmatrix} \frac{-l_{12}l_{22}-l_{11}l_{21}}{l_{11}l_{22}-l_{12}l_{21}} - 1 & \frac{l_{12}^2+l_{11}^2}{l_{11}l_{22}-l_{12}l_{21}} \\ \frac{-l_{22}^2-l_{21}^2}{l_{11}l_{22}-l_{12}l_{21}} & \frac{l_{22}l_{12}+l_{11}l_{21}}{l_{11}l_{22}-l_{12}l_{21}} - 1 \end{bmatrix}. \tag{44}$$

Consequently,

$$\mathcal{A}_L^P = \omega \begin{bmatrix} \left|\frac{l_{12}l_{22}+l_{11}l_{21}}{l_{11}l_{22}-l_{12}l_{21}} + 1\right| & \left|\frac{l_{12}^2+l_{11}^2}{l_{11}l_{22}-l_{12}l_{21}}\right| \\ \left|\frac{l_{22}^2+l_{21}^2}{l_{11}l_{22}-l_{12}l_{21}}\right| & \left|\frac{l_{22}l_{12}+l_{11}l_{21}}{l_{11}l_{22}-l_{12}l_{21}} - 1\right| \end{bmatrix}. \tag{45}$$

One can prove easily that for all real number m, the inequality $2 \le |1+m| + |1-m|$ is satisfied. We deduce that $\mathrm{tr}(\mathcal{A}_L^P) \ge 2\omega > 1$. On the other hand,

$$\begin{aligned} \det \mathcal{A}_L^P &= \omega^2 \left(\left| \left(\frac{l_{12}l_{22}+l_{11}l_{21}}{l_{11}l_{22}-l_{12}l_{21}}\right)^2 - 1 \right| - \frac{(l_{22}^2+l_{21}^2)(l_{12}^2+l_{11}^2)}{(l_{11}l_{22}-l_{12}l_{21})^2} \right) \\ &= \frac{\omega^2\left[\left|(l_{12}l_{22}+l_{11}l_{21})^2-(l_{11}l_{22}-l_{12}l_{21})^2\right| - (l_{22}^2+l_{21}^2)(l_{12}^2+l_{11}^2)\right]}{(l_{11}l_{22}-l_{12}l_{21})^2} \\ &= \frac{\omega^2\left[\left|(l_{12}l_{22}+l_{11}l_{21})^2-(l_{11}l_{22}-l_{12}l_{21})^2\right| - [l_{22}^2l_{12}^2+l_{21}^2l_{12}^2+l_{22}^2l_{11}^2+l_{21}^2l_{11}^2]\right]}{(l_{11}l_{22}-l_{12}l_{21})^2}. \end{aligned} \tag{46}$$

Now, observing that

$$(l_{12}l_{22} + l_{11}l_{21})^2 = l_{12}^2l_{22}^2 + l_{11}^2l_{21}^2 + 2l_{12}l_{22}l_{11}l_{21} \le l_{12}^2l_{22}^2 + l_{11}^2l_{21}^2 + l_{12}^2l_{21}^2 + l_{22}^2l_{11}^2$$

and

$$(l_{11}l_{22} - l_{12}l_{21})^2 = l_{11}^2l_{22}^2 + l_{12}^2l_{21}^2 - 2l_{11}l_{22}l_{12}l_{21} \le l_{12}^2l_{22}^2 + l_{11}^2l_{21}^2 + l_{12}^2l_{21}^2 + l_{22}^2l_{11}^2.$$

We deduce that

$$\left|(l_{12}l_{22} + l_{11}l_{21})^2 - (l_{11}l_{22} - l_{12}l_{21})^2\right| \le l_{12}^2l_{22}^2 + l_{11}^2l_{21}^2 + l_{12}^2l_{21}^2 + l_{22}^2l_{11}^2,$$

which implies that $\det \mathcal{A}_L^P \le 0$. This inequality in combination with the inequality $\mathrm{tr}(\mathcal{A}_L^P) > 1$ implies that at least one of the eigenvalues of $\mathcal{A}_L^P$ is a real number larger than 1.

■

4. TRANSFORMATIONS OF LINEAR SYSTEMS INTO NONNEGATIVE SYSTEMS

The previous section motivates the main results of the present and the next section. In this section, we establish that any discrete-time exponentially stable time-invariant linear system can be transformed into a nonnegative and exponentially stable time-invariant system through a linear time-varying change of coordinates. In Section 5, we will use this result to construct interval observers for linear systems.

Theorem 3
Consider the system

$$x_{k+1} = \mathcal{A}x_k \, , k \in \mathbb{N}, \tag{47}$$

with $x_k \in \mathbb{R}^n$, where $\mathcal{A} \in \mathbb{R}^{n\times n}$ is a constant Schur stable matrix. Then there exists a time-varying change of coordinates $h_k = \mathcal{R}_k x_k$, where $(\mathcal{R}_k)$ is a sequence of invertible matrices in $\mathbb{R}^{n\times n}$ such that there exists a constant $c > 0$ such that for all $k \in \mathbb{N}$, $|\mathcal{R}_k| + |\mathcal{R}_k^{-1}| \le c$, which transforms (47) into a positive and exponentially stable linear system.

Proof. The proof splits up into two steps. Firstly, we recall that any real matrix admits a real Jordan canonical form. Secondly, we transform systems in Jordan canonical form into positive systems.
Step 1: Jordan canonical forms.

From [19, Section 1.8], we deduce that for some integers $r \in \{0, 1, ..., n\}$, $s \in \{0, 1, ..., n-1\}$ there exists a linear time-invariant change of coordinates

$$g_k = \mathcal{P} x_k, \tag{48}$$

which transforms (47) into

$$g_{k+1} = \overline{\mathcal{J}} g_k, \tag{49}$$

with

$$\overline{\mathcal{J}} = \text{diag}\{\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_s\} \in \mathbb{R}^{n \times n}, \tag{50}$$

where the matrices $\mathcal{J}_i$ are partitioned into two groups: the first r matrices are associated with the r real eigenvalues of multiplicity n_i of $\mathcal{A}$ and the others are associated with the imaginary eigenvalues of multiplicity m_i of $\mathcal{A}$. Therefore $n = \sum_{i=1}^{r} n_i + \sum_{r+1}^{s} 2m_i$ and, for $i = 1$ to r,

$$\mathcal{J}_i = \begin{bmatrix} -\mu_i & 1 & \dots & 0 \\ 0 & -\mu_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -\mu_i \end{bmatrix} \in \mathbb{R}^{n_i \times n_i}, \tag{51}$$

where the μ_i's are real numbers and, for $i = r + 1$ to s,

$$\mathcal{J}_i = \begin{bmatrix} \mathcal{M}_i & I_2 & 0 & \dots & 0 \\ 0 & \mathcal{M}_i & I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I_2 \\ 0 & \dots & \dots & 0 & \mathcal{M}_i \end{bmatrix} \in \mathbb{R}^{2m_i \times 2m_i}, \tag{52}$$

with

$$\mathcal{M}_i = \begin{bmatrix} -\kappa_i & \omega_i \\ -\omega_i & -\kappa_i \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{53}$$

and

$$I_2 = \text{diag}\{1, 1\} \in \mathbb{R}^{2\times 2}, \tag{54}$$

where the ω_i's are non-zero real numbers.
Step 2: time-varying change of coordinates. We consider the system (49). From Lemmas 4 and 5, we deduce that, for any system

$$a_{k+1} = \mathcal{J}_i a_k, \tag{55}$$

there exist a nonnegative matrix $\mathcal{H}_i$ with a spectral radius smaller than 1 and a sequence $(\mathcal{Q}_{k,i})$ of invertible matrices bounded in norm by 1 and with inverses bounded in norm by 1 such that the change of coordinates

$$b_k = \mathcal{Q}_{k,i} a_k \tag{56}$$

gives

$$b_{k+1} = \mathcal{H}_i b_k. \tag{57}$$

Next, we consider the change of coordinates

$$h_k = \overline{\mathcal{Q}}_k g_k, \tag{58}$$

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with

$$\overline{\mathcal{Q}}_k = \text{diag}\{\mathcal{Q}_{k,1}, \mathcal{Q}_{k,2}, ..., \mathcal{Q}_{k,s}\} \in \mathbb{R}^{n\times n}. \tag{59}$$

Then

$$h_{k+1} = \overline{\mathcal{Q}}_{k+1}\overline{\mathcal{J}}g_k = \overline{\mathcal{Q}}_{k+1}\overline{\mathcal{J}}\,\overline{\mathcal{Q}}_k^{-1}h_k = \overline{\mathcal{H}}h_k \tag{60}$$

with

$$\overline{\mathcal{H}} = \text{diag}\{\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_s\} \in \mathbb{R}^{n\times n}.$$

Finally, we conclude by observing that the change of coordinates

$$h_k = \mathcal{R}_k x_k, \tag{61}$$

with $\mathcal{R}_k = \overline{\mathcal{Q}}_k\mathcal{P}$ gives the nonnegative exponentially stable time-invariant system

$$h_{k+1} = \overline{\mathcal{H}}h_k$$

and that $\mathcal{R}_k^{-1} = \mathcal{P}^{-1}\overline{\mathcal{Q}}_k^{-1}$, which implies that the sequences $(\mathcal{R}_k)$ and $(\mathcal{R}_k^{-1})$ are bounded in norm. ■

5. TIME-VARYING INTERVAL OBSERVERS

The result of this section shows how Theorem 3 can be used when it comes to constructing interval observers for linear systems with output.

Theorem 4
Consider the system

$$\begin{array}{rcl} x_{k+1} & = & \alpha x_k + w_k \,, k \in \mathbb{N}, \\ y_k & = & Cx_k + v_k, \end{array} \tag{62}$$

with $x_k \in \mathbb{R}^n$, the output $y_k \in \mathbb{R}^q$, $w_k \in \mathbb{R}^n$, $v_k \in \mathbb{R}^q$, where $\alpha \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{q\times n}$ are matrices such that there exists $K \in \mathbb{R}^{n\times q}$ such that the matrix $\mathcal{A} = \alpha + KC$ is Schur stable. Let the sequence (w_k) and (v_k) be bounded by known sequences (w_k^+), (w_k^-), (v_k^+), (v_k^-): for all integer $k \geq 0$,

$$w_k^- \leq w_k \leq w_k^+ \;,\; v_k^- \leq v_k \leq v_k^+. \tag{63}$$

Then there exist a sequence of invertible real matrices $(\mathcal{R}_k)$ and a real number $c > 0$ such that for all $k \in \mathbb{N}$, $|\mathcal{R}_k| + |\mathcal{R}_k^{-1}| \leq c$ and

$$\mathcal{R}_{k+1}\mathcal{A}\mathcal{R}_k^{-1} = \mathcal{E}, \tag{64}$$

where $\mathcal{E} \in \mathbb{R}^{n\times n}$ is a nonnegative Schur stable matrix. Let $\mathcal{S}_k = \mathcal{R}_k^{-1}$ for all $k \in \mathbb{N}$. Then the system

$$\left\{\begin{array}{l} z_{k+1}^+ = \mathcal{E}z_k^+ - \mathcal{R}_{k+1}Ky_k + \mathcal{R}_{k+1}^+w_k^+ - \mathcal{R}_{k+1}^-w_k^- + (\mathcal{R}_{k+1}K)^+v_k^+ - (\mathcal{R}_{k+1}K)^-v_k^-, \\ z_{k+1}^- = \mathcal{E}z_k^- - \mathcal{R}_{k+1}Ky_k + \mathcal{R}_{k+1}^+w_k^- - \mathcal{R}_{k+1}^-w_k^+ + (\mathcal{R}_{k+1}K)^+v_k^- - (\mathcal{R}_{k+1}K)^-v_k^+, \end{array}\right. \tag{65}$$

associated with the initial conditions

$$z_{k_0}^+ = \mathcal{R}_{k_0}^+x_{k_0}^+ - \mathcal{R}_{k_0}^-x_{k_0}^- \;,\; z_{k_0}^- = \mathcal{R}_{k_0}^+x_{k_0}^- - \mathcal{R}_{k_0}^-x_{k_0}^+ \tag{66}$$

and the bounds

$$x_k^+ = \mathcal{S}_k^+z_k^+ - \mathcal{S}_k^-z_k^- \;,\; x_k^- = \mathcal{S}_k^+z_k^- - \mathcal{S}_k^-z_k^+ \tag{67}$$

is an interval observer for system (62).

Proof. The sequence $(\mathcal{R}_k)$ and the matrix $\mathcal{E}$ are provided by Theorem 3. Next, we consider vectors $x_{k_0}, x_{k_0}^+, x_{k_0}^-, z_{k_0}^+, z_{k_0}^-$ in $\mathbb{R}^n$ such that

$$x_{k_0}^- \leq x_{k_0} \leq x_{k_0}^+ \tag{68}$$

and

$$z_{k_0}^{+} = \mathcal{R}_{k_0}^{+}x_{k_0}^{+} - \mathcal{R}_{k_0}^{-}x_{k_0}^{-} \ , \ z_{k_0}^{-} = -\mathcal{R}_{k_0}^{-}x_{k_0}^{+} + \mathcal{R}_{k_0}^{+}x_{k_0}^{-}. \tag{69}$$

Since the matrices $\mathcal{R}_k^+$ and $\mathcal{R}_k^-$ are nonnegative matrices, it follows that

$$\mathcal{R}_{k_0}^{+}x_{k_0}^{-} \leq \mathcal{R}_{k_0}^{+}x_{k_0} \leq \mathcal{R}_{k_0}^{+}x_{k_0}^{+} \ , \ \mathcal{R}_{k_0}^{-}x_{k_0}^{-} \leq \mathcal{R}_{k_0}^{-}x_{k_0} \leq \mathcal{R}_{k_0}^{-}x_{k_0}^{+}.$$

We deduce that

$$-\mathcal{R}_{k_0}^{-}x_{k_0}^{+} + \mathcal{R}_{k_0}^{+}x_{k_0}^{-} \leq (\mathcal{R}_{k_0}^{+} - \mathcal{R}_{k_0}^{-})x_{k_0} \leq \mathcal{R}_{k_0}^{+}x_{k_0}^{+} - \mathcal{R}_{k_0}^{-}x_{k_0}^{-}.$$

These inequalities rewrite

$$z_{k_0}^{-} \leq \mathcal{R}_k x_{k_0} \leq z_{k_0}^{+}. \tag{70}$$

Next, let us consider the solutions (x_k), (z_k^+), (z_k^-) of the systems

$$\left\{\begin{array}{lcl} x_{k+1} & = & \mathcal{A}x_k - Ky_k + w_k + Kv_k, \\ z_{k+1}^{+} & = & \mathcal{E}z_k^{+} - \mathcal{R}_{k+1}Ky_k + \mathcal{R}_{k+1}^{+}w_k^{+} - \mathcal{R}_{k+1}^{-}w_k^{-} + (\mathcal{R}_{k+1}K)^{+}v_k^{+} \\ & & -(\mathcal{R}_{k+1}K)^{-}v_k^{-}, \\ z_{k+1}^{-} & = & \mathcal{E}z_k^{-} - \mathcal{R}_{k+1}Ky_k + \mathcal{R}_{k+1}^{+}w_k^{-} - \mathcal{R}_{k+1}^{-}w_k^{+} + (\mathcal{R}_{k+1}K)^{+}v_k^{-} \\ & & -(\mathcal{R}_{k+1}K)^{-}v_k^{+}, \end{array}\right. \tag{71}$$

with initial conditions x_{k_0}, $z_{k_0}^{+}$, $z_{k_0}^{-}$ such that (68) and (69) are satisfied. Then

$$\begin{array}{lcl} \mathcal{R}_{k+1}x_{k+1} & = & \mathcal{R}_{k+1}[\mathcal{A}x_k - Ky_k + w_k + Kv_k] \\ & = & \mathcal{E}\mathcal{R}_k x_k - \mathcal{R}_{k+1}Ky_k + \mathcal{R}_{k+1}w_k + \mathcal{R}_{k+1}Kv_k. \end{array}$$

It follows that

$$\begin{array}{lcl} z_{k+1}^{+} - \mathcal{R}_{k+1}x_{k+1} & = & \mathcal{E}[z_k^{+} - \mathcal{R}_k x_k] + \mathcal{R}_{k+1}^{+}w_k^{+} - \mathcal{R}_{k+1}^{-}w_k^{-} - (\mathcal{R}_{k+1}^{+} - \mathcal{R}_{k+1}^{-})w_k \\ & & +(\mathcal{R}_{k+1}K)^{+}v_k^{+} - (\mathcal{R}_{k+1}K)^{-}v_k^{-} - ((\mathcal{R}_{k+1}K)^{+} - (\mathcal{R}_{k+1}K)^{+})v_k \\ & = & \mathcal{E}[z_k^{+} - \mathcal{R}_k x_k] + p_k, \end{array}$$

with

$$p_k = \mathcal{R}_{k+1}^{+}(w_k^{+} - w_k) + \mathcal{R}_{k+1}^{-}(w_k - w_k^{-}) + (\mathcal{R}_{k+1}K)^{+}(v_k^{+} - v_k) + (\mathcal{R}_{k+1}K)^{-}(v_k - v_k^{-})$$

and, similarly,

$$\mathcal{R}_{k+1}x_{k+1} - z_{k+1}^{-} \quad = \quad \mathcal{E}[\mathcal{R}_k x_k - z_k^{-}] + q_k,$$

with

$$q_k = \mathcal{R}_{k+1}^{+}(w_k - w_k^{-}) + \mathcal{R}_{k+1}^{-}(w_k^{+} - w_k) + (\mathcal{R}_{k+1}K)^{+}(v_k - v_k^{-}) + (\mathcal{R}_{k+1}K)^{-}(v_k^{+} - v_k).$$

Since the matrix $\mathcal{E}$ is nonnegative, for all integer k, $p_k \geq 0$, $q_k \geq 0$ and according to (70), $\mathcal{R}_{k_0}x_{k_0} - z_{k_0}^{-} \geq 0$, $z_{k_0}^{+} - \mathcal{R}_{k_0}x_{k_0} \geq 0$, we deduce from Lemma 1 that, for all $k \geq k_0$,

$$0 \leq z_k^{+} - \mathcal{R}_k x_k \ , \ 0 \leq \mathcal{R}_k x_k - z_k^{-}.$$

Therefore, for all $k \geq k_0$,

$$\begin{array}{c} \mathcal{S}_k^{+}z_k^{-} \leq \mathcal{S}_k^{+}\mathcal{R}_k x_k \leq \mathcal{S}_k^{+}z_k^{+}, \\ \mathcal{S}_k^{-}z_k^{-} \leq \mathcal{S}_k^{-}\mathcal{R}_k x_k \leq \mathcal{S}_k^{-}z_k^{+}. \end{array}$$

It follows readily that, for all $k \geq k_0$,

$$\mathcal{S}_k^{+}z_k^{-} - \mathcal{S}_k^{-}z_k^{+} \leq \mathcal{S}_k^{+}\mathcal{R}_k x_k - \mathcal{S}_k^{-}\mathcal{R}_k x_k \leq \mathcal{S}_k^{+}z_k^{+} - \mathcal{S}_k^{-}z_k^{-}.$$

Since $\mathcal{R}_k^{-1} = \mathcal{S}_k = \mathcal{S}_k^{+} - \mathcal{S}_k^{-}$, we deduce that, for all $k \geq k_0$,

$$\mathcal{S}_k^{+}z_k^{-} - \mathcal{S}_k^{-}z_k^{+} \leq x_k \leq \mathcal{S}_k^{+}z_k^{+} - \mathcal{S}_k^{-}z_k^{-}.$$

Finally, we deduce from (71) that, in the absence of w_k^{+} and w_k^{-},

$$z_{k+1}^{+} - z_{k+1}^{-} = \mathcal{E}(z_k^{+} - z_k^{-}).$$

Since $\mathcal{E}$ is Schur stable, we can conclude. ■

6. ILLUSTRATIVE EXAMPLES

In this section, we illustrate Theorem 1 and Theorem 4 through two examples.

6.1. Nonlinear system

We apply Theorem 1 to the nonlinear system without output

$$x_{k+1} = \frac{1}{4}\sin(x_k) + x_k w_k \ , k \in \mathbb{N}, \tag{72}$$

with $x_k \in \mathbb{R}$, where $w_k \in \mathbb{R}$ represents disturbances. Let $\mathcal{F}(x,w) = \frac{1}{4}\sin(x) + xw$.

To verify the conditions in Assumption 1, we select the function $\mathcal{F}_c : \mathbb{R}^4 \to \mathbb{R}$

$$\mathcal{F}_c(a,b,c,d) = \frac{1}{4}a + \frac{1}{4}(\sin(b) - b), + \varpi(a)\varpi(c) + \varpi(b)\mho(c) + \mho(a)\varpi(d) + \mho(b)\mho(d), \tag{73}$$

with the functions ϖ and $\mho$ defined in the Section 2.1, which is such that

$$\mathcal{F}(x,w) = \mathcal{F}_c(x,x,w,w), \forall x \in \mathbb{R} \ , \ w \in \mathbb{R}$$

and is nondecreasing with respect to the variables a, c and nonincreasing with respect to the variables b, d. Next, we analyze the stability properties of the system

$$\left\{ \begin{array}{rcl} a_{k+1} & = & \frac{1}{4}a_k + \frac{1}{4}(\sin(b_k) - b_k), \\ b_{k+1} & = & \frac{1}{4}b_k + \frac{1}{4}(\sin(a_k) - a_k), \end{array} \right. \tag{74}$$

by using the positive definite quadratic function

$$V(a,b) = a^2 + b^2 \ .$$

From

$$V(a_{k+1}, b_{k+1}) = \left[\frac{1}{4}a_k + \frac{1}{4}(\sin(b_k) - b_k)\right]^2 + \left[\frac{1}{4}b_k + \frac{1}{4}(\sin(a_k) - a_k)\right]^2, \tag{75}$$

we deduce that

$$\begin{array}{rcl} V(a_{k+1}, b_{k+1}) & \leq & \frac{1}{8}\left[a_k^2 + (\sin(b_k) - b_k)^2\right] + \frac{1}{8}\left[b_k^2 + (\sin(a_k) - a_k)^2\right] \\ & \leq & \frac{1}{8}\left[a_k^2 + 4b_k^2\right] + \frac{1}{8}\left[b_k^2 + 4a_k^2\right] \\ & \leq & \frac{5}{8}V(a_k, b_k) . \end{array} \tag{76}$$

Since V is a positive definite quadratic function, we conclude that the system (74) admits the origin as a globally exponentially stable equilibrium point.

According to Theorem 1, system (72) admits the following interval observer:

$$\begin{array}{rcl} z_{k+1}^+ & = & \frac{1}{4}z_k^+ + \frac{1}{4}(\sin(z_k^-) - z_k^-) + \varpi(z_k^+)\varpi(w_k^+) + \varpi(z_k^-)\mho(w_k^+) + \mho(z_k^+)\varpi(w_k^-) \\ & & + \mho(z_k^-)\mho(w_k^-) \ , \\ z_{k+1}^- & = & \frac{1}{4}z_k^- + \frac{1}{4}(\sin(z_k^+) - z_k^+) + \varpi(z_k^-)\varpi(w_k^-) + \varpi(z_k^+)\mho(w_k^-) + \mho(z_k^-)\varpi(w_k^+) \\ & & + \mho(z_k^+)\mho(w_k^+) \ , \\ z_{k_0}^+ & = & x_{k_0}^+ \ , \ z_{k_0}^- \ = \ x_{k_0}^- \ , \\ x_k^+ & = & z_k^+ \ , \ x_k^- \ = \ z_k^- \ . \end{array} \tag{77}$$

We present the following simulations.

Figure 1 gives the solution x and the bounds provided by the interval observer in the absence of disturbances.

Figure 2 gives the solution x and the bounds provided by the interval observer in the presence of disturbances:

$$w_k = \frac{4}{3}\cos(k)^2,$$

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with the bounds

$$w_k^- = \frac{2}{9}\cos(k)^2,\ w_k^+ = \frac{1}{3}[2\cos(k)^2 + 1].$$

In both cases the selected initial conditions are

$$x(1) = 1,\ z^+(1) = \frac{5}{2},\ z^-(1) = \frac{3}{10}.$$

The simulations confirm the mathematical result: in the absence of disturbance, there is convergence of the two bounds of the solution and in the presence of disturbance, the bounds do not necessarily converge to the same value.

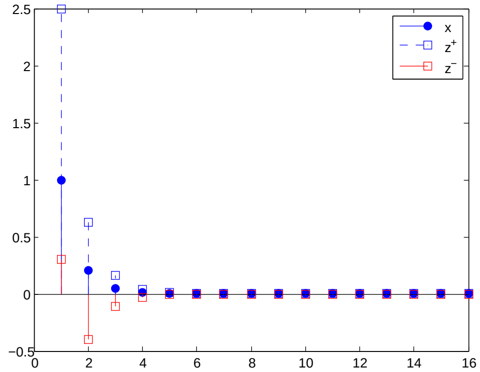

Figure 1. Solution in the absence of disturbances.

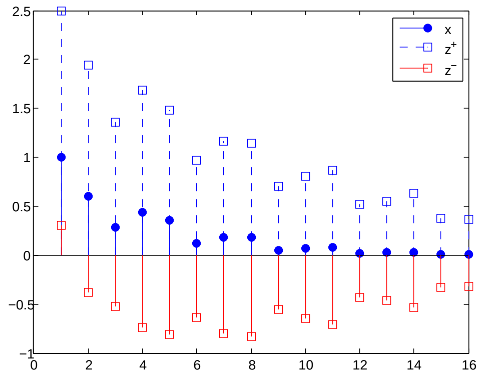

Figure 2. Solution in the presence of disturbances.

6.2. *Linear system*

We illustrate the construction presented in Section 5 by constructing step by step a time-varying interval observer for the linear system without output

$$x_{k+1} = \mathcal{A}x_k + w_k\ , k \in \mathbb{N}\ , \tag{78}$$

with

$$\mathcal{A} = \frac{1}{2}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}. \tag{79}$$

Since the eigenvalues of $\mathcal{A}$ are $-\frac{1}{2}$ and $-\frac{1}{2} \pm \frac{i}{2}$, the spectral radius of $\mathcal{A}$ is $\frac{\sqrt{2}}{2} < 1$. Therefore Theorem 4 applies to (78). We follow now the proof of this theorem.
Step 1 : Transformation of $\mathcal{A}$ into a matrix of Jordan form.

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The equality

$$\mathcal{P}\mathcal{A}\mathcal{P}^{-1} = \mathcal{J}\ , \tag{80}$$

with

$$\mathcal{J} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad \mathcal{P} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{81}$$

is satisfied. The Jordan matrix $\mathcal{J}$ is not nonnegative.

Step 2 : Construction of a diagonalizing time-varying change of coordinates.

For the particular case we consider, the matrix $\overline{\mathcal{Q}}_k$ in (59) is

$$\overline{\mathcal{Q}}_k = \begin{bmatrix} (-1)^k & 0 & 0 \\ 0 & \cos\left(\frac{5\pi}{4}k\right) & \sin\left(\frac{5\pi}{4}k\right) \\ 0 & -\sin\left(\frac{5\pi}{4}k\right) & \cos\left(\frac{5\pi}{4}k\right) \end{bmatrix}, \tag{82}$$

the matrix $\mathcal{R}_k = \overline{\mathcal{Q}}_k\mathcal{P}$ is

$$\mathcal{R}_k = \begin{bmatrix} (-1)^k & 0 & 0 \\ -\cos\left(\frac{5\pi}{4}k\right) & \cos\left(\frac{5\pi}{4}k\right) & \sin\left(\frac{5\pi}{4}k\right) \\ \sin\left(\frac{5\pi}{4}k\right) & -\sin\left(\frac{5\pi}{4}k\right) & \cos\left(\frac{5\pi}{4}k\right) \end{bmatrix} \tag{83}$$

and

$$\mathcal{S}_k = \mathcal{R}_k^{-1} = \begin{bmatrix} (-1)^k & 0 & 0 \\ (-1)^k & \cos\left(\frac{5\pi}{4}k\right) & -\sin\left(\frac{5\pi}{4}k\right) \\ 0 & \sin\left(\frac{5\pi}{4}k\right) & \cos\left(\frac{5\pi}{4}k\right) \end{bmatrix}. \tag{84}$$

One can check readily that, for all $k \in \mathbb{N}$,

$$|\mathcal{R}_k| + |\mathcal{S}_k| < 6 \tag{85}$$

and

$$\mathcal{E} = \mathcal{R}_{k+1}\mathcal{A}\mathcal{R}_k^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}. \tag{86}$$

The matrix $\mathcal{E}$ is a nonnegative matrix whose spectral radius is smaller than 1.

Step 3 : Interval observer.

According to Theorem 4, the system (78) admits the following interval observer

$$\begin{array}{rcl} z_{k+1}^+ & = & \mathcal{E}z_k^+ + \mathcal{R}_{k+1}^+ w_k^+ - \mathcal{R}_{k+1}^- w_k^-\ , \\ z_{k+1}^- & = & \mathcal{E}z_k^- + \mathcal{R}_{k+1}^+ w_k^- - \mathcal{R}_{k+1}^- w_k^+\ , \\ z_{k_0}^+ & = & \mathcal{R}_{k_0}^+ x_{k_0}^+ - \mathcal{R}_{k_0}^- x_{k_0}^-\ , \\ z_{k_0}^- & = & \mathcal{R}_{k_0}^+ x_{k_0}^- - \mathcal{R}_{k_0}^- x_{k_0}^+\ , \\ x_k^+ & = & \mathcal{S}_k^+ z_k^+ - \mathcal{S}_k^- z_k^-\ , \\ x_k^- & = & \mathcal{S}_k^+ z_k^- - \mathcal{S}_k^- z_k^+\ , \end{array} \tag{87}$$

with $\mathcal{E}$ defined in (86),

$$\mathcal{R}_k^+ = \begin{bmatrix} \varpi\left((-1)^k\right) & 0 & 0 \\ \varpi\left(-\cos\left(\frac{5\pi}{4}k\right)\right) & \varpi\left(\cos\left(\frac{5\pi}{4}k\right)\right) & \varpi\left(\sin\left(\frac{5\pi}{4}k\right)\right) \\ \varpi\left(\sin\left(\frac{5\pi}{4}k\right)\right) & \varpi\left(-\sin\left(\frac{5\pi}{4}k\right)\right) & \varpi\left(\cos\left(\frac{5\pi}{4}k\right)\right) \end{bmatrix}, \tag{88}$$

$$\mathcal{S}_k^+ = \begin{bmatrix} \varpi\left((-1)^k\right) & 0 & 0 \\ \varpi\left((-1)^k\right) & \varpi\left(\cos\left(\frac{5\pi}{4}k\right)\right) & \varpi\left(-\sin\left(\frac{5\pi}{4}k\right)\right) \\ 0 & \varpi\left(\sin\left(\frac{5\pi}{4}k\right)\right) & \varpi\left(\cos\left(\frac{5\pi}{4}k\right)\right) \end{bmatrix}, \tag{89}$$

$$\mathcal{R}_k^- = -\begin{bmatrix} \mho\left((-1)^k\right) & 0 & 0 \\ \mho\left(-\cos\left(\frac{5\pi}{4}k\right)\right) & \mho\left(\cos\left(\frac{5\pi}{4}k\right)\right) & \mho\left(\sin\left(\frac{5\pi}{4}k\right)\right) \\ \mho\left(\sin\left(\frac{5\pi}{4}k\right)\right) & \mho\left(-\sin\left(\frac{5\pi}{4}k\right)\right) & \mho\left(\cos\left(\frac{5\pi}{4}k\right)\right) \end{bmatrix}, \tag{90}$$

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and

$$\mathcal{S}_k^- = -\begin{bmatrix} \mho\left((-1)^k\right) & 0 & 0 \\ \mho\left((-1)^k\right) & \mho\left(\cos\left(\frac{5\pi}{4}k\right)\right) & \mho\left(-\sin\left(\frac{5\pi}{4}k\right)\right) \\ 0 & \mho\left(\sin\left(\frac{5\pi}{4}k\right)\right) & \mho\left(\cos\left(\frac{5\pi}{4}k\right)\right) \end{bmatrix}, \tag{91}$$

We present simulations.

Figure 1 shows the second component x and the bounds provided by the interval observer in the absence of disturbances.

Figure 2 shows the second component x and the bounds provided by the interval observer in the presence of disturbances:

$$w_k = \begin{pmatrix} 0 \\ 2\sin(3k) \\ 0 \end{pmatrix},$$

with the bounds

$$w_k^+ = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \; w_k^- = \begin{pmatrix} 0 \\ 2\sin(3k) - 1 \\ 0 \end{pmatrix}.$$

In both cases the selected initial conditions are

$$x(1) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \; x^+(1) = \begin{pmatrix} 1.5 \\ 6 \\ 4 \end{pmatrix}, \; x^-(1) = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}.$$

The behavior of the other components of the solutions is similar. The simulations confirm the theoretic result: in the absence of disturbance, there is convergence of the two bounds of the solution and in the presence of disturbance, the bounds do not necessarily converge to the same value.

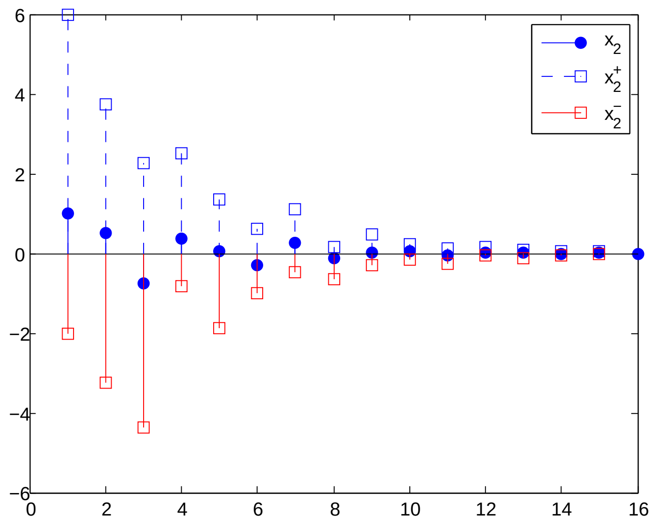

Figure 3. Solution in the absence of disturbances.

7. CONCLUSION

We have proposed a technique of construction of time-invariant interval observers for a family of nonlinear discrete-time time-invariant systems and a technique of construction of time-varying

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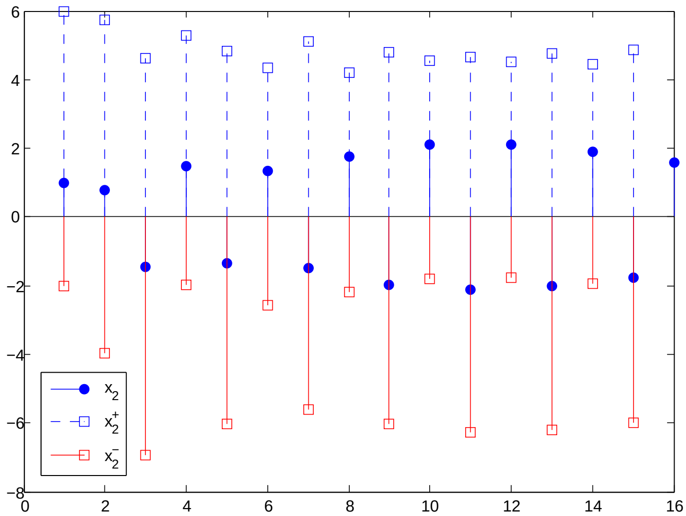

Figure 4. Solution in the presence of disturbances.

interval observers for discrete-time linear time-invariant systems. Much remains to be done. A discrete-time version of the contribution [16], which is devoted to linear systems with delays, can be expected. Extensions of the results of the present work to families of discrete-time nonlinear systems with delays can be expected too.

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8. APPENDIX

Lemma 3
Let

$$\mathcal{M} = \begin{bmatrix} -\kappa & \omega \\ -\omega & -\kappa \end{bmatrix} \tag{92}$$

where κ and ω are two real numbers such that $\kappa^2 + \omega^2 > 0$. Let α be any real number such that

$$\sin(\alpha) = -\tfrac{\omega}{\sqrt{\kappa^2+\omega^2}} \ , \ \cos(\alpha) = -\tfrac{\kappa}{\sqrt{\kappa^2+\omega^2}} \tag{93}$$

and let, for all $j \in \mathbb{N}$,

$$\mathcal{L}_j = \begin{bmatrix} \cos(\alpha j) & \sin(\alpha j) \\ -\sin(\alpha j) & \cos(\alpha j) \end{bmatrix} . \tag{94}$$

Then, for all $k \in \mathbb{N}$, the equality

$$\mathcal{L}_{k+1} \mathcal{M} \mathcal{L}_k^{-1} = \sqrt{\kappa^2 + \omega^2} I_2 \tag{95}$$

is satisfied.

Proof. The equalities

$$\begin{array}{rcl} \sin(\alpha(k+1)) & = & \sin(\alpha k)\cos(\alpha) + \cos(\alpha k)\sin(\alpha) \ , \\ \cos(\alpha(k+1)) & = & \cos(\alpha k)\cos(\alpha) - \sin(\alpha k)\sin(\alpha) \ , \end{array} \tag{96}$$

imply that

$$\begin{array}{rcl} \mathcal{L}_{k+1} & = & \frac{1}{\sqrt{\kappa^2+\omega^2}} \begin{bmatrix} -\kappa\cos(\alpha k) + \omega\sin(\alpha k) & -\kappa\sin(\alpha k) - \omega\cos(\alpha k) \\ \kappa\sin(\alpha k) + \omega\cos(\alpha k) & -\kappa\cos(\alpha k) + \omega\sin(\alpha k) \end{bmatrix} \\ & = & \frac{1}{\sqrt{\kappa^2+\omega^2}} \begin{bmatrix} \cos(\alpha k) & \sin(\alpha k) \\ -\sin(\alpha k) & \cos(\alpha k) \end{bmatrix} \begin{bmatrix} -\kappa & -\omega \\ \omega & -\kappa \end{bmatrix} . \end{array} \tag{97}$$

Therefore

$$\mathcal{L}_{k+1} \quad = \quad \tfrac{1}{\sqrt{\kappa^2+\omega^2}} \mathcal{L}_k \mathcal{M}^\top \ . \tag{98}$$

From the equality

$$\mathcal{M}^\top \mathcal{M} = \left(\kappa^2 + \omega^2\right) I_2 \tag{99}$$

we deduce that

$$\mathcal{L}_{k+1}\mathcal{M} = \sqrt{\kappa^2 + \omega^2}\mathcal{L}_k \,. \tag{100}$$

This allows us to conclude. ■

Lemma 4
We consider the system

$$a_{k+1} = \mathcal{J} a_k \,, k \in \mathbb{N} \tag{101}$$

with

$$\mathcal{J} = \begin{bmatrix} \mathcal{M} & I_2 & 0 & \dots & 0 \\ 0 & \mathcal{M} & I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I_2 \\ 0 & \dots & \dots & 0 & \mathcal{M} \end{bmatrix} \in \mathbb{R}^{2p\times 2p}, \tag{102}$$

with $\mathcal{M}$ defined in (92). Then there exists a constant α such that the time-varying change of coordinates

$$b_k = \overline{\mathcal{L}}_k a_k \tag{103}$$

with

$$\overline{\mathcal{L}}_k = \text{diag}\{\mathcal{L}_{k-p+1}, \mathcal{L}_{k-p+2}, ..., \mathcal{L}_{k-1}, \mathcal{L}_k\} \in \mathbb{R}^{2p\times 2p} \tag{104}$$

with, for all integer j, $\mathcal{L}_j$ defined in (94), transforms the system (101) into the system

$$b_{k+1} = \sqrt{\kappa^2 + \omega^2} b_k + \mathcal{K} b_k, \tag{105}$$

with

$$\mathcal{K} = \begin{bmatrix} 0 & I_2 & 0 & \dots & 0 \\ 0 & 0 & I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I_2 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{2p\times 2p}. \tag{106}$$

Proof. Let α be any real number such that

$$\sin(\alpha) = -\tfrac{\omega}{\sqrt{\kappa^2+\omega^2}} \,, \ \cos(\alpha) = -\tfrac{\kappa}{\sqrt{\kappa^2+\omega^2}} \,. \tag{107}$$

According to the definition of b_k in (103) and (101),

$$b_{k+1} = \overline{\mathcal{L}}_{k+1}\mathcal{J} a_k = \overline{\mathcal{L}}_{k+1}\mathcal{J}\overline{\mathcal{L}}_k^{-1} b_k = \overline{\mathcal{L}}_{k+1}\mathcal{J}\overline{\mathcal{L}}_k^{\top} b_k \,. \tag{108}$$

One can check readily that

$$\mathcal{J}\overline{\mathcal{L}}_k^\top = \begin{bmatrix} \mathcal{M}\mathcal{L}_{k-p+1}^\top & \mathcal{L}_{k-p+2}^\top & 0 & \dots & 0 \\ 0 & \mathcal{M}\mathcal{L}_{k-p+2}^\top & \mathcal{L}_{k-p+3}^\top & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \mathcal{L}_k^\top \\ 0 & \dots & \dots & 0 & \mathcal{M}\mathcal{L}_k^\top \end{bmatrix}. \tag{109}$$

Now, from (100), we deduce that

$$\overline{\mathcal{L}}_{k+1} = \frac{1}{\sqrt{\kappa^2+\omega^2}} \begin{bmatrix} \mathcal{L}_{k-p+1}\mathcal{M}^\top & 0 & \dots & \dots & 0 \\ 0 & \mathcal{L}_{k-p+2}\mathcal{M}^\top & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & \mathcal{L}_{k-1}\mathcal{M}^\top & 0 \\ 0 & \dots & \dots & 0 & \mathcal{L}_k\mathcal{M}^\top \end{bmatrix} . \tag{110}$$

It follows that

$$\overline{\mathcal{L}}_{k+1}\mathcal{J}\overline{\mathcal{L}}_k^\top = \frac{1}{\sqrt{\kappa^2+\omega^2}} \begin{bmatrix} \varphi_{k,1} & \nu_{k,1} & 0 & \dots & 0 \\ 0 & \varphi_{k,2} & \nu_{k,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \nu_{k,p-1} \\ 0 & \dots & \dots & 0 & \varphi_{k,p} \end{bmatrix} \tag{111}$$

with

$$\varphi_{k,i} = \mathcal{L}_{k-p+i}\mathcal{M}^\top\mathcal{M}\mathcal{L}_{k-p+i}^\top \tag{112}$$

and

$$\nu_{k,i} = \mathcal{L}_{k-p+i}\mathcal{M}^\top\mathcal{L}_{k-p+i+1}^\top . \tag{113}$$

Since $\mathcal{M}^\top\mathcal{M} = (\kappa^2+\omega^2)I_2$ and $\mathcal{L}_i\mathcal{L}_i^\top = I_2$, we deduce that

$$\varphi_{k,i} = (\kappa^2+\omega^2)\mathcal{L}_{k-p+i}\mathcal{L}_{k-p+i}^\top = (\kappa^2+\omega^2)I_2 . \tag{114}$$

Now, observe that (100) implies that

$$\mathcal{L}_i\mathcal{M}^\top\mathcal{L}_{i+1}^\top = \mathcal{L}_i\mathcal{M}^\top \frac{1}{\sqrt{\kappa^2+\omega^2}}\mathcal{M}\mathcal{L}_i^\top . \tag{115}$$

Using again the equalities $\mathcal{M}^\top\mathcal{M} = (\kappa^2+\omega^2)I_2$ and $\mathcal{L}_i\mathcal{L}_i^\top = I_2$, we obtain

$$\mathcal{L}_i\mathcal{M}^\top\mathcal{L}_{i+1}^\top = \sqrt{\kappa^2+\omega^2}\mathcal{L}_i\mathcal{L}_i^\top = \sqrt{\kappa^2+\omega^2}I_2 . \tag{116}$$

We deduce that

$$\overline{\mathcal{L}}_{k+1}\mathcal{J}\overline{\mathcal{L}}_k^\top = \begin{bmatrix} \sqrt{\kappa^2+\omega^2}I_2 & I_2 & 0 & \dots & 0 \\ 0 & \sqrt{\kappa^2+\omega^2}I_2 & I_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & I_2 \\ 0 & \dots & \dots & 0 & \sqrt{\kappa^2+\omega^2}I_2 \end{bmatrix} . \tag{117}$$

This allows to conclude. ■

Lemma 5
We consider the system

$$a_{k+1} = \mathcal{J}_a a_k \ , k \in \mathbb{N} \tag{118}$$

with

$$\mathcal{J}_a = \begin{bmatrix} -\mu & 1 & \dots & 0 \\ 0 & -\mu & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -\mu \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{119}$$

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where μ is a positive real number. Then, the time-varying change of coordinates

$$b_k = \mathcal{G}_k a_k \tag{120}$$

with

$$\mathcal{G}_k = \text{diag}\{(-1)^k, (-1)^{k+1}, ..., (-1)^{k+n-1}\} \in \mathbb{R}^{n\times n} \tag{121}$$

gives

$$b_{k+1} = \mathcal{J}_b b_k \ , k \in \mathbb{N} \tag{122}$$

with

$$\mathcal{J}_b = \begin{bmatrix} \mu & 1 & \dots & 0 \\ 0 & \mu & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \mu \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{123}$$

Proof. According to (120) and (118),

$$b_{k+1} \quad = \quad \mathcal{G}_{k+1}\mathcal{J}_a a_k \ = \ \mathcal{G}_{k+1}\mathcal{J}_a\mathcal{G}_k^{-1} b_k \ . \tag{124}$$

Since $\mathcal{G}_k^{-1} = \mathcal{G}_k$ and $\mathcal{G}_{k+1} = -\mathcal{G}_k$, the sequence b_k satisfies

$$b_{k+1} = -\mathcal{G}_k\mathcal{J}_a\mathcal{G}_k b_k \ . \tag{125}$$

Now, we have

$$\mathcal{J}_a\mathcal{G}_k = \begin{bmatrix} \mu(-1)^{k+1} & (-1)^{k+1} & \dots & 0 \\ 0 & \mu(-1)^{k+2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & (-1)^{k+n-1} \\ 0 & \dots & 0 & \mu(-1)^{k+n} \end{bmatrix} . \tag{126}$$

Therefore

$$\mathcal{G}_k\mathcal{J}_a\mathcal{G}_k \quad = \quad - \begin{bmatrix} \mu & 1 & \dots & 0 \\ 0 & \mu & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & \mu \end{bmatrix} . \tag{127}$$

This equality and (125) allow to conclude. ■

Lemma 6
Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function of class C^1. Then there exists a function $f_c : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ nondecreasing with respect to each of its m first variables and nonincreasing with respect to each of its m last variables such that, for all $\mathfrak{x} \in \mathbb{R}^m$, the equality

$$f_c(\mathfrak{x}, \mathfrak{x}) = f(\mathfrak{x}) \tag{128}$$

is satisfied.

Proof. The function

$$\Gamma(s) = \sup_{|z|\leq\sqrt{s}} F(z) + 1 + s, \tag{129}$$

where $F : \mathbb{R}^m \to \mathbb{R}$ is defined by

$$F(\mathfrak{x}) = |\mathfrak{x}||f(\mathfrak{x})| + \left|\frac{\partial f}{\partial \mathfrak{x}}(\mathfrak{x})\right|, \tag{130}$$

is continuous, positive and nondecreasing over $[0, +\infty)$. Then the function

$$\xi(s) = \frac{1}{2}\int_s^{+\infty} \frac{1}{e^m\Gamma(m)} dm \tag{131}$$

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is well-defined, decreasing, continuously differentiable, such that, for all $s \geq 0$, $\xi(s) > 0$ and

$$\xi(s) \leq \frac{1}{2\Gamma(s)} \int_s^{+\infty} \frac{1}{e^m} dm \leq \frac{1}{2\Gamma(s)}, \tag{132}$$

$$|\xi'(s)| \leq \frac{1}{2\Gamma(s)}. \tag{133}$$

Now, we observe that, for all $\mathfrak{x} \in \mathbb{R}^m$,

$$f(\mathfrak{x}) = f_1(\mathfrak{x}) + f_2(\mathfrak{x}) \tag{134}$$

with

$$f_1(\mathfrak{x}) = \frac{4\sum_{j=1}^m \mathfrak{x}_j}{\xi(|\mathfrak{x}|^2)}, \; f_2(\mathfrak{x}) = \frac{f_3(\mathfrak{x})}{\xi(|\mathfrak{x}|^2)}, \; f_3(\mathfrak{x}) = \xi(|\mathfrak{x}|^2) f(\mathfrak{x}) - 4\sum_{j=1}^m \mathfrak{x}_j. \tag{135}$$

Moreover, (131) and (132) imply that, for all $i \in \{1, ..., m\}$ and for all $\mathfrak{x} \in \mathbb{R}^m$,

$$\begin{array}{rcl} \frac{\partial f_3}{\partial \mathfrak{x}_i}(\mathfrak{x}) & = & 2\xi'(|\mathfrak{x}|^2)\mathfrak{x}_i f(\mathfrak{x}) + \xi(|\mathfrak{x}|^2)\frac{\partial f}{\partial \mathfrak{x}_i}(\mathfrak{x}) - 4 \\ & \leq & 2|\xi'(|\mathfrak{x}|^2)||\mathfrak{x}_i||f(\mathfrak{x})| + \xi(|\mathfrak{x}|^2)\left|\frac{\partial f}{\partial \mathfrak{x}_i}(\mathfrak{x})\right| - 4 \\ & \leq & \frac{1}{\Gamma(|\mathfrak{x}|^2)}|\mathfrak{x}_i||f(\mathfrak{x})| + \frac{1}{2\Gamma(|\mathfrak{x}|^2)}\left|\frac{\partial f}{\partial \mathfrak{x}_i}(\mathfrak{x})\right| - 4 \\ & \leq & \frac{F(\mathfrak{x})}{\Gamma(|\mathfrak{x}|^2)} - 4 \\ & \leq & -3, \end{array} \tag{136}$$

where the last inequality is deduced from the definition of Γ. Therefore f_3 is nonincreasing with respect to each of its variables.

Now, we define two functions:

$$\phi_1(a,b) = \frac{4\sum_{j=1}^m \varpi(a_j)}{\xi\left(\sum_{j=1}^m \varpi(a_j)^2 + \sum_{j=1}^m \mho(b_j)^2\right)} + \frac{4\sum_{j=1}^m \mho(a_j)}{\xi\left(\sum_{j=1}^m \varpi(b_j)^2 + \sum_{j=1}^m \mho(a_j)^2\right)}, \tag{137}$$

$$\phi_2(a,b) = \frac{\varpi(f_3(b))}{\xi\left(\sum_{j=1}^m \varpi(a_j)^2 + \sum_{j=1}^m \mho(b_j)^2\right)} + \frac{\mho(f_3(b))}{\xi\left(\sum_{j=1}^m \varpi(b_j)^2 + \sum_{j=1}^m \mho(a_j)^2\right)}. \tag{138}$$

Using the fact that, for all $\mathfrak{x} \in \mathbb{R}^m$,

$$|\mathfrak{x}|^2 = \sum_{j=1}^m \mathfrak{x}_j^2 = \sum_{j=1}^m \varpi(\mathfrak{x}_j)^2 + \sum_{j=1}^m \mho(\mathfrak{x}_j)^2 \tag{139}$$

we deduce that, for all $\mathfrak{x} \in \mathbb{R}^m$,

$$f_1(\mathfrak{x}) = \frac{4\sum_{j=1}^m \varpi(\mathfrak{x}_j)}{\xi\left(\sum_{j=1}^m \varpi(\mathfrak{x}_j)^2 + \sum_{j=1}^m \mho(\mathfrak{x}_j)^2\right)} + \frac{4\sum_{j=1}^m \mho(\mathfrak{x}_j)}{\xi\left(\sum_{j=1}^m \varpi(\mathfrak{x}_j)^2 + \sum_{j=1}^m \mho(\mathfrak{x}_j)^2\right)}. \tag{140}$$

Therefore, for all $\mathfrak{x} \in \mathbb{R}^m$, $\phi_1(\mathfrak{x}, \mathfrak{x}) = f_1(\mathfrak{x})$ and ϕ_1 is nondecreasing with respect to each of its m first variables and nonincreasing with respect to each of its m last variables. We also observe that,

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for all $\mathfrak{x} \in \mathbb{R}^m$,

$$f_2(\mathfrak{x}) = \frac{\varpi(f_3(\mathfrak{x}))}{\xi\left(\sum_{j=1}^{m} \varpi(\mathfrak{x}_j)^2 + \sum_{j=1}^{m} \mho(\mathfrak{x}_j)^2\right)} + \frac{\mho(f_3(\mathfrak{x}))}{\xi\left(\sum_{j=1}^{m} \varpi(\mathfrak{x}_j)^2 + \sum_{j=1}^{m} \mho(\mathfrak{x}_j)^2\right)}, \tag{141}$$

which implies that, for all $\mathfrak{x} \in \mathbb{R}^m$, $\phi_2(\mathfrak{x}, \mathfrak{x}) = f_2(\mathfrak{x})$. Since f_3 is nonincreasing with respect to each of its variables, then ϕ_2 is nondecreasing with respect to each of its m first variables and nonincreasing with respect to each of its m last variables.

We are ready to conclude. We observe that

$$f(\mathfrak{x}) = f_1(\mathfrak{x}) + f_2(\mathfrak{x}) = f_c(\mathfrak{x}, \mathfrak{x}) \tag{142}$$

with $f_c(a, b) = \phi_1(a, b) + \phi_2(a, b)$. Thus the function f_c is nondecreasing with respect to each of its m first variables and nonincreasing with respect to each of its m last variables. ■