Chapter 1
Background on Nonlinear Systems

Abstract We review some basic concepts from the theory of ordinary differential equations and nonlinear control systems, as well as several notions of stability, including the input-to-state stability paradigm. An important feature is the distinction between uniform and non-uniform stability for time-varying systems. We also include an overview of the problem of stabilization of nonlinear systems, including the "virtual" obstacles to stabilization imposed by Brockett's Necessary Condition. Brockett's Criterion motivates our use of time-varying feedbacks to stabilize both autonomous and time-varying systems. We illustrate these notions in several examples. In later chapters, we revisit these notions using strict Lyapunov functions.

1.1 Preliminaries

Throughout this book, we use the following standard notation and classical results. We let $\mathbb{N}$ denote the set of natural numbers $\{1, 2, \ldots\}$, $\mathbb{Z}$ the set of all integers, and $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$. Also, $\mathbb{R}$ (resp., $\mathbb{R}^n$) denotes the set of all real numbers (resp., real n-tuples for any $n \in \mathbb{N}$). We use the following norms for vectors $x = (x_1, ..., x_n) \in \mathbb{R}^n$:

$$|x|_\infty = \max_{1 \leq i \leq n} |x_i|, \quad |x|_1 = \sum_{i=1}^{n} |x_i|, \quad \text{and} \quad |x|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}.$$

Unless we indicate otherwise, the norm on $\mathbb{R}^n$ is $|\cdot|_2$ which we often denote by $|\cdot|$. For a measurable essentially bounded[1] function $u : \mathcal{I} \to \mathbb{R}^p$ on an interval $\mathcal{I} \subseteq \mathbb{R}$, we let $|u|_\mathcal{I}$ denote its essential supremum, which we

[1] Readers who are not familiar with Lebesgue measure theory can replace "measurable essentially bounded" with "bounded and piecewise continuous" throughout our work, in which case the essential supremum is just the sup norm.

indicate by $|u|_\infty$ when $\mathcal{I} = \mathbb{R}$. For real matrices A, we use the matrix norm $||A|| = \sup\{|Ax| : |x| = 1\}$, and $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. Given an interval $\mathcal{I} \subseteq \mathbb{R}$ and a function $x : \mathcal{I} \to \mathbb{R}^n$ that is differentiable (Lebesgue) almost everywhere, we use $\dot{x}$ or $\dot{x}(t)$ to denote its derivative $\frac{dx}{dt}(t)$.

For each $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, a real-valued function defined on an open subset of Euclidean space is called C^k provided its partial derivatives exist and are continuous up to order k. A C^0 function is one that is continuous, and a C^∞ function is one that is a smooth function, that is, it has continuous partial derivatives of any finite order. We use the same C^k notation for vector fields on $\mathbb{R}^n$. We present all of our results under those differentiability assumptions that lead to the shortest and clearest proofs. Throughout the book, increasing means strictly increasing and similarly for decreasing.

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field and $h : \mathbb{R}^n \to \mathbb{R}$ is smooth, the *Lie derivatives* of h in the direction of f are defined recursively by

$$L_f h(x) \doteq \frac{\partial h}{\partial x}(x) f(x) \quad \text{and} \quad L_f^k h(x) = L_f(L_f^{k-1} h)(x) \quad \forall k \geq 2.$$

Recall the following classes of comparison functions. We say that a C^0 function $\gamma : [0,\infty) \to [0,\infty)$ belongs to class $\mathcal{K}$ and write $\gamma \in \mathcal{K}$ provided it is increasing and $\gamma(0) = 0$. We say that it belongs to class $\mathcal{K}_\infty$ if, in addition, $\gamma(r) \to \infty$ as $r \to \infty$. We say that a C^0 function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of class $\mathcal{KL}$ provided for each fixed $s \geq 0$, the function $\beta(\cdot, s)$ belongs to class $\mathcal{K}$, and for each fixed $r \geq 0$, the function $\beta(r, \cdot)$ is non-increasing and $\beta(r, s) \to 0$ as $s \to \infty$. The following lemma is well-known:

Lemma 1.1. *(Barbalat's Lemma) If $\phi : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $[0,\infty)$ and*

$$\lim_{t\to\infty} \int_0^t \phi(m)\, \mathrm{d}m$$

exists and is finite, then $\lim_{t\to\infty} \phi(t) = 0$.

We also use Young's Inequality, which says that

$$ab \leq \frac{1}{p}|a|^p + \frac{p-1}{p}|b|^q$$

for all $a \in \mathbb{R}$ and $b \in \mathbb{R}$, and all $p > 1$ and $q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

1.2 Families of Nonlinear Systems

The basic families of dynamics are autonomous systems, nonautonomous systems, and systems with inputs. We review these basic families next for the case where the dynamics are given by families of ordinary differential equations. We then discuss their analogs in discrete time. In later chapters, we consider more general systems with multiple time scales, such as hybrid time-varying systems. In general, we allow nonlinear systems, meaning the dynamics are nonlinear in the state variable.

1.2.1 Nonautonomous Systems

A general nonautonomous ordinary differential equation consists of a finite number of first-order one-dimensional differential equations:

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, x_2, ..., x_n) \\ \dot{x}_2 = f_2(t, x_1, x_2, ..., x_n) \\ \quad\vdots \\ \dot{x}_n = f_n(t, x_1, x_2, ..., x_n) \end{cases} \tag{1.1}$$

where t is the time, and each function f_i is in general nonlinear in all of its arguments. The variables x_i are called *states*, and n is called the *dimension* of the system. The differential equations characterize the evolution of the states with respect to time. Frequently, we write (1.1) more compactly as

$$\dot{x} = f(t, x). \tag{1.2}$$

The state vector $x = (x_1, x_2, ..., x_n)$ is valued in a given open set $\mathcal{X} \subseteq \mathbb{R}^n$. Given a constant $t_0 \geq 0$, $x_0 \in \mathcal{X}$, and a constant $t_{\max} > t_0$, the corresponding initial value problem $\text{IVP}(t_{\max}, t_0, x_0)$ for (1.2) is that of determining an absolutely continuous function $y : [t_0, t_{\max}) \to \mathcal{X}$ such that $\dot{y}(t) = f(t, y(t))$ for almost all $t \in [t_0, t_{\max})$ and $y(t_0) = x_0$. We assume that the vector field $f : [0, \infty) \times \mathcal{X} \to \mathbb{R}^n$ is measurable in t and of class C^1 in x, meaning the function $x \mapsto f(t, x)$ is C^1 for each $t \geq 0$. We further assume that for each compact set $K \subseteq \mathcal{X}$, there is a locally integrable function α_K so that

$$\left|\frac{\partial f}{\partial x}(t, x)\right| \leq \alpha_K(t) \text{ for all } x \in K \text{ and } t \geq 0.$$

