BACKSTEPPING FOR NONLINEAR SYSTEMS WITH DELAY IN THE INPUT REVISITED*

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Abstract. In this paper, a new solution to the problem of globally asymptotically stabilizing a nonlinear system in feedback form with a known pointwise delay in the input is proposed. The result covers a family of systems wider than those studied in the literature and endows with control laws with a single delay, in contrast to those given in previous works which include two distinct pointwise delays or distributed delays. The strategy of design is based on the construction of an appropriate Lyapunov-Krasovskii functional. An illustrative example ends the paper.

Key words. delay, nonlinear, backstepping, Lyapunov-Krasovskii.

AMS subject classifications. 93D15, 93C10, 93D05

1. Introduction.

Tme-delay systems represent an important family of systems spanning a wide range of application including network control, population dynamics, biological systems to cite only a few. Most of the literature on systems with delay is devoted to linear systems (see, for instance [4], [24], [22] and the references therein). Nevertheless, especially in the last two decades, some important results for nonlinear systems with delay have appeared. In particular, extensions to systems with delays of the two techniques of recursive design of control laws called *backstepping* and *forwarding* have been obtained. Forwarding in [10] and the backstepping in [3], [23] have been adapted to important families of systems with pointwise delays and delay-free inputs. It is worth mentioning that the problem of stabilizing nonlinear systems with time delayed inputs is also of interest due to the transport and measurement delays that naturally arise in control applications (see, e.g., [22]). Although such a problem appears as being difficult, a few papers present extensions of the forwarding approach to the case of retarded inputs ([13], [2], [27] and [19]). For backstepping, the situation is different: although backstepping is one of the most popular techniques of design of stabilizing control laws for nonlinear systems, which has been largely developed in the literature (see, for instance, [15], [12], [16], [1], [14] and the references therein), to the best of our knowledge, only two contributions [17], [11] are devoted to the problem of extending the backstepping approach to the case where there are delays in the inputs. More precisely, stabilization is achieved in [11], via a control law with distributed terms over some time interval and in [17], stabilization is achieved via a control law with two pointwise delays.

Motivated by the precedent considerations, the present work continues the efforts to extend the *backstepping approach* to nonlinear systems with a delay in the input. Since it owes a great deal to [17], we briefly describe its main result. Systems of the

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form:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))z(t) \ , \\ \dot{z}(t) & = & u(t-\tau) + h(x(t-\tau), z(t-\tau)) \ , \end{array} \right. \tag{1.1}$$

with $x \in \Re^n$, $z \in \Re$, where $u \in \Re$ is the input and $\tau \geq 0$ is the delay and appropriate initial conditions, are considered. The existence of a control law $z_s(x)$ which globally asymptotically stabilizes the system $\dot{x}(t) = f(x(t)) + g(x(t))z_s(x(t))$ is assumed. Technical assumptions are introduced to guarantee that the system

$$\dot{x}(t) = f(x(t)) + g(x(t))z_s(x(t-\tau)) \tag{1.2}$$

admits also the origin as a globally asymptotically stable equilibrium point. Basically, these assumptions are growth conditions on the functions f and g which prevent the finite escape time phenomenon from happening and make it possible to construct a strict Lyapunov-Krasovskii functional for the system (1.2). With these considerations in mind, the introduction of the operator $Z : C_{in} \mapsto \Re$ defined on the dynamics of (1.1) by

$$Z(t) = z(t) - z_s(x(t-\tau)) \ , \tag{1.3}$$

which plays the role of a change of coordinates, leads to a Lyapunov functional U and a control law

$$\begin{array}{rcl} u(t-\tau) & = & -\varepsilon[z(t-\tau) - z_s(x(t-2\tau))] - h(x(t-\tau), z(t-\tau)) \\ & & + \frac{\partial z_s}{\partial x}(x(t-\tau))[f(x(t-\tau)) + g(x(t-\tau))z(t-\tau)] \end{array} \tag{1.4}$$

with $\varepsilon > 0$, which ensures that, for all $t \geq \tau$, the derivative of U along the trajectories of the closed-loop system is smaller than a negative definite function of $x(t)$, $z(t)$. To the best of the authors' knowledge, this pioneer technique made it possible for the first time to globally stabilize nonlinear systems in feedback form by retarded feedbacks of class C^1 without using distributed terms. However, it has three limitations: (i) in the formula of the control law (1.4), both $x(t)$ and $x(t-2\tau)$ are present, although there is only the delay τ in the input of (1.1) through $u(t-\tau)$, (ii) the derivative of the Lyapunov-Krasovskii functional along the trajectories of the closed loop system is negative definite only for values of time larger than τ and therefore this functional does not offer all the advantages inherent to classical Lyapunov-Krasovskii functionals (see for instance [25] and [7] for more information on the usefulness of Lyapunov-Krasovskii functionals) (iii) finally, the function h in (1.1), which depends on the retarded values $x(t-\tau)$, $z(t-\tau)$, cannot depend on $x(t)$, $z(t)$. If it does, the technique of construction *does not apply.*

In the present work, we overcome these limitations. In particular, we wish to point out that the main result of stabilization we are deriving applies to systems of the family

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))z(t) \ , \\ \dot{z}(t) & = & u(t-\tau) + h_1(x(t), z(t)) + h_2(x(t-\tau), z(t-\tau)) \ , \end{array} \right. \tag{1.5}$$

with $x \in \Re^n$, $z \in \Re$, $u \in \Re$, under appropriate initial conditions. It is worth mentioning that such a class of systems is important due to the presence of the term without delay $h_1(x(t), z(t))$ and because this family encompasses the family of the systems of the form

$$\left\{\begin{array}{lcl} \dot{x}_1(t) & = & x_2(t)\ , \\ & \vdots & \\ \dot{x}_{n-1}(t) & = & x_n(t)\ , \\ \dot{x}_n(t) & = & z(t), \\ \dot{z}(t) & = & u(t-\tau) + h_1(x(t), z(t))\ , \end{array}\right. \tag{1.6}$$

with $x = (x_1, ..., x_n)$, which may result from the attempt to linearize a single-input single-output system including a single delay only in the input, which cannot be completed when $\tau > 0$ because the term $h_1(x(t), z(t))$ cannot be removed through a change of feedback. Interestingly, the contribution [5] presents results for the systems (1.6), under various assumptions on the growth properties of the term h_1. In contrast to the feedbacks of the present work, the control laws of [5] depend on the past values of the controls.

The Lyapunov based technique we will expose relies on the use of an *operator* of a *new type*. It is reminiscent of the one introduced for the first time in [20] and [21] and also shares some features with the one used in [8] and [9] to reduce a system with delay to another one without delay. However neither the approach of [20] nor the one of [8] and [9] can be applied to stabilize systems of the form (1.5): the assumptions imposed in [20] are not satisfied by (1.5) (notice in particular that the main result of [20] is a result of exponential stabilization whereas (1.5) is not necessarily locally exponentially stabilizable) and the contributions [8] and [9] are not concerned with retarded inputs. The operator leads to the construction of a Lyapunov-Krasovskii functional whose derivative along the trajectories of (1.5) can be made negative definite by an appropriate choice of state feedback of class C^1. Using this strict Lyapunov-Krasovskii functional, we shall prove for a family of systems with additive disturbances that the control laws we propose give to the systems the desirable ISS property (see for instance [26], [16] for a detailed presentation of the notion of ISS) with respect to additive disturbances.

The remaining of the paper is organized as follows. In Section 2, we introduce the general family of systems that will be studied and the assumptions that will be used throughout the paper. Next, the main result is stated and proved in Section 3. A result of ISS robustness is established in Section 4. A second-order example in Section 5 illustrates the control design of the previous section. Some concluding remarks in Section 6 complete the work.

Notations and definitions.

- Denote $|\cdot|$ the Euclidean norm of matrices and vectors of any dimension.
- Given $\phi : I \to \Re^p$ defined on an interval I, denote its (essential) supremum over I by $|\phi|_I$.
- Let p be any positive integer. We denote $C_{\text{in}} = C([-\tau, 0], \Re^p)$ the set of all continuous $\Re^p$-valued functions defined on a given interval $[-\tau, 0]$.
- For a continuous function $\varphi : [-\tau, +\infty) \to \Re^k$, for all $t \geq 0$, the function φ_t defined by $\varphi_t(\theta) = \varphi(t+\theta)$ for all $\theta \in [-\tau, 0]$ is sometimes called translation operator.
- A function $\kappa : [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{K}_\infty$ if $\kappa(0) = 0$, κ is continuous, increasing and unbounded.
- The notations will be simplified whenever no confusion can arise from the context.

2. Particular family of systems in feedback form. We consider the nonlinear systems:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))z(t) \ , \\ \dot{z}(t) & = & u(t-\tau) + h_1(x(t), z(t)) + h_2(x(t-\tau), z(t-\tau)) \ , \end{array} \right. \tag{2.1}$$

with $x \in \Re^n$, $z \in \Re$, where $u \in \Re$ is the input, where $\tau > 0$ is a constant, where f, g, h_1, h_2 are functions of class C^1, under appropriate initial conditions. The main goal of the section is to develop a method for deriving stabilizing feedbacks for systems (2.1). To achieve it, we introduce a set of assumptions:

Assumption H1. *There exist a positive definite, radially unbounded function V of class C^1 and a function $z_s(x)$ of class C^2 such that $z_s(0) = 0$ and the function*

$$W : \Re^n \to \Re \ , \ W(x) = -\frac{\partial V}{\partial x}(x)F(x) \ , \tag{2.2}$$

where

$$F : \Re^n \to \Re^n \ , \ F(x) = f(x) + g(x)z_s(x) \ , \tag{2.3}$$

is positive definite.

Assumption H2. *There exist a constant $c_1 \geq 0$ and a nonnegative locally Lipschitz function G_1 such that the function*

$$H : \Re^n \times \Re \to \Re \ , \ H(x, Z) = -\frac{\partial z_s}{\partial x}(x)[F(x) + g(x)Z] + h_1(x, Z + z_s(x)) \tag{2.4}$$

satisfies, for all $x \in \Re^n, Z \in \Re$, the inequality

$$|H(x, Z)| \leq c_1|Z| + G_1(x) \ . \tag{2.5}$$

There exist constants $c_2 > 0$ and $c_3 > 0$ such that, for all $x \in \Re^n$, the inequalities

$$\left| \frac{\partial V}{\partial x}(x)g(x) \right|^2 \leq c_2 W(x) \ , \tag{2.6}$$

$$G_1(x)^2 \leq c_3 W(x) \ , \tag{2.7}$$

are satisfied.

Assumption H3. *The delay τ and the constants c_i, $i = 1, 2, 3$, in Assumption H2 satisfy*

$$\tau < \left\{ \begin{array}{c} \min\left\{ \frac{1}{2\sqrt{2}c_1}, \frac{1}{\sqrt{6c_2c_3}} \right\} \quad \textit{if} \quad c_1 > 0 \ , \\ \frac{1}{\sqrt{6c_2c_3}} \quad \textit{if} \quad c_1 = 0 \ . \end{array} \right. \tag{2.8}$$

3. Main result. We are ready to state and prove the following result:

THEOREM 3.1. *Consider the system (2.1). Assume that it satisfies Assumptions H1 to H3. Then for all $L \in \Re$ such that*

$$L \in \left[c_1 - \frac{1}{2\sqrt{2}\tau}, 0 \right) \tag{3.1}$$

the system (2.1) in closed-loop with

$$u(x,z) = Le^{\tau L}(z - z_s(x)) + e^{\tau L}\left[\frac{\partial z_s}{\partial x}(x)(f(x)+g(x)z) - h_1(x,z)\right] - h_2(x,z) \tag{3.2}$$

admits the origin as a globally asymptotically stable equilibrium point.

Discussion of Theorem 3.1.

– Assumption H1 is one of the fundamental assumptions of the backstepping method: it ensures the stabilizability of the x-subsystem of (2.1) with z as virtual input.

– Since it is assumed that $z_s(x)$ of class C^2, the feedback (3.2) is of class C^1.

– There are in Assumption H2 growth restrictions on the functions f, g, h_1. They allow us performing our Lyapunov design. However, they are not simple technical assumptions which might be removed without changing the result: in [17], it is proved that there are some systems (1.1), which do not satisfy Assumption H2 and for which there exists no feedback $u(x(t-\tau), z(t-\tau))$ so that the finite escape time phenomenon does not occur.

– Since in the absence of delay, globally asymptotically stabilizing feedbacks can be determined through the backstepping approach, the result of [18], which is a result of robustness with respect to the presence of a delay in the input, applies in some cases when extra assumptions, which pertain in particular to the growth of $\frac{\partial f}{\partial x}$, $\frac{\partial g}{\partial x}$ $\frac{\partial h_1}{\partial x}$, $\frac{\partial h_1}{\partial z}$, $\frac{\partial^2 z_s}{\partial^2 x}$, are satisfied.

– A restriction on the size of τ must be imposed, otherwise the problem does not always admit a solution. Indeed, the very simple two-dimensional linear system

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & x(t) + z(t)\ , \\ \dot{z}(t) & = & u(t-\tau) + z(t)\ , \end{array}\right. \tag{3.3}$$

cannot be locally asymptotically stabilized by a control $u(x,z)$ of class C^1 when τ is larger than a certain constant. However, there is a subfamily of systems (2.1), for which for any delay $\tau > 0$, one can select a "fictitious" feedback z_s such that Assumptions H1 to H3 are satisfied. Typically, these systems have no term h_1 and are such that, for all $\delta > 0$, there exists z_s satisfying Assumptions H1 and H2 such that $\left|\frac{\partial z_s}{\partial x}(x)\right| \leq \delta$ for all $x \in \Re^n$.

– One can easily check that if the system (2.1) is linear and asymptotically stabilizable, then there always exists a quadratic Lyapunov function V and a linear function z_s such that Assumptions H1 and H2 are satisfied.

– Assumptions H1 to H3 do not imply that the system (2.1) with $\tau = 0$ admits an exponentially stabilizable linear approximation at the origin: they are satisfied for instance by the two-dimensional system

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & x(t)z(t)\ , \\ \dot{z}(t) & = & u(t-\tau)\ , \end{array}\right. \tag{3.4}$$

studied in [17] with $V(x) = \ln(1+x^2)$, $z_s(x) = -\frac{\omega x^2}{1+x^2}$, $\omega > 0$, where τ is sufficiently small relative to ω. This is a remarkable feature of Theorem 3.1 because most of the stabilization results for systems with delays apply only to systems that can be locally exponentially stabilized by a feedback of class C^1.

Proof. To begin with, we observe that Assumption H3 ensures that there exist negative constants L such that the inequality (3.1) is satisfied. Now, to ease the control design, we perform the change of coordinate $Z = z - z_s(x)$ and take u under the form $u(t-\tau) = v(x(t-\tau), z(t-\tau) - z_s(x(t-\tau))) - h_2(x(t-\tau), z(t-\tau))$, where v is a function to be determined later. With these transformations, the system (2.1) rewrites as:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & F(x(t)) + g(x(t))Z(t) \ , \\ \dot{Z}(t) & = & v(x(t-\tau), Z(t-\tau)) + H(x(t), Z(t)) \ , \end{array} \right. \tag{3.5}$$

with F defined in (2.3) and H defined in (2.4). Then we start the construction of a Lyapunov functional candidate by introducing the operator $\alpha : C_{in} \mapsto \Re$ defined by

$$\alpha(\phi_x, \phi_Z) = \int_{-\tau}^{0} e^{-L(m+\tau)} v(\phi_x(m), \phi_Z(m)) dm \ , \tag{3.6}$$

where L is the constant in Theorem 3.1. Along the trajectories of (3.5), it satisfies

$$\alpha(x_t, Z_t) = \int_{t-\tau}^{t} e^{L(t-m-\tau)} v(x(m), Z(m)) dm \ , \tag{3.7}$$

and its time-derivative, denoted simply $\dot{\alpha}(t)$, satisfies

$$\dot{\alpha}(t) = L\alpha(x_t, Z_t) + e^{-\tau L} v(x(t), Z(t)) - v(x(t-\tau), Z(t-\tau)) \ . \tag{3.8}$$

It follows that the operator $\beta : C_{in} \mapsto \Re$ defined by

$$\beta(\phi_x, \phi_Z) = \phi_Z(0) + \alpha(\phi_x, \phi_Z) \tag{3.9}$$

is such that, for all $t \geq 0$,

$$\dot{\beta}(t) = L\alpha(x_t, Z_t) + e^{-\tau L} v(x(t), Z(t)) + H(x(t), Z(t)) \ . \tag{3.10}$$

Selecting the function

$$v(x, Z) