1. Disturbance attenuation for discrete-time feedforward nonlinear systems

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Summary.

In this paper the inverse optimal stabilization problem is solved for nonlinear nonaffine control discrete-time systems which are globally stable when uncontrolled. Stabilizing feedback laws and nonquadratic cost functionals are constructed. The result is applied to feedforward systems.

1.1 Introduction

Inverse optimal control problems for linear system have been studied more than thirty years ago by Kalman [7] and next by Anderson and Moore [1]. However, inverse optimality for nonlinear systems is a more recent subject studied by Moylan and Anderson [16] and later by Freeman and Kokotović [2], [3]. Other works have followed [5, Section 3.5], [4, 8, 9]. This area of the nonlinear control theory arises from the wish to determine for nonlinear systems stabilizing feedbacks having the good performances of those resulting from the optimization theory without having to solve the Hamilton-Jacobi-Bellman equation which is not always a feasible task. In the inverse optimal approach, a stabilizing feedback is designed first and then it is shown to be optimal for a cost functional of the form

$$\int_0^{\infty} \left(l(x) + u^\top R(x) u \right) dt$$

where $l(x)$ is positive and and $R(x)$ is positive definite. In other words, the functions $l(x)$ and $R(x)$ are *a posteriori* determined from the data of a particular stabilizing feedback, rather than *a priori* chosen by the designer i.e. regardless of any stabilizing feedback.

So far, to the best knowledge of our, all the works on this subject are concerned with continuous-time systems. The purpose of the present paper is to address the problem of inverse optimal stabilization for general classes of nonlinear discrete-time systems. More precisely, its aim is twofold. In a first part, we show that the stabilizing feedbacks designed in [13, Appendix A] and in [14, Section 2] for nonlinear discrete-time systems

$$x_{i+1} = f(x_i) + g(x_i, u_i)u_i \tag{1.1}$$

with $f(0) = 0$, $x \in \mathbf{R}^n$ and $u \in \mathbf{R}$ which are globally stable when $u = 0$ also minimize cost functionals of the form

$$J = \sum_{i=0}^{\infty} \left(\alpha(x_i, u_i) + \beta(x_i, u_i)|u_i|^2 \right) .$$

The functions $\alpha(x, u)$ and $\beta(x, u)$ of the cost functional we consider depend on u, although in inverse optimal control for continous-time systems, no such cost functional are in general considered. This is due to the specificities of discrete-time systems and in Section 1.3 we show how to infer from these costs (ater a slight modification) disturbance attenuation properties. In a second part, we apply the results of the first to a particular class of discrete-time feedforward systems i.e. systems which admit a representation of the form:

$$\begin{aligned} z_{i+1} &= \mathcal{F}(z_i) + \psi(z_i, \xi_i) + g_2(z_i, \xi_i, u_i)u_i + m_1(z_i, \xi_i, u_i, d_i)d_i \\ \xi_{i+1} &= a(\xi_i) + g_1(z_i, \xi_i, u_i)u_i + m_2(z_i, \xi_i, u_i, d_i)d_i \end{aligned}$$

where $m_1(z_i, \xi_i, u_i, d_i)d_i$ and $m_2(z_i, \xi_i, u_i, d_i)d_i$ are disturbances, where $z_{i+1} = \mathcal{F}(z_i)$ is globally stable, $\xi_{i+1} = a(\xi_i)$ is globally asymptotically stable and locally exponentially stable $\psi(z, 0) = 0$ for all z. Observe that the study of this class of systems is, from a practical point of view, appealing. On the one hand, many physical systems are described by feedforward equations (Cart-pendulum system, Ball and Beam with friction term, PVTOL) and, on the other hand the technique provides us with bounded feedbacks for null-controllable linear systems. These systems, in continuous-time, have been studied for the first time by A. Teel in [19] where a family of stabilizing feedbacks is displayed. Many extentions this pioneer work have followed: see [13, 5, 15, 12, 18] where new family of controls, control Lyapunov functions and output feedback results are given. A discrete-time version of the main result of [13] is proved in [14]. In continuous time, it is shown in [17, Section 6.2.2] that the forwarding design applied to affine systems has stability margins. This result is proved via inverse optimal results. This result has no direct equivalent in discrete-time because, on the one hand, even affine discrete-time feedforward systems cannot be rendered passive in the classical sense and, on the other hand, only bounded stabilizing feedbacks are available: see [14]. However, our work owes a great deal to the technique of proof of this result as long as to those of [8], where is stressed the link there is between inverse optimal control and disturbance attenuation, and [20] where a disturbance attenuation result for continuous-time feedforward system is given. To consider a slightly larger class of systems than the one studied in [14], we propose a discrete-time version of the main result of [5] to construct a Lyapunov function enabling us to prove disturbance attenuation properties of the closed-loop systems.

Preliminaries and definitions

1. Throughout the paper we assume that the functions encountered are sufficiently smooth.

2. We denote by χ_i the solution of the discrete-time system:

$$\chi_{i+1} = \mathcal{H}(\chi_i) \tag{1.2}$$

with the initial condition $\chi_0 = \chi$.

3. For a sequence χ_i solution of (1.2) and a function $\mathcal{V}(\chi)$ we denote by $\Delta\mathcal{V}$ the term $\mathcal{V}(\chi_{i+1}) - \mathcal{V}(\chi_i)$.

4. A function $\mathcal{V}(\chi_i)$ is positive definite if

$$\mathcal{V}(\chi) > 0 \, , \, \forall \chi \neq 0 \, .$$

5. The *inverse optimal stabilization problem* for discrete-time systems (1.1) is solvable if there exist positive real-valued functions $\alpha(x, u)$ and $\beta(x, u)$ such that there exists a feedback law $u(x)$ which globally asymptotically stabilizes (1.1) and at the same time minimizes the cost functional

$$J = \int_0^\infty (\alpha(x, u) + \beta(x, u)u^2)dt.$$

1.2 Inverse optimal control

In this section, we study the inverse optimal problem for the discrete-time nonlinear systems (1.1) and introduce the following assumptions:

- **H1.** There exists a proper, positive definite function $V(x)$ and such that $V(f(x)) - V(x) \leq 0$.
- **H1'.** There exists a positive definite function $V(x)$ such that $V(f(x)) - V(x) \leq 0$.

H2. The sets

$$\Omega = \{x \in \mathbf{R}^n : V(f^{i+1}(x)) = V(f^i(x)) \ , \ i = 0, 1, 2, \dots\}$$

$$S = \{x \in \mathbf{R}^n : \frac{\partial V}{\partial x}(f^{i+1}(x))g(f^i(x), 0) = 0 \ , \ i = 0, 1, 2, \dots\}$$

are such that

$$\Omega \bigcap S = \{0\}.$$

Remark. The system (1.1) is a single-input system. Generalizations of our results to the case of multi-input systems can be carried out but it turns out that the proofs are then much more intricate.

1.2.1 Globally asymptotically stabilizing feedback

Let us recall a result which is an immediate consequence of [13, Lemma II.4] or of the feedback design of [14, Section2] and is an extension of [11, Corollary 3.1].

Theorem 1.2.1. *Consider the discrete-time systems (1.1). Assume that Assumptions H1 and H2 are satisfied. Then for all function* $\mu(x) > 0$, *there exists a smooth function* $\phi(x)$ *such that the following feedback control*

$$\overline{u}(x) = -\phi(x)h(x, 0), \qquad 0 < \phi(x) \leq \mu(x) \tag{1.3}$$

$$h(x, u) = \int_0^1 \frac{\partial V}{\partial x}(f(x) + g(x, u)u\theta)g(x, u)d\theta \tag{1.4}$$

globally asymptotically stabilizes the system (1.1).

Theorem 1.2.2. *Consider the discrete-time systems (1.1). Assume that Assumptions H1' and H2 are satisfied. Then, for all function* $\mu(x) > 0$ *such that there exists a smooth function* $\phi(x)$ *such that all the solutions of (1.1) in closed-loop with the following feedback control*

$$\overline{u}(x) = -\phi(x)h(x, 0), \qquad 0 < \phi(x) \leq \mu(x) \tag{1.5}$$

$$h(x, u) = \int_0^1 \frac{\partial V}{\partial x}(f(x) + g(x, u)u\theta)g(x, u)d\theta \tag{1.6}$$

are bounded, the system (1.1) is globally asymptotically stabilizes by the feedbacks (1.5).

Discussion of Theorem 1.2.1 and Theorem 1.2.2.

i) Observe that the feedbacks (1.3), (1.5) are given by explicit formulas and not as the implicit solutions of nonlinear algebraic equations which do not necessarily admit a solution as those proposed in [10] are.

ii) Theorem 1.2.1 is a discrete-time nonaffine version of the Jurdjevic-Quinn theorem [6]: Assumption H1 guarantees the global stability of (1.1) with $u = 0$ and the technical Assumption H2 guarantees that a detectability property which allows to conclude by invoking the LaSalle invariance principle with arbitrarily small feedbacks is satisfied.

iii) The main difference there is between Theorem 1.2.1 and Theorem 1.2.2 is clear: the first theorem requires the knowledge of a proper function $V(x)$ but not the second.

1.2.2 Optimal criterion design for discrete-time systems

Let us state the main result.

Theorem 1.2.3. *Consider the system (1.1). Assume that the assumptions of Theorem 1.2.1 or Theorem 1.2.2 are satisfied. Then, for all function $\mu(x) > 0$ the inverse optimal stabilization problem is solved by the control law*

$$\overline{u}(x) \ = \ -\phi(x)h(x,0), \qquad 0 < \phi(x) \leq \mu(x) \tag{1.7}$$

with $h(x,u)$ given in (1.4) and the cost functional

$$J \ = \ \sum_{i=0}^{\infty}(\alpha(x_i,u_i) + \beta(x_i,u_i)u_i^2) \tag{1.8}$$

with

$$\alpha(x,u) \ = \ V(x) - V(f(x)) + \frac{1}{2}\phi(x)\rho(u)h(x,0)^2 \tag{1.9}$$

$$\beta(x,u) \ = \ k(x,u) + \frac{\rho(u)}{2\phi(x)} - h(x,0)\frac{1-\rho(u)}{u} \tag{1.10}$$

where $\rho(u)$ is a strictly positive function given by an explicit formula and such that $\rho(0) = 1$ and $k(x,u)$ is defined by

$$-h(x,u) \ = \ -h(x,0) + k(x,u)u \ . \tag{1.11}$$

Proof. Theorem 1.2.1 or Theorem 1.2.2 provide us with a globally asymptotically stabilizing feedback $\overline{u}(x)$ of the form (1.3) or (1.5). Let us consider the following criterion

$$S = -\sum_{i=0}^{\infty}(V(x_{i+1}) - V(x_i)) + \sum_{i=0}^{\infty}\frac{\rho(u_i)}{2\phi(x_i)}(u_i - \overline{u}(x_i))^2 \ .$$

Observe that, since $\rho(u)$ is a positive function, $u = \overline{u}(x)$ minimizes S because, when a globally asymptotically feedback is applied,

$$S \ = \ V(x_0) + \sum_{i=0}^{\infty}\frac{\rho(u_i)}{2\phi(x_i)}(u_i - \overline{u}(x_i))^2 \ .$$

The function S rewrites as:

$$\begin{aligned} S = &-\sum_{i=0}^{\infty}(V(f(x_i)) - V(x_i)) - \sum_{i=0}^{\infty}h(x_i,u_i)u_i \\ &+\sum_{i=0}^{\infty}\left[\frac{\rho(u_i)}{2\phi(x_i)}u_i^2 - \frac{\rho(u_i)}{\phi(x_i)}\overline{u}(x_i)u_i + \frac{\rho(u_i)}{2\phi(x_i)}\overline{u}(x_i)^2\right] \ . \end{aligned}$$

Using (1.11), we get:

$$\begin{aligned} S = &-\sum_{i=0}^{\infty}(V(f(x_i)) - V(x_i)) \\ &+\sum_{i=0}^{\infty}k(x_i,u_i)u_i^2 - \sum_{i=0}^{\infty}h(x_i,0)u_i \\ &+\sum_{i=0}^{\infty}\left[\frac{\rho(u_i)}{2\phi(x_i)}u_i^2 + h(x_i,0)\rho(u_i)u_i + \frac{\rho(u_i)}{2\phi(x_i)}\overline{u}(x_i)^2\right] \ . \end{aligned}$$

Regrouping the terms differently, we obtain:

$$\begin{aligned} S = &\sum_{i=0}^{\infty}\left[(V(x_i) - V(f(x_i))) + \frac{1}{2}\phi(x_i)\rho(u_i)h(x_i,0)^2\right] \\ &+\sum_{i=0}^{\infty}\left[k(x_i,u_i) + \frac{\rho(u_i)}{2\phi(x_i)} - h(x_i,0)\frac{1-\rho(u_i)}{u_i}\right]u_i^2 \ . \end{aligned}$$

Therefore $S = J$ where J is the function defined in (1.8). Assumption **H1** guarantees that $\alpha(x,u)$ is positive. The proof is completed by using the following lemma

Lemma 1.2.1. *Functions $\phi(x)$ and $\rho(u)$ such that $\beta(x,u)$ is a strictly positive function can be determined.*

Proof. Since $\rho(u)$ is smooth and such that $\rho(0) = 1$, the function $\left|\frac{1-\rho(u)}{u}\right|$ is bounded on a neighborhood of the origin. It follows readily that the expression of a function $\phi(x)$ such that, for all x, u

$$\frac{\rho(u)}{4\phi(x)} - h(x,0)\frac{1-\rho(u)}{u} \;\geq\; 0$$

can be given. Next, let us prove that we may determine $\phi(x)$ and $\rho(u)$ so that

$$k(x,u) + \frac{\rho(u)}{4\phi(x)} \geq 0. \tag{1.12}$$

Determining explicit formulas of positive functions $\lambda(x), \mu(u)$ such that $|k(x,u)| \leq \lambda(x)\mu(u)$ and $\mu(0) \leq 1$ is always a feasible task. When this inequality holds, (1.12) is satisfied if

$$\frac{1}{\phi(x)} \;\geq\; 4\lambda(x)\frac{1}{\rho(u)}\mu(u).$$

which is met with $\phi(x) \leq \frac{1}{4\lambda(x)+1}$ and $\rho(u) \geq \mu(u)$. This allows us to conclude our proof.

1.3 Disturbance attenuation for discrete-time systems

1.3.1 Inverse optimal $\mathcal{H}_\infty$ problem

In this part we consider the system

$$x_{i+1} \;=\; f(x_i) + g(x_i,u_i)u_i + m(x_i,u_i,d_i)d_i \tag{1.13}$$

with $f(0) = 0$ and $x \in \mathbf{R}^n$, $u \in \mathbf{R}$, $d \in \mathbf{R}$. We assume that $m(x,u,d)$ is known and that d_i is an unknown sequence of class L^2.

Define a function $h_d(x,u,d)$ as:

$$\begin{aligned} h_d(x,u,d)d = & \int_0^1 \Big[\frac{\partial V}{\partial x}(f(x)+g(x,u)u\theta+m(x,u,d)d\theta) \\ & \qquad \times g(x,u)\Big]\, d\theta \\ & - \int_0^1 \left[\frac{\partial V}{\partial x}(f(x)+g(x,u)u\theta)g(x,u)\right] d\theta \\ & + \int_0^1 \Big[\frac{\partial V}{\partial x}(f(x)+g(x,u)u\theta+m(x,u,d)d\theta) \\ & \qquad \times m(x,u,d)d\Big]\, d\theta \end{aligned} \tag{1.14}$$

and $k_d(x,u,d)$ by

$$-\,h_d(x,u,d) \;=\; -\,h_d(x,u,0) \,+\, k_d(x,u,d)d \tag{1.15}$$

Lemma 1.3.1. *Consider the system (1.13). Assume that the assumptions of Theorem 1.2.1 or Theorem 1.2.2 are satisfied. Then, for all function $\mu(x) > 0$ the following problem is solved:*

The control law

$$\overline{u}(x) = -\phi(x)h(x,0) , \qquad 0 < \phi(x) \le \mu(x) \tag{1.16}$$

with $h(x,u)$ given in (1.4) minimizes the cost functional

$$J_d = \sup_{\{d_i \in \mathcal{D}\}} \sum_{i=0}^{\infty} \left(\alpha(x_i,u_i) + \beta(x_i,u_i)u_i^2 - \beta_d(x_i,u_i,d_i)d_i^2\right) \tag{1.17}$$

with $\mathcal{D}$ the set such that all the solutions of the system (1.1) in closed-loop with $\overline{u}(x)$ with distubances $(d_i) \in \mathcal{D}$ go to the origin with

$$\begin{aligned}\alpha(x,u) = V(x) - V(f(x)) + \tfrac{1}{2}\phi(x)\rho(u)h(x,0)^2 \\ +\tfrac{1}{2}\phi_d(x,u)h_d(x,u,0)^2\end{aligned} \tag{1.18}$$

$$\beta(x,u) = k(x,u) + \frac{\rho(u)}{2\phi(x)} - h(x,0)\frac{1-\rho(u)}{u} \tag{1.19}$$

where $\rho(u)$ is a strictly positive function given by an explicit formula and such that $\rho(0) = 1$, $k(x,u)$ is the function defined in (1.11),

$$\begin{aligned}\beta_d(x,u,d) = -k_d(x,u,d) + \tfrac{\rho_d(d)}{2\phi_d(x,u)} \\ -h_d(x,u,0)\tfrac{1-\rho_d(d)}{d} + \tfrac{\rho_d(d)-1}{2d^2\phi_d(x,u)}\overline{d}(x,u)^2\end{aligned} \tag{1.20}$$

where $\rho_d(d)$ is a strictly positive function given by an explicit formula and such that $\rho(0) = 1$, $k_d(x,u,d)$ and $h_d(x,u,d)$ are the functions given in respectively (1.15), (1.14) and $\overline{d}(x,u) = \phi_d(x,u)h_d(x,u,0)$ where $\phi_d(x,u)$ is a stricly positive function given by an explicit formula.

Proof. Theorem 1.2.1 or Theorem 1.2.2 provide us with a globally asymptotically stabilizing feedback $\overline{u}(x)$ of the form (1.3) or (1.5) when $d_i = 0$ for all i. Similarly, let

$$\overline{d}(x,u) = \phi_d(x,u)h_d(x,u,0) \tag{1.21}$$

where $\phi_d(x,u)$ is a stricly positive function to be chosen later. Let us consider the following criterion

$$\begin{aligned}S_d = -\sum_{i=0}^{\infty}(V(x_{i+1}) - V(x_i)) + \sum_{i=0}^{\infty}\frac{\rho(u_i)}{2\phi(x_i)}(u_i - \overline{u}(x_i,u_i))^2 \\ -\sum_{i=0}^{\infty}\frac{\rho_d(d_i)}{2\phi_d(x_i,u_i)}(d_i - \overline{d}(x_i,u_i))^2 \ .\end{aligned}$$

Observe that, since $\rho(u)$ is a positive function, $u = \overline{u}(x)$ minimizes S when $(d_i) \in \mathcal{D}$ because, when a globally asymptotically stabilizing feedback is applied,

$$S_d = V(x_0) + \sum_{i=0}^{\infty} \frac{\rho(u_i)}{2\phi(x_i)}(u_i - \overline{u}(x_i, u_i))^2 - \sum_{i=0}^{\infty} \frac{\rho_d(d_i)}{2\phi_d(x_i, u_i)}(d_i - \overline{d}(x_i, u_i))^2 .$$

The function S_d rewrites as:

$$\begin{aligned} S_d = &- \sum_{i=0}^{\infty} (V(f(x_i)) - V(x_i)) - \sum_{i=0}^{\infty} h(x_i, u_i)u_i - \sum_{i=0}^{\infty} h_d(x_i, u_i, d_i)d_i \\ &+ \sum_{i=0}^{\infty} \left[\frac{\rho(u_i)}{2\phi(x_i)} u_i^2 - \frac{\rho(u_i)}{\phi(x_i)} \overline{u}(x_i)u_i + \frac{\rho(u_i)}{2\phi(x_i)} \overline{u}(x_i)^2 \right] \\ &- \sum_{i=0}^{\infty} \left[\frac{\rho_d(d_i)}{2\phi_d(x_i, u_i)} d_i^2 - \frac{\rho_d(d_i)}{\phi_d(x_i, u_i)} \overline{d}(x_i, u_i)d_i + \frac{\rho_d(d_i)}{2\phi_d(x_i, u_i)} \overline{d}(x_i, u_i)^2 \right] . \end{aligned}$$

According to (1.11) and (1.15), we have:

$$\begin{aligned} S_d = &- \sum_{i=0}^{\infty} (V(f(x_i)) - V(x_i)) + \sum_{i=0}^{\infty} k(x_i, u_i)u_i^2 - \sum_{i=0}^{\infty} h(x_i, 0)u_i \\ &+ \sum_{i=0}^{\infty} \left[\frac{\rho(u_i)}{2\phi(x_i)} u_i^2 + h(x_i, 0)\rho(u_i)u_i + \frac{\rho(u_i)}{2\phi(x_i)} \overline{u}(x_i)^2 \right] \\ &+ \sum_{i=0}^{\infty} k_d(x_i, u_i, d_i)d_i^2 - \sum_{i=0}^{\infty} h_d(x_i, u_i, 0)d_i \\ &- \sum_{i=0}^{\infty} \left[\frac{\rho_d(d_i)}{2\phi_d(x_i, u_i)} d_i^2 - \frac{\rho_d(d_i)}{\phi_d(x_i, u_i)} \overline{d}(x_i, u_i)d_i + \frac{\rho_d(d_i)}{2\phi_d(x_i, u_i)} \overline{d}(x_i, u_i)^2 \right] . \end{aligned}$$

Regrouping the terms differently, we obtain:

$$\begin{aligned} S_d = \sum_{i=0}^{\infty} &\left[(V(x_i) - V(f(x_i))) + \frac{1}{2}\phi(x_i)\rho(u_i)h(x_i, 0)^2 \right. \\ &\left. + \frac{\rho_d(0)}{2\phi_d(x_i, u_i)} \overline{d}(x_i, u_i)^2 \right] \\ &+ \sum_{i=0}^{\infty} \left[k(x_i, u_i) + \frac{\rho(u_i)}{2\phi(x_i)} - h(x_i, 0)\frac{1 - \rho(u_i)}{u_i} \right] u_i^2 \\ &- \sum_{i=0}^{\infty} \left[-k_d(x_i, u_i, d_i) + \frac{\rho_d(d_i)}{2\phi_d(x_i, u_i)} \right. \\ &\left. - h_d(x_i, u_i, 0)\frac{1 - \rho_d(d_i)}{d_i} + \frac{\rho_d(d_i) - 1}{2d_i^2\phi_d(x_i, u_i)} \overline{d}(x_i, u_i)^2 \right] d_i^2 . \end{aligned}$$

Therefore $S = J$ where J is the function defined in (1.8). Assumption **H1** guarantees that $\alpha(x,u)$ is positive. The proof is completed by using the following lemma

Lemma 1.3.2. *Functions $\phi_d(x,u)$ and $\rho_d(d)$ such that $\beta_d(x,u,d)$ is a strictly positive function can be determined.*

Proof. The proof is similar to the proof of Lemma 1.2.1.

1.3.2 An $\mathcal{L}_2$ disturbance attenuation result

In this section, we introduce assumptions ensuring that for a system (1.13) all the sequences $d_i \in L^2$ belong to $\mathcal{D}$.

H3. There exists a function $B(x,u)$ such that

$$|\beta_d(x,u,d)| \leq B(x,u)$$

H4. There exists $c > 0$ such that,

$$\alpha(x,\overline{u}(x)) + \beta(x,\overline{u}(x))\overline{u}(x)^2 \geq c\frac{|x|^2}{1+|x|^2} \tag{1.22}$$

We state the main result of the section.

Theorem 1.3.1. *Assume that the system (1.13) satisfies the assumption of Theorem 1.2.3 when $d_i = 0$ for all i and the assumptions H3 and H4. Then there exists $C > 0$ such that for all sequence $d_i \in L^2$, the solution of (1.13) with $x_0 = 0$ satisfies:*

$$\sum_{i=0}^{\infty} |x_i|^2 \leq C \sum_{i=0}^{\infty} d_i^2$$

Proof. When $u = \overline{u}(x)$,

$$\begin{aligned} J_d = & \sup_{\{d_i \in \mathcal{D}\}} \sum_{i=0}^{\infty} \big[\alpha(x_i,\overline{u}(x_i)) + \beta(x_i,\overline{u}(x_i))\overline{u}(x_i)^2 \\ & \qquad -\beta_d(x_i,\overline{u}(x_i),d_i)d_i^2\big] \\ = & V(x_0) \end{aligned} \tag{1.23}$$

So when the initial condition is at the origin, we deduce that for all $(d_i) \in \mathcal{D}$,

$$\sum_{i=0}^{\infty} \left[\alpha(x_i,\overline{u}(x_i)) + \beta(x_i,\overline{u}(x_i))\overline{u}(x_i)^2\right] \leq \sum_{i=0}^{\infty} \beta_d(x_i,\overline{u}(x_i),d_i)d_i^2 \tag{1.24}$$

Using Assumption H3 and Assumption H4, we deduce that there exists $C > 0$ (which can be explicitly determined) such that

$$\sup_{\{d_i \in \mathcal{D}\}} \sum_{i=0}^{\infty} |x_i|^2 \ \leq \ C \sup_{\{d_i \in \mathcal{D}\}} \sum_{i=0}^{\infty} d_i^2$$

Moreover, according to Assumption H4 and the fact that $|\beta_d(x, \overline{u}(x), d)|$ is smaller than a function independent from d, all the sequences $d_i \in L_2$ belong to $\mathcal{D}$.

This concludes our proof.

1.4 Feedforward discrete-time nonlinear systems

In this section, we particularize the results of Section 1.3 to the class of the discrete-time feedforward systems i.e. systems having the following representation:

$$\begin{aligned} z_{i+1} &= \mathcal{F}(z_i) + \psi(z_i, \xi_i) + g_1(z_i, \xi_i, u_i)u_i \\ \xi_{i+1} &= a(\xi_i) + g_2(z_i, \xi_i, u_i)u_i \end{aligned} \tag{1.25}$$

with $z_i \in \mathbf{R}^{n_z}$, $\xi_i \in \mathbf{R}^{n_\xi}$, $u \in \mathbf{R}$. We introduce the following assumptions

A1. There exists a proper, positive definite function $W_1(\cdot)$ which is zero at the origin and such that for all z,

$$W_1(\mathcal{F}(z)) - W_1(z) \ \leq \ 0 \ .$$

A2. There exist a function $W_2(\cdot)$ positive definite radially unbounded zero at the origin and a function $\nu(\cdot)$ positive definite and zero at the origin such that

$$W_2(a(\xi)) - W_2(\xi) \ \leq \ -\nu(\xi) \ .$$

Moreover both $W_2(\cdot)$ and $\nu(\cdot)$ are lower bounded on a neighborhood of the origin by a positive definite quadratic function.

A3. There exist two differentiable positive functions $\gamma_0(\xi)$ and $\gamma_1(\xi)$ zero at the origin and such that

$$|\psi(z, \xi)| \ \leq \ \gamma_0(\xi) + \gamma_1(\xi) W_1(z)$$

A4. The following inequality is satisfied:

$$\left| \frac{\partial W_1}{\partial z}(z) \right| \ \leq \ 1 \ .$$

Discussion of the assumptions.

• The assumptions A1 to A4 are the standard assumptions of the forwarding approach. We conjecture that they can be relaxed in the time-varying context as they are relaxed in [15] in the continuous-time context.

• The family of systems (1.25) is slightly larger than the one studied in [14] since it is not required on $z_{i+1} = \mathcal{F}(z_i)$ to be linear.

• As an immediate consequence of A2, we have that the system $\xi_{i+1} = a(\xi_i)$ is globally asymptotically stable and locally exponentially stable.

• If is known a function $\mathcal{W}_1(z)$ such that

$$\left| \frac{\partial \mathcal{W}_1}{\partial z}(z) \right| \leq \mathcal{L}(\mathcal{W}_1(z))$$

where $\mathcal{L}(\cdot)$ is a positive function such that $\frac{1}{1+\mathcal{L}(\cdot)} \notin L^1$, then Assumption A4 is satisfied by

$$W_1(z) = \int_0^{\mathcal{W}_1(z)} \frac{1}{\mathcal{L}(s)+1} ds \ .$$

If $\mathcal{W}_1(z)$ is a quadratic form, the corresponding function $W_1(z)$ satisfies a linear growth property and Assumption A3 is a linear growth assumption imposed on the coupling term.

In order to derive inverse optimal controls from the result of the previous part, we first prove that the assumptions A1 to A4 ensure the Lyapunov stability of the free system associated with (1.25). We construct a candidate Lyapunov function depending on the given functions $W_1(\cdot)$ and $W_2(\cdot)$.

The construction we adopt mimics the one proposed in [5] (see also [15]) for continuous-time feedforward systems. An alternative construction is given in [14] in a slightly more restrictive context.

1.4.1 Stability of the uncontrolled system

Let us introduce the notations: $x = (z, \xi)^\top$,

$$f(z,\xi) = \begin{pmatrix} \mathcal{F}(z) + \psi(z,\xi) \\ a(\xi) \end{pmatrix},$$

$$g(z,\xi,u) = \begin{pmatrix} g_1(z,\xi,u) \\ g_2(z,\xi,u) \end{pmatrix} .$$

Theorem 1.4.1. *Assume that the system (1.25) satisfies the Assumptions* **A1** *to* **A4**. *Then this system is Lyapunov stable when $u = 0$ and there exists a function zero at the origin, positive definite and of the form*

$$V(x) \;=\; W_1(z) + \varPhi(z,\xi) + W_2(\xi) \tag{1.26}$$

such that $\varDelta V \;\le\; 0$ *when* $u = 0$.

Remark. One may easily deduce from the forthcoming proof a family of Lyapunov functions for (1.25): for all real-valued functions $k(\cdot), l(\cdot)$ of class $\mathcal{K}^\infty$ there exists a cross-term $\varPhi_{kl}(z,\xi)$ such that $V_{kl}(x) \;=\; l(W_1(z)) + \varPhi_{kl}(z,\xi) + k(W_2(\xi))$ is proper, positive definite zero at zero and such that $\varDelta V_{kl} \;\le\; 0$.

Proof. First, let us determine a cross term function $\varPhi(z,\xi)$ such that the candidate Lyapunov function given in (1.26) satisfies $\varDelta V \;\le\; 0$. The expression of the variation $\varDelta V$ is

$$\begin{aligned}\varDelta V = {} & W_1(\mathcal{F}(z_i) + \psi(z_i,\xi_i)) - W_1(z_i) \\ & + \varPhi(z_{i+1},\xi_{i+1}) - \varPhi(z_i,\xi_i) + W_2(a(\xi_i)) - W_2(\xi_i)\ .\end{aligned}$$

Since for all z, ξ

$$W_1(\mathcal{F}(z) + \psi(z,\xi)) - W_1(\mathcal{F}(z)) \;=\; \int_0^1 \left(\frac{\partial W_1}{\partial z}(\mathcal{F}(z) + \psi(z,\xi)\theta)\psi(z,\xi) \right) d\theta$$

Assumption A1 implies that $\varDelta V$ is negative if

$$\varPhi(z_{i+1},\xi_{i+1}) - \varPhi(z_i,\xi_i) \;=\; -\int_0^1 \left(\frac{\partial W_1}{\partial z}(\mathcal{F}(z_i) + \psi(z_i,\xi_i)\theta)\psi(z_i,\xi_i) \right) d\theta\ .$$

Let us denote the right hand side of this expression by $-q(z_i,\xi_i)$. It straightforwardly follows from (1.27) that

$$\varPhi(z,\xi) \;=\; \sum_{i=0}^{\infty} q(z_i,\xi_i) \tag{1.27}$$

provided that the right hand side of (1.27) is a well-defined function.

The next part of the proof consists of showing that this power series converges. From the definition of $q(z,\xi)$ and Assumptions A3 and A4 we successively obtain:

$$\begin{aligned}|q(z,\xi)| &\le \left| \int_0^1 \frac{\partial W_1}{\partial z}(\mathcal{F}(z) + \psi(z,\xi)\theta)\psi(z,\xi) d\theta \right| \\ &\le |\psi(z,\xi)| \;\le\; \gamma_0(\xi) + \gamma_1(\xi) W_1(z)\ .\end{aligned}$$

From Assumption A2, we deduce that there exists a positive and increasing function $\widetilde{\gamma}(\xi)$ zero at the origin and a strictly positive real number r such that:

$$|q(z_i,\xi_i)| \;\leq\; \widetilde{\gamma}(\xi)e^{-ri}[W_1(z_i)+1]\;. \tag{1.28}$$

To conclude, we prove that the sequence $W(z_i)$ is bounded.

Using Assumptions A3 and A4, we deduce that

$$\begin{aligned} W_1(z_{i+1}) - W_1(z_i) &\leq W_1(\mathcal{F}(z_i)+\psi(z_i,\xi_i)) - W_1(\mathcal{F}(z_i)) \\ &\leq \gamma_0(\xi_i) + \gamma_1(\xi_i)W_1(z_i)\;. \end{aligned}$$

Using Assumption A2, we deduce that there exist a function $\Gamma_1(\xi)$ smooth, positive, zero at zero and $r>0$ such that:

$$\frac{W_1(z_{i+1})+1}{W_1(z_i)+1} \;\leq\; 1+\Gamma_1(\xi)e^{-ri}\;.$$

It follows that

$$\ln(W_1(z_{i+1})+1) \;\leq\; \ln(W_1(z)+1) + \sum_{j=0}^{i}\ln\left[1+\Gamma_1(\xi)e^{-rj}\right]$$

which implies that there exists $\Gamma_2(\xi)$ such that for all integer l,

$$W_1(z_l) \;\leq\; (W_1(z)+1)\Gamma_2(\xi)\;. \tag{1.29}$$

To show that $V(z,\xi)$ defined in (1.26) is a positive function, consider

$$\begin{aligned} W_1(z)+\Phi(z,\xi) &= W_1(z) + \sum_{i=0}^{\infty}[W_1(\mathcal{F}(z_i)+\psi(z_i,\xi_i)) \\ &\qquad -W_1(\mathcal{F}(z_i))] \\ &= W_1(z) + \sum_{i=0}^{\infty}[W_1(z_{i+1}) - W_1(\mathcal{F}(z_i))]\;. \end{aligned}$$

For all integer $J>0$, we have

$$\begin{aligned} W_1(z)+\Phi(z,\xi) \;=\; & \sum_{i=0}^{J}[W_1(z_i) - W_1(\mathcal{F}(z_i))] + W_1(z_{J+1}) \\ & + \sum_{i=J+1}^{\infty}[W_1(z_{i+1}) - W_1(\mathcal{F}(z_i))]\;. \end{aligned}$$

According to Assumption A1, the term $\sum_{i=0}^{J}[W_1(z_i) - W_1(\mathcal{F}(z_i))]$ is positive.

On the other hand, using Assumptions A3 and A4, one can prove easily that:

$$\lim_{J\to+\infty}\sum_{i=J+1}^{\infty}[W_1(z_{i+1})-W_1(\mathcal{F}(z_i))] \;=\; 0\;.$$

It follows that $W_1(z)+\Phi(z,\xi)$ is positive. Since $\Phi(z,0)=0$ for all z, it straightforwardly follows that $V(z,\xi)$ is a positive definite function.

Remark. It is worth noting that surprisingly, Assumptions A1 to A4 do not guarantee that $V(x)$ is radially unbounded whereas similar assumptions in the continuous-time context ensure that the corresponding function $V(x)$ is radially unbounded.

1.4.2 Disturbance attenuation property of feedforward systems

Consider the system

$$\begin{aligned} z_{i+1} &= \mathcal{F}(z_i)+\psi(z_i,\xi_i)+g_2(z_i,\xi_i,u_i)u_i \\ &\quad +m_1(z_i,\xi_i,u_i,d_i)d_i \\ \xi_{i+1} &= a(\xi_i)+g_1(z_i,\xi_i,u_i)u_i+m_2(z_i,\xi_i,u_i,d_i)d_i \end{aligned} \tag{1.30}$$

with $z_i\in\mathbf{R}^{n_z}$, $\xi_i\in\mathbf{R}^{n_\xi}$, $u\in\mathbf{R}$, $d\in\mathbf{R}$. Let us state a disturbance attenuation result for feedforward systems.

Corollary 1.4.1. *Assume that the system (1.30) satisfies the Assumptions A1 to A4. Assume that the cross term of the function $V(x)$ provided by Theorem 1.4.1 is continuously differentiable and that Assumption H2 is satisfied. Then the inverse optimal stabilization problem is solvable. Moreover, if the system (1.30) satisfies the the assumptions of Theorem 1.2.3 when $d_i=0$ for all i and the assumptions H3 and H4. Then there exists $C>0$ such that for all sequence $d_i\in L^2$, the solution of (1.13) with $x_0=0$ satisfies:*

$$\sum_{i=0}^{\infty}|x_i|^2 \;\le\; C\sum_{i=0}^{\infty}d_i^2$$

Remark. Observe that Corollary 1.4.1 can be applied repeatedly.

Proof. The solvability of the inverse optimal stabilization problem is an immediate consequence of Theorem 1.2.3 and Theorem 1.4.1 when $V(x)$ is a proper function. When $V(x)$ is not a proper function then Assumption H1' and not Assumption H1 is satisfied. To apply Theorem 1.2.2, we have to prove that there exist feedbacks of the form (1.3) which do not destabilize the system. This can be done by using the arguments similar to those invoked above in the proof of Theorem 1.4.1. The disturbance attenuation result is a consequence of Theorem 1.3.1. This concludes our proof.

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