By classical results (reviewed, e.g., in [161]), these properties ensure that for each $t_0 \geq 0$ and $x_0 \in \mathcal{X}$, there exists a $t_{\max} > t_0$ so that $\text{IVP}(t_{\max}, t_0, x_0)$ has a solution $t \mapsto x(t, t_0, x_0)$ with the following uniqueness and maximality property: If $\tilde{t} > t_0$ and $\text{IVP}(\tilde{t}, t_0, x_0)$ admits a solution $z(t)$, then $\tilde{t} \leq t_{\max}$ and $z(t) = x(t, t_0, x_0)$ for all $t \in [t_0, \tilde{t})$. If $x(t, t_0, x_0)$ can be uniquely defined for all $t \geq t_0$ for all initial conditions $x(t_0, t_0, x_0) = x_0$, then we call (1.1) *forward complete*. Since f depends on time, the systems (1.2) are also called *time-varying systems*.

An equilibrium point $x^* = (x_1^*, ..., x_n^*)$ of (1.2) is defined to be a vector in $\mathbb{R}^n$ for which $f(t, x^*) = 0$ for all $t \geq 0$. Frequently, the equilibrium point is the origin $x^* = 0$. If a system $\dot{X} = g(t, X)$ admits a solution $X_s(t)$, then, through the time-varying change of variable $x = X - X_s(t)$, we can transform the system $\dot{X} = g(t, X)$ into a new time-varying x dynamics

$$\dot{x} = f(t, x), \quad \text{where} \quad f(t, x) = g(t, x + X_s(t)) - \dot{X}_s(t)$$

which admits $x^* = 0$ as an equilibrium point. This transformation is used to analyze the asymptotic behavior of a system with respect to a specific solution $X_s(t)$, i.e., tracking. Frequently, the time-varying systems in engineering applications are *periodic with respect to* t, meaning there is a constant $w > 0$ (called a *period*) such that f satisfies

$$f(t + w, x) = f(t, x)$$

for all (t, x) in its domain.

1.2.2 Autonomous Systems

If the right side of (1.1) or (1.2) is independent of the time variable t, then the systems are called *autonomous* or *time-invariant* systems. Naturally, they are written in compact form as

$$\dot{x} = f(x) \tag{1.3}$$

and their flow maps are denoted by $x(t, x_0)$. In this case, we view $t \mapsto x(t, x_0)$ as being defined on some maximal interval $\mathcal{I} \subseteq \mathbb{R}$, possibly depending on the initial state x_0. If $t \mapsto x(t, x_0)$ is uniquely defined on $\mathbb{R}$ for all $x_0 \in \mathcal{X}$, then we call (1.3) *complete*. The family of systems (1.3) is the simplest we consider in this book. However, the behavior of the solutions of (1.3) is a very general subject and by no means simple. No general prediction of the asymptotic behavior of the solutions exists as soon as the dimension n of the system is larger than 2. Rather, such a classification exists only for systems of dimension 1 and 2, by the celebrated Poincaré-Bendixson Theorem [23, 153].

1.2.3 Systems with Inputs

The general *time-varying continuous time control system* is

$$\dot{x} = f(t, x, u) \tag{1.4}$$

or, equivalently,

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, x_2, ..., x_n, u_1, ...u_p) \\ \dot{x}_2 = f_2(t, x_1, x_2, ..., x_n, u_1, ...u_p) \\ \quad \vdots \\ \dot{x}_n = f_n(t, x_1, x_2, ..., x_n, u_1, ...u_p). \end{cases} \tag{1.5}$$

The variables $u_1, ..., u_p$ are called *inputs*. The state and input vectors are valued in a given open set $\mathcal{X} \subseteq \mathbb{R}^n$ and a given set $U \subseteq \mathbb{R}^p$, respectively. When discussing systems with inputs, we assume that $[0, \infty) \times \mathcal{X} \times U \ni (t, x, u) \mapsto f(t, x, u) \in \mathbb{R}^n$ is piecewise continuous in t and of class C^1 in (x, u). We refer to the preceding conditions as our usual (or standing) assumptions on (1.5). We also let $\mathcal{M}(U)$ denote the set of all measurable essentially bounded functions $u : [0, \infty) \to U$; i.e., inputs that are bounded in $|\cdot|_\infty$. Solutions of (1.5) are obtained by replacing $(u_1, u_2, \ldots, u_p)$ with an element $u \in \mathcal{M}(U)$. For all $u \in \mathcal{M}(U)$, $x_0 \in \mathcal{X}$, and $t_0 \geq 0$, we let $x(t, t_0, x_0, u)$ denote the solution of (1.4) with u as input that satisfies $x(t_0, t_0, x_0, u) = x_0$, defined on its maximal interval $[t_0, b)$. If $t \mapsto x(t, t_0, x_0, u)$ is uniquely defined on $[t_0, \infty)$ for all $t_0 \geq 0$, $x_0 \in \mathcal{X}$, and $u \in \mathcal{M}(U)$, then we call (1.4) *forward complete*. By an equilibrium state of (1.4), we mean a vector $x^* \in \mathbb{R}^n$ that admits a vector $u^* \in U$ such that $f(t, x^*, u^*) = 0$ for all $t \geq 0$. If the system (1.5) can be written in the form

$$\dot{x} \;=\; \mathcal{F}(t, x) + \mathcal{G}(t, x)u$$

for some vector fields $\mathcal{F}$ and $\mathcal{G}$, then we say that (1.5) is *affine in controls* or *control affine*.

Inputs are essential in nonlinear control theory. One of the principal aims of control theory is to provide functions $u(t, x)$ such that all or some of the solutions of the system $\dot{x} = f(t, x, u(t, x))$ possess a desired property. In this situation, we refer to $u(t, x)$ as a *controller* or a *feedback*, and the feedback controlled system $\dot{x} = f(t, x, u(t, x))$ as a *closed-loop system*. Inputs can also represent disturbances, which are uncertainties that may modify the behavior of the solutions (often in an undesirable way). Then, the problem of quantifying the effect of disturbances $u(t)$ on the solutions of (1.4) arises.

1.2.4 Discrete Time Dynamics

The general family of time-varying discrete time systems with inputs admits the representation

$$x_{k+1} = f(k, x_k, u_k) \tag{1.6}$$

or, equivalently,

$$\begin{cases} x_{k+1,1} = f_1(k, x_{k,1}, x_{k,2}, \ldots, x_{k,n}, u_{k,1}, ...u_{k,p}) \\ x_{k+1,2} = f_2(k, x_{k,1}, x_{k,2}, \ldots, x_{k,n}, u_{k,1}, ...u_{k,p}) \\ \quad\vdots \\ x_{k+1,n} = f_n(k, x_{k,1}, x_{k,2}, \ldots, x_{k,n}, u_{k,1}, ...u_{k,p}). \end{cases} \tag{1.7}$$

The variables $u = (u_1, ..., u_p)$ are again called *inputs*, which are now *sequences* $u_k = (u_{k,1}, ...u_{k,p})$ that take their values in some subset $U \subseteq \mathbb{R}^p$ for each time $k \in \{0, 1, 2, \ldots\}$. We use k for the time indices to emphasize that they are discrete instants rather than being on a continuum. We let $\mathcal{D}(U)$ denote the set of all such input sequences. The state vector $x_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,n})$ at each instant k is assumed to be valued in a given open set $\mathcal{X} \subseteq \mathbb{R}^n$.

Since the solutions of (1.7) are given recursively, there is no need to impose the regularity on f that we assumed in the continuous time case. However, when discussing discrete time systems, we always assume that the recursion defining the solutions is *forward complete*, meaning that solutions of (1.7) exist for all integers $k \geq 0$, all initial conditions $x(k_0) = x_0 \in \mathcal{X}$, and all $u \in \mathcal{D}(U)$. As in the continuous time case, $x(k, k_0, x_0, u)$ then denotes the unique solution of (1.4) that satisfies $x(k_0, k_0, x_0, u) = x_0$ for all $u \in \mathcal{D}(U)$, $x_0 \in \mathcal{X}$, and $k \geq k_0 \geq 0$. We define equilibrium states for (1.6) and time-invariant discrete time systems analogously to the definitions for continuous time systems.

Discrete time dynamics are of significant interest in engineering applications. In fact, when time-varying continuous time systems with inputs are implemented in labs, this is often done using sampling, which leads to dynamics of the form (1.6). Discrete time systems are also important from the theoretical point of view, including cases where (1.6) is a sub-dynamics of a larger *hybrid* time-varying system that has mixtures of continuous and discrete parts and prescribed mechanisms for switching between the parts.

It is possible to define time-varying systems in a unifying, behavioral way that includes both continuous and discrete time systems. This was done in [161, Chap. 2]. However, strict Lyapunov function constructions for continuous and discrete time systems are often very different, so we treat continuous time and discrete time systems separately in most of the sequel.

1.3 Notions of Stability

Stability, instability, asymptotic stability, exponential stability and input-to-state stability are of utmost importance for nonlinear control systems. Stability formalizes the following intuition: an equilibrium point of a system is stable if any solution with any initial state close to the equilibrium point stays close to the equilibrium point forever. *Asymptotic* stability formalizes the following: an equilibrium point is asymptotically stable if it is stable and all solutions starting near the equilibrium point converge to the equilibrium point as time goes to infinity.

An equilibrium point of a system is *exponentially* stable if it is asymptotically stable and if the solutions are smaller in norm than a positive function of time that exponentially decays to zero. Finally, *input-to-state* stability roughly says that an equilibrium point of a system with inputs is asymptotically stable for the zero input and, in the presence of a bounded input, the solutions are bounded and asymptotically smaller in norm than a function of the sup norm of the input. We use the following abbreviations and acronyms:

LAS	local asymptotic stability
UGAS	uniform global asymptotic stability
GAS	global asymptotic stability
LES	local exponential stability
GES	global exponential stability
UGES	uniform global exponential stability
ISS	input-to-state stability

We also use ISS to mean input-to-state stable, and similarly for the other stability notions. We now make the various stability notions mathematically precise. We focus on continuous time systems but one can define these notions for discrete time systems in an analogous way. For any constants $\rho > 0$, $r \in \mathbb{N}$, and $q \in \mathbb{R}^r$, we use the notation $\rho\mathcal{B}_r(q) \doteq \{x \in \mathbb{R}^r : |x - q| \leq \rho\}$, which we denote simply by $\rho\mathcal{B}_r$ when $q = 0$.

1.3.1 Stability

Assume that the system (1.2) admits the origin 0 as an equilibrium point. This equilibrium point is *stable* provided for each constant $\varepsilon > 0$, there exists a constant $\delta(\varepsilon) > 0$ such that for each initial state $x_0 \in \mathcal{X} \cap \delta(\varepsilon)\mathcal{B}_n$ and each initial time $t_0 \geq 0$, the unique solution $x(t, t_0, x_0)$ satisfies $|x(t, t_0, x_0)| \leq \varepsilon$ for all $t \geq t_0$. Otherwise we call the equilibrium *unstable.*

1.3.2 Asymptotic and Exponential Stability

Assume that the system (1.2) admits the origin 0 as an equilibrium point. *Uniform globally asymptotic stability (UGAS)* of the equilibrium 0 means that there exists a function $\beta \in \mathcal{KL}$ such that for each initial state $x_0 \in \mathcal{X}$ and each initial time $t_0 \geq 0$, the solution $x(t, t_0, x_0)$ for (1.2) satisfies

$$\big|x(t, t_0, x_0)\big| \leq \beta\big(|x_0|, t - t_0\big) \;\; \forall t \geq t_0 \geq 0. \tag{1.8}$$

In this case, we also say that the system is UGAS to 0, or simply UGAS, and similarly for the other stability notions. When the system is autonomous, this property is called *global asymptotic stability (GAS)*. If there exists a function $\beta \in \mathcal{KL}$ and a constant $\bar{c} > 0$ independent of t_0 such that (1.8) holds for all initial conditions $x_0 \in \bar{c}\mathcal{B}_n \cap \mathcal{X}$, then we call the system *uniformly asymptotically stable*. Hence, uniform asymptotic stability of the equilibrium implies that it is stable and that there exists a constant $\bar{c} > 0$ such that for each initial state $x_0 \in \mathcal{X} \cap \bar{c}\mathcal{B}_n$ and each initial time $t_0 \geq 0$, the solution $x(t, t_0, x_0)$ satisfies $\lim_{t \to +\infty} x(t, t_0, x_0) = 0$. When the system is time-invariant, we call the preceding property *(local) asymptotic stability (LAS)*.

When (1.2) admits the origin 0 as an equilibrium point, we call the equilibrium point *uniformly exponentially stable* provided there exist positive constants K_1, K_2, and r such that for each initial state $x_0 \in \mathcal{X} \cap r\mathcal{B}_n$ and each $t_0 \geq 0$, the corresponding solution $x(t, t_0, x_0)$ satisfies $|x(t, t_0, x_0)| \leq K_1 e^{-K_2(t-t_0)}$ for all $t \geq t_0$. When the system is autonomous, we call this property *local exponential stability (LES)* or, if r can be taken to be $+\infty$, *global exponential stability (GES)*. The special case of uniformly exponentially stability where we can take $r = +\infty$ is called *uniform global exponential stability (UGES)*. More generally, we say that an equilibrium point x^* of (1.2) (which may or may not be zero) is GES provided the dynamics of $x(t) - x^*$ is GES, and similarly for the other stability notions.

1.3.3 Input-to-State Stability

The input-to-state stability (ISS) condition for (1.4) is the requirement that there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for each $u \in \mathcal{M}(U)$ and each initial condition $x(t_0) = x_0 \in \mathcal{X}$, the solution $x(t, t_0, x_0, u)$ of (1.4) with input vector u satisfies

$$\big|x(t, t_0, x_0, u)\big| \leq \beta\big(|x_0|, t - t_0\big) + \gamma\big(|u|_{[t_0, t]}\big) \;\; \forall t \geq t_0 \, . \tag{1.9}$$

The ISS paradigm plays a fundamental role in nonlinear control, as do its extensions to systems with outputs; see [165] for an extensive discussion.

One immediate consequence of (1.9) is that if (1.4) admits an input $u \in \mathcal{M}(U)$ and an initial condition for which the corresponding trajectory is unbounded, then the system cannot be ISS. This gives a method for testing whether a system is ISS. In Chap. 2, we use this alternative method:

Lemma 1.2. *Assume that (1.4) has state space $\mathcal{X} = \mathbb{R}^n$. Let $\delta \in \mathcal{M}(U)$ be any non-zero input, let $L \in \mathbb{R}^{n \times n}$ be invertible, and set $z(t, t_0, z_0) = Lx(t, t_0, L^{-1}z_0, \delta)$ for each $t \geq t_0 \geq 0$ and $z_0 \in \mathbb{R}^n$. If there is an index $k \in \{1, 2, \ldots, n\}$ such that the kth component z_k of $z(t, t_0, z_0)$ satisfies $\frac{\partial}{\partial t} z_k(t, t_0, z_0) = 0$ for all $t \geq t_0 \geq 0$ and all $z_0 \in \mathbb{R}^n$, then (1.4) is not ISS.*

Proof. Suppose the contrary. Then the dynamics

$$\dot{z} = Lf(t, L^{-1}z, u) \tag{1.10}$$

is easily shown to be ISS as well.[2] Pick $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$\big|z(t, t_0, z_0)\big| \;\le\; \beta\big(|z_0|, t - t_0\big) + \gamma\big(|\delta|_\infty\big) \tag{1.11}$$

along all trajectories of (1.10). By assumption,

$$\big|z(t, t_0, z_0)\big| \;\ge\; \big|z_k(t, t_0, z_0)\big| \;=\; \big|z_{0,k}\big|$$

for all $t \ge t_0 \ge 0$ and $z_0 \in \mathbb{R}^n$, so we get a contradiction by picking

$$z_{0,k} = 2\gamma(|\delta|_\infty)$$

and letting $t \to +\infty$ in (1.11). □

If a system (1.4) is ISS, then necessarily the system

$$\dot{x} = f(t, x, 0) \tag{1.12}$$

is UGAS. However, if (1.12) is UGAS, then it does not follow that (1.4) is ISS. The one-dimensional system

$$\dot{x} = -\arctan(x) + u \tag{1.13}$$

illustrates this. When $u = 0$, the system (1.13) becomes $\dot{x} = -\arctan(x)$ which is GAS. However, (1.13) is not ISS because the bounded input $u = 2$ results in the system

$$\dot{x} = 2 - \arctan(x)$$

which has unbounded solutions.

On the other hand, the system (1.13) is *integral input-to-state stable (iISS)* [160]. For a general nonlinear system (1.4), the iISS condition says that there exist functions $\underline{\gamma}, \bar{\gamma} \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that for each $u \in \mathcal{M}(U)$ and each initial condition $x(t_0) = x_0$, the unique solution $x(t, t_0, x_0, u)$ of (1.4) with input vector u satisfies

$$\underline{\gamma}(|x(t, t_0, x_0, u)|) \le \beta(|x_0|, t - t_0) + \int_{t_0}^{t} \bar{\gamma}(|u(m)|)\mathrm{d}m \tag{1.14}$$

for all $t \ge t_0$. The fact that (1.13) is iISS will follow from the Lyapunov characterizations for ISS and iISS that we discuss in Chap. 2.[3] The ISS property is essentially global. Indeed, any system (1.4) such that the corresponding

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[2] For example, if (1.4) has the ISS Lyapunov function V, then (1.10) has the ISS Lyapunov function $\tilde{V}(t, z) \doteq V(t, L^{-1}z)$; see Chap. 2 for the relevant definitions.

[3] In fact, (1.13) admits the iISS Lyapunov function $V(x) = x\arctan(x)$ and therefore is iISS.

system (1.12) admits the origin as a locally uniformly asymptotically stable equilibrium point is locally ISS, meaning there exists a neighborhood $\mathcal{G}$ of the equilibrium so that an ISS estimate holds along trajectories remaining in $\mathcal{G}$; this follows from the local Lipschitzness of the dynamics in the state.[4]

1.3.4 Linear Systems and Linearizations

Stability analysis is considerably simpler for linear systems than for nonlinear systems. For example, for a linear system

$$\dot{x} = Ax \tag{1.15}$$

with a constant matrix A, the properties GAS, LAS, GES, and LES are all equivalent, and they are satisfied if and only if all eigenvalues of A have negative real parts, in which case A is called *Hurwitz*. The solutions of (1.15) have the form $x(t) = e^{At}x_0$. Frequently, the local behavior of a nonlinear system $\dot{x} = f(x)$ can be analyzed using the fact that an equilibrium point x^* of a time-invariant nonlinear system is LES if and only if its variational matrix $A \doteq Df(x^*)$ is Hurwitz [161]. This can be equivalently formulated by saying that the equilibrium point of a nonlinear system is LES if and only if its linear approximation at the equilibrium point is LES.

Even when the variational matrix is not Hurwitz, the linearization can still provide important information. One important result in that direction is the following one from [131, p.120]:

Theorem 1.1. *(Hartman-Grobman Theorem) Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a neighborhood of the origin, and let $f : \mathcal{X} \to \mathbb{R}^n$ be C^1 with equilibrium point* 0*. Assume that $A \doteq Df(0)$ has no eigenvalue with zero real part. Then we can find a homeomorphism H of an open neighborhood $\mathcal{V}_1$ of the origin into an open neighborhood $\mathcal{V}_2$ of* 0 *such that for each $x_0 \in \mathcal{V}_1$, there is an interval $\mathcal{I}$ containing* 0 *for which $H(x(t, x_0)) = e^{At}H(x_0)$ for all $t \in \mathcal{I}$.*

Here $x(t, x_0)$ is the flow of $\dot{x} = f(x)$ in the usual ODE sense.

1.3.5 Uniformity vs. Non-uniformity

For time-varying systems, asymptotic stability and uniform asymptotic stability are different. The one-dimensional linear time-varying system

[4] Given $n, p \in \mathbb{N}$, an interval $\mathcal{I} \subseteq \mathbb{R}$, and a subset $\mathcal{X} \subseteq \mathbb{R}^n$, we say that a function $g : \mathcal{I} \times \mathcal{X} \to \mathbb{R}^p$ is *locally Lipschitz in $x \in \mathcal{X}$* provided for each compact subset $K \subseteq \mathcal{X}$, there is a constant L_K so that $|g(t, x) - g(t, x')| \leq L_K|x - x'|$ for all $t \in \mathcal{I}$ and $x, x' \in K$. If L_K can be chosen independently of K, then we say that g is Lipschitz in $x \in \mathcal{X}$.

$$\dot{x} = -\tfrac{x}{1+t} \tag{1.16}$$

is GAS in the sense that its solutions are

$$x(t, t_0, x_0) = x_0 \frac{1 + t_0}{1 + t}$$

and therefore go to zero when t goes to infinity. However, it is not UGAS. To prove this, we proceed by contradiction. Suppose that there exists a function $\beta \in \mathcal{KL}$ such that for all $t \geq t_0 \geq 0$, the inequality

$$\left|x(t, t_0, x_0)\right| \leq \beta\big(|x_0|, t - t_0\big) \tag{1.17}$$

is satisfied. Choosing $x_0 = 1$ and $t = 2t_0 + 1$, we have

$$\frac{1}{2} \; = \; \frac{1 + t_0}{2 + 2t_0} \; \leq \; \beta(1, t_0 + 1). \tag{1.18}$$

Since $\beta(1, t_0 + 1)$ goes to zero when t_0 goes to infinity, the inequality (1.18) leads to a contradiction.

1.3.6 Basin of Attraction

The *region of attraction* (also called the *basin of attraction*) of a LAS equilibrium point of a system is the set of all initial states that generate solutions of the system that converge to the equilibrium point. Often, it is not sufficient to determine that a given system has an asymptotically stable equilibrium point. Rather, it is important to find the region of attraction or an approximation of this region. Such approximations can be found using Lyapunov functions. We revisit the problem of estimating the basin of attraction in Sect. 2.5.

1.4 Stabilization

Consider the classical problem of constructing a control law $u_s(t, x)$ such that the origin of (1.4) is asymptotically stable. Later, we will see how this problem can often be handled by Lyapunov function constructions. When the problem is restricted to local stabilization, techniques based on the stabilization of the linear approximation of (1.4) at the origin are frequently used.

However, when UGAS is desirable, linear techniques usually cannot be used. Then, nonlinear design techniques called *backstepping* and *forwarding* apply, provided the system admits a special structure. Backstepping applies to lower triangular systems

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, x_2) \\ \dot{x}_2 = f_2(t, x_1, x_2, x_3) \\ \quad \vdots \\ \dot{x}_n = f_n(t, x_1, ..., x_n, u). \end{cases} \tag{1.19}$$

These systems are called *feedback* systems. Forwarding applies to systems having the upper triangular form

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, ..., x_n, u) \\ \dot{x}_2 = f_2(t, x_2, ..., x_n, u) \\ \quad \vdots \\ \dot{x}_n = f_n(t, x_n, u) \end{cases} \tag{1.20}$$

which are called *feedforward* systems. We discuss backstepping in detail in Chap. 7.

When a nonlinear system admits a linear approximation around an equilibrium point that is not exponentially stabilizable, it may not be easy to tell whether the equilibrium point is locally asymptotically stabilizable. Besides, in some cases, an equilibrium point is asymptotically stabilizable by a C^1 time-varying feedback but not stabilizable by a C^1 *time-invariant* state feedback. For example, this phenomenon occurs for the origin of

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 u_1. \end{cases} \tag{1.21}$$

The fact that the origin of this system is not asymptotically stabilizable by a C^1 time-invariant feedback can be proven using the following necessary condition from [18]:

Theorem 1.2. *(Brockett's Stabilization Theorem) Consider a system*

$$\dot{x} = f(x, u) \tag{1.22}$$

with $f \in C^1$. Assume that there exist an equilibrium point x_* and a C^1 feedback $u_s(x)$ such that the system

$$\dot{x} = f\big(x, u_s(x)\big)$$

admits x_* as a LAS equilibrium point. Then the image of the map f contains some neighborhood of x_*.

The system (1.21) does not satisfy the necessary condition of Brockett's Theorem at the origin, because for any $\varepsilon \neq 0$, there is no pair (x, u) such that

$$(u_1, u_2 u_1) = (0, \varepsilon)$$

and for any open neighborhood of the origin $\mathcal{V} \subseteq \mathbb{R}^2$, there exists $\varepsilon \neq 0$ such that $(0, \varepsilon) \in \mathcal{V}$. On the other hand, it can be globally stabilized by a *time-varying* C^1 feedback; see p.19. These considerations show one reason why time-varying systems are important.

1.5 Examples

In many cases, one can use Lyapunov functions to establish the various stability properties. However, in the following examples, we establish the stability properties using other techniques. In later chapters, we primarily use Lyapunov function methods to establish stability. As we will see later, strict Lyapunov functions have the advantage that they can also be used to quantify the effects of uncertainty, especially when they are given in explicit closed form.

1.5.1 Stable System

An example of a nonlinear system that is stable but not asymptotically stable is given by the two-dimensional pendulum dynamics

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -\frac{g}{l}\sin(\theta) \end{cases} \tag{1.23}$$

where g and l are positive real numbers. To simplify, we assume

$$\frac{g}{l} = 1.$$

The local stability of the origin can be proved as follows. Let $\varepsilon \in \left(0, \frac{1}{4}\right]$. Consider the non-negative function

$$H(\theta, \omega) \ = \ 1 - \cos(\theta) + \frac{1}{2}\omega^2.$$

Let $\delta(\varepsilon) = \frac{1}{8}\varepsilon^2$. Take any solution $(\theta(t), \omega(t))$ of (1.23) with any initial condition satisfying $|(\theta(0), \omega(0))|_\infty \leq \delta(\varepsilon)$. Since $\delta(\varepsilon) \leq 1/128$, we get

$$H\big(\theta(0), \omega(0)\big) \ \leq \ |\theta(0)| + \frac{1}{2}\omega^2(0) \ \leq \ 2\delta(\varepsilon).$$

Simple calculations yield

$$\frac{d}{dt}H\big(\theta(t), \omega(t)\big) = 0 \ \ \forall t \geq 0.$$

Hence, (1.23) cannot be asymptotically stable. On the other hand, since H is constant along the trajectories of (1.23),

$$1 - \cos\big(\theta(t)\big) + \frac{1}{2}\omega^2(t) \;=\; H\big(\theta(t), \omega(t)\big) \;\leq\; \frac{1}{4}\varepsilon^2 \;\leq\; \frac{1}{64}$$

for all $t \geq 0$. Therefore, $|\omega(t)| \leq \varepsilon$ for all $t \geq 0$. Also, since $1 - \cos(\theta(t)) \leq \frac{1}{64}$ for all $t \geq 0$ and $|\theta(0)| \leq \pi/4$, we deduce that $|\theta(t)| \leq \pi/4$ for all $t \geq 0$, which implies that

$$\frac{1}{4}\theta^2(t) \;\leq\; 1 - \cos(\theta(t)) \;\leq\; \frac{1}{4}\varepsilon^2$$

for all $t \geq 0$. This gives $|(\theta(t), \omega(t))|_\infty \leq \varepsilon$ for all $t \geq 0$, which is the desired stability estimate.

Remark 1.1. One can also construct unstable autonomous systems all of whose trajectories converge to the origin. An example of this phenomenon is

$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)}, \quad \dot{x}_2 = \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)}. \qquad (1.24)$$

For the proof that (1.24) satisfies the requirements, see [54, pp. 191-194].

1.5.2 Locally Asymptotically Stable System

When a friction term is added to (1.23), the system becomes

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -\dfrac{g}{l}\sin(\theta) - \dfrac{k}{m}\omega \end{cases} \qquad (1.25)$$

where k and m are positive real numbers. The origin of (1.25) is a LES equilibrium point that is not GAS because the system admits multiple equilibrium points.

The proof that the origin of (1.25) is LES is a consequence of the fact that its linear approximation at the origin is

$$\begin{cases} \dot{\theta}_e = \omega_e \\ \dot{\omega}_e = -\dfrac{g}{l}\theta_e - \dfrac{k}{m}\omega_e \,, \end{cases} \qquad (1.26)$$

which is an exponentially stable linear system because the eigenvalues of the matrix

$$\begin{bmatrix} 0 & 1 \\ -\dfrac{g}{l} & -\dfrac{k}{m} \end{bmatrix} \qquad (1.27)$$

have negative real parts.

1.5.3 Globally Asymptotically Stable System

The two-dimensional system

$$\begin{cases} \dot{S} = D(S_e - S) - \dfrac{KS}{L+S}x \\ \dot{x} = \left(\dfrac{KS}{L+S} - D\right)x \end{cases} \tag{1.28}$$

with positive constant parameters D, K, L, and S_e has the invariant domain $\mathcal{X} = (0,\infty) \times (0,\infty)$. It is a simplified model of a bio-reactor with dilution rate D, input nutrient concentration S_e, and Monod growth rate

$$\mu(S) = \frac{KS}{L+S};$$

see [153] for generalizations. Assume that

$$K > D \quad \text{and} \quad S_e > \frac{DL}{K-D}. \tag{1.29}$$

We show that the equilibrium point

$$(S_*, x_*) = \left(\frac{DL}{K-D}, S_e - \frac{DL}{K-D}\right) \tag{1.30}$$

for (1.28) is GAS and LES.

The variable $Z = S + x - S_e$ satisfies

$$\dot{Z} = -DZ. \tag{1.31}$$

We easily deduce that all of the trajectories of (1.28) enter

$$B = (0, 2S_e) \times (0, 2S_e).$$

One can readily check that (S_*, x_*) and $(S_e, 0)$ are the unique equilibrium points of (1.28) in the closure $\overline{B}$ of B. Also, (S_*, x_*) is the unique LES equilibrium point in $\overline{B}$ (by considering the linearization of (1.28) around (S_*, x_*), and using (1.29) to show that $(S_e, 0)$ is not an asymptotically stable equilibrium, because $\dot{x} > 0$ when $S > S_e$ and S is near S_e).

We next consider any trajectory $(S(t), x(t))$ of (1.28) with any initial condition in B and prove that it converges asymptotically to (S_*, x_*). This will show that (S_*, x_*) is a GAS equilibrium of (1.28) with state space $\mathcal{X} = B$. Our analysis uses basic results from dynamic systems theory; see, e.g., [53, 153].

Let Ω denote the ω-limit set of this trajectory. One can easily prove that $\Omega \neq \{(S_e, 0)\}$ because

$$\frac{KS_e}{L + S_e} - D > 0.$$

We claim that $(S_e, 0) \notin \Omega$. To prove this claim, we proceed by contradiction. Suppose that $(S_e, 0) \in \Omega$. Since $\Omega \neq \{(S_e, 0)\}$, the well-known Butler-McGehee Theorem (e.g., from [153, p.12]), applied to the hyperbolic rest point $(S_e, 0)$, provides a value $S_c \neq S_e$ such that $(S_c, 0)$ belongs to Ω. This is impossible because

$$Z(t) = S(t) + x(t) - S_e \ \rightarrow \ 0.$$

Therefore $(S_e, 0) \notin \Omega$. Similarly, one can prove that there is no point of the form $(S_p, 0) \in \overline{B}$ in Ω.

Therefore, we deduce from the Poincaré-Bendixson Trichotomy [153, p.9] that either $\Omega = \{(S_*, x_*)\}$ or it is a periodic orbit which does not contain any point of the form $(S_p, 0)$. Suppose Ω is a periodic orbit, and set

$$f_1(S, \xi) = D(S_e - S) - \frac{KS}{L + S} e^{\xi} \ \text{ and } \ f_2(S, \xi) = \frac{KS}{L + S} - D.$$

Then, the system

$$\begin{cases} \dot{S} = f_1(S, \xi) \\ \dot{\xi} = f_2(S, \xi) \,, \end{cases} \tag{1.32}$$

which is deduced from (1.28) through the change of coordinate $\xi = \ln x$, also admits a periodic trajectory. On the other hand,

$$\frac{\partial f_1}{\partial S}(S, \xi) + \frac{\partial f_2}{\partial \xi}(S, \xi) < 0,$$

so Dulac's Criterion [53] implies that (1.32) admits no periodic orbit. This contradiction shows that Ω is reduced to (S_*, x_*), as claimed.

1.5.4 UGAS Time-Varying System

The one-dimensional linear time-varying system

$$\dot{x} = -\sin^2(t)x \tag{1.33}$$

admits the origin as a UGAS equilibrium point. For all $t \geq t_0$ and initial states x_0, its solutions are

$$x(t,t_0,x_0) = \exp\left(-\int_{t_0}^{t} \sin^2(m)\mathrm{d}m\right) x_0 \ ,$$

which satisfy

$$\begin{aligned} |x(t,t_0,x_0)| &= \exp\left(-\frac{1}{2}(t-t_0) + \frac{1}{4}(\sin(2t) - \sin(2t_0))\right) |x_0| \\ &\leq \beta(|x_0|, t-t_0) \end{aligned} \tag{1.34}$$

with $\beta(r,s) = e^{-\frac{1}{2}s+\frac{1}{2}}r$. The function β is of class $\mathcal{KL}$.

1.5.5 Systems in Chained Form

We have seen that the origin of the system (1.21) is not asymptotically stabilizable by any feedback of class C^1 that is independent of t. However, the origin of this system can be globally uniformly asymptotically stabilized by time-varying control laws of class C^1. To prove this, let us choose

$$\begin{aligned} u_1 &= -x_1 + \sin(t)[\cos(t)x_1 + x_2] \\ u_2 &= -\sin(t) - \cos(t). \end{aligned} \tag{1.35}$$

This choice yields the chained form system

$$\begin{cases} \dot{x}_1 &= -x_1 + \sin(t)\big[\cos(t)x_1 + x_2\big] \\ \dot{x}_2 &= \big[-\sin(t) - \cos(t)\big]\big[-x_1 + \sin(t)(\cos(t)x_1 + x_2)\big]. \end{cases} \tag{1.36}$$

It follows that the time derivative of $\zeta \doteq \cos(t)x_1 + x_2$ satisfies

$$\begin{aligned} \dot{\zeta} &= -\sin(t)x_1 + \cos(t)\big[-x_1 + \sin(t)(\cos(t)x_1 + x_2)\big] \\ &\quad + \big[-\sin(t) - \cos(t)\big]\big[-x_1 + \sin(t)(\cos(t)x_1 + x_2)\big] \\ &= -\sin^2(t)\zeta. \end{aligned} \tag{1.37}$$

We showed in Sect. 1.5.4 that for all $t \geq t_0$ and any initial state (x_{10}, x_{20}),

$$|\zeta(t,t_0,\zeta_0)| \ \leq \ \exp\left(-\frac{1}{2}(t-t_0) + \frac{1}{2}\right) |\zeta_0| \ , \tag{1.38}$$

where $\zeta_0 = \cos(t_0)x_{10} + x_{20}$. On the other hand, we have

$$\dot{x}_1 = -x_1 + \sin(t)\zeta. \tag{1.39}$$

We deduce that for all $t \geq t_0$ and any initial condition (x_{10}, x_{20}),

$$\begin{aligned} |\zeta(t,t_0,\zeta_0)| &\leq e^{-\frac{1}{2}(t-t_0)+\frac{1}{2}}|\zeta_0| \text{ and} \\ |x_1(t,t_0,x_{10},x_{20})| &\leq e^{-(t-t_0)}|x_{10}| \\ &\quad +2e^{\frac{1}{2}}\left(e^{-\frac{1}{2}(t-t_0)} - e^{-(t-t_0)}\right)|\zeta_0|. \end{aligned} \tag{1.40}$$

Therefore,

$$|x_2(t,t_0,x_{10},x_{20})| \leq e^{-(t-t_0)}|x_{10}| + 3e^{\frac{1}{2}}e^{-\frac{1}{2}(t-t_0)}(|x_{10}|+|x_{20}|), \tag{1.41}$$

because $|x_2| \leq |x_1| + |\zeta|$ everywhere. These inequalities give the announced result.

1.6 Comments

The ISS paradigm was first announced by Sontag in [156]. This was a significant development, because it merged the state space framework of Lyapunov with the input-output operator approach of Zames. ISS enjoys invariance under coordinate changes, and can be stated in various equivalent forms including energy-like estimates that generalize the standard Lyapunov decay condition. Sontag and Wang characterized ISS by proving that a system is ISS if and only if it admits an ISS Lyapunov function [169]; see our discussion on ISS Lyapunov functions in the next chapter. This characterization simplifies the task of checking that a system is ISS. Another important property of ISS is the following ISS superposition principle [168]:

Theorem 1.3. *A time-invariant system*

$$\dot{x} = f(x,u), \quad x \in \mathbb{R}^n, \;\; u \in \mathbb{R}^m \tag{1.42}$$

is ISS if and only if the following are true: its zero-system $\dot{x} = f(x,0)$ *is stable and (1.42) satisfies the asymptotic gain property.*

The asymptotic gain property is the requirement that there exists a function $\gamma \in \mathcal{K}_\infty$ such the flow map $x(t,x_0,u)$ of (1.42) satisfies

$$\limsup_{t\to+\infty} |x(t,x_0,u)| \leq \gamma(|u|_\infty)$$

for all $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{M}(\mathbb{R}^m)$. It is tempting to surmise that the GAS property of $\dot{x} = f(x,0)$ (i.e., 0-GAS of (1.42)) guarantees boundedness of all trajectories of (1.42) under disturbances that converge to 0. This is true if (1.42) is a linear time-invariant system $\dot{x} = Ax + Bu$. In fact, 0-GAS linear time-invariant systems satisfy the *converging-input converging state (CICS) property* which says that trajectories converge to zero when the inputs do [165]. However, this does not carry over to nonlinear systems because as noted in [165], the system $\dot{x} = -x + (x^2+1)u$ has divergent solutions when $u(t) = (2t+2)^{-1/2}$.

One can also give a superposition principle for iISS, using the following *bounded energy frequently bounded state (BEFBS) property* :

$$\begin{gathered}\exists \sigma \in \mathcal{K}_\infty \text{ such that :}\\ \int_0^{+\infty} \sigma(|u(s)|)\mathrm{d}s < \infty \;\;\Rightarrow\;\; \liminf_{t\to+\infty} |x(t,x_0,u)| < \infty.\end{gathered} \tag{BEFBS}$$

In fact, (1.42) is iISS if and only if it satisfies the BEFBS property and is 0-GAS [6].

During the past ten years, ISS has been generalized in several different directions. There are now notions of ISS for hybrid systems, which involve discrete and continuous subsystems and rules for switching between the subsystems [47, 48]. There are also analogs of ISS for systems with outputs

$$\dot{x} = f(x,u), \;\; y = H(x), \tag{1.43}$$

such as input-to-output stability (IOS), which is the requirement that there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that

$$|y(t)| \;\leq\; \beta\big(|x(0)|,t\big) + \gamma\big(|u|_{[0,t]}\big)$$

along all trajectories of the system [171]. One then shows that a system is IOS if and only if it admits an IOS Lyapunov function; see Sect. 6.7 for the relevant definitions and results on constructing explicit IOS Lyapunov functions for time-varying systems.

Some other output stability concepts for (1.43) include *input/output-to-state stability (IOSS)* and *output-to-state stability (OSS)* which are the requirements that there are functions $\gamma_i \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that

$$|x(t)| \;\leq\; \beta\big(|x(0)|,t\big) + \gamma_1\big(|u|_{[0,t]}\big) + \gamma_2\big(|y|_{[0,t]}\big) \tag{IOSS}$$

and

$$|x(t)| \;\leq\; \beta\big(|x(0)|,t\big) + \gamma_3\big(|y|_{[0,t]}\big) \tag{OSS}$$

along all trajectories of (1.43), respectively. The IOSS and OSS properties can be characterized in terms of the existence of Lyapunov functions as well [73]. Input-measurement-to-error stability (IMES) is a significant generalization of ISS for systems

$$\dot{x} = f(x,u), \;\; y = h(x), \;\; w = g(x) \tag{1.44}$$

with error outputs $y = h(x)$ and measurement outputs $w = g(x)$ [165]. The IMES property says that there exist $\beta \in \mathcal{KL}$ and functions $\sigma, \gamma \in \mathcal{K}$ such that

$$|y(t)| \;\leq\; \beta\big(|x(0)|,t\big) + \sigma\big(|w|_{[0,t]}\big) + \gamma\big(|u|_{[0,t]}\big) \tag{IMES}$$

along all trajectories of (1.44). However, to our knowledge, there is no smooth Lyapunov characterization for IMES available.

Backstepping is discussed in detail in [149]. See also Chap. 7. Some pioneering results on backstepping include [19, 31, 179].

The proof of Brockett's Stabilization Theorem uses basic facts from degree theory, combined with a homotopy argument. Here is a sketch of the proof; see [161, Sect. 5.9] for details. Using degree theory results from [15], one first proves the following:

Lemma 1.3. *Let $\rho > 0$ be a given constant and $H : [0,1] \times \rho\mathcal{B}_n \to \mathbb{R}^n$ be a continuous function such that the following hold:*

1. $H(1,x) = -x$ for all x; and
2. $H(t,x) \neq 0$ for all $x \in \mathrm{boundary}(\rho\mathcal{B}_\mathrm{n})$.

Then there is a constant $\varepsilon > 0$ such that the image of $\rho\mathcal{B}_n \ni x \mapsto H(0,x)$ contains $\varepsilon\mathcal{B}_n$.

Brockett's Theorem follows by applying Lemma 1.3 to

$$H(t,x) \doteq \begin{cases} f(x,u_s(x)), & \text{if } t=0 \\ -x, & \text{if } t=1 \\ \frac{1}{t}\left[\phi\left(\frac{t}{1-t},x\right)-x\right], & \text{if } 0<t<1 \end{cases}$$

where ϕ is the flow map for the closed-loop system $\dot{x} = f(x,u_s(x))$ and $\rho > 0$ is chosen so that $\rho\mathcal{B}_n$ is in the domain of attraction of the closed-loop system. Brockett's Criterion is a far reaching result because it implies that no system of the form

$$\dot{x} = u_1 g_1(x) + \ldots + u_m g_m(x) = G(x)u$$

with $m < n$ and

$$\mathrm{rank}[g_1(0),\ldots,g_m(0)] \ = \ m$$

admits a C^1 pure state stabilizing feedback $u_s(x)$; see [163] for the simple proof. Hence, no totally nonholonomic mechanical system is C^1 stabilizable by a pure state feedback.

In [146], Samson provided important general results that use time-varying feedback to help overcome the obstructions imposed by Brockett's Criterion. See also [65], which uses backstepping to build a time-varying feedback stabilizer for a two degrees-of-freedom mobile robot. By [27], the system $\dot{x} = f(x,u)$ is stabilizable by a time-varying continuous feedback $u = k(t,x)$ when it is drift free (meaning $f(x,0) \equiv 0$) and completely controllable.

Another approach to stabilizing the system is to look for a *dynamic* stabilizer, meaning a locally Lipschitz dynamics

$$\dot{z} = A(z,x)$$

and a locally Lipschitz function $k(z,x)$ such that the combined system

$$\begin{cases} \dot{x} = f(x, k(z,x)) \\ \dot{z} = A(z,x) \end{cases}$$

is GAS. See [160] for a detailed discussion on dynamic stabilizers for linear systems. However, a dynamic feedback for $\dot{x} = f(x,u)$ may fail to exist, even if the system is completely controllable. An example from [173] where this happens is

$$\dot{x} = f(x,u) = \begin{bmatrix} (4 - x_2^2)u_2^2 \\ e^{-x_1} + x_2 - 2e^{-x_1}\sin^2(u_1) \end{bmatrix}, \quad x \in \mathbb{R}^2, \ u \in \mathbb{R}^2. \tag{1.45}$$

The fact that (1.45) is completely controllable (and therefore GAC to $\mathcal{A} = \{0\}$) was shown in [173], which also shows that it is impossible to pick paths converging to the origin in such a way that this selection is continuous as a function of the initial states. Since the flow map of any dynamic stabilizer would give a continuous choice of paths converging to 0, no dynamic stabilizer for (1.45) can exist, even if we drop the requirement that the state of the regulator converges to zero. As a special case, (1.45) cannot admit a continuous time-varying feedback $u = k(t,x)$. This does not contradict the existence theory [27] for time-varying feedbacks because (1.45) has drift.

Yet another approach to circumventing the "virtual obstacles" to feedback stabilization imposed by Brockett's Condition involves nonsmooth analysis and discontinuous feedbacks. See for example [94] where a nonsmooth (but time-invariant) feedback was constructed for Brockett's Nonholonomic Integrator using a generalized Lie derivative, which involves a proximal subgradient [22] and a semi-concave control-Lyapunov function (CLF). Discontinuous feedbacks complicate the analysis because they give differential equations with discontinuous right hand sides. Discontinuous dynamics can sometimes be handled using Filippov solutions, sample-and-hold solutions, or Euler solutions [94, 162].

In addition to "virtual" obstacles, there are also "topological" obstacles to time-invariant feedback stabilization. If a time-invariant system $\dot{x} = f(x,u)$ evolving on some manifold $\mathcal{M}$ is globally asymptotically controllable to a singleton equilibrium and has a continuous stabilizing feedback $k(x)$, then Milnor's Theorem [115] implies that $\mathcal{M}$ is diffeomorphic to Euclidean space. This follows because $k(x)$ would guarantee the existence of a smooth CLF that could be taken as a Morse function with a unique critical point, and manifolds admitting such Morse functions are known to be diffeomorphic to Euclidean space [163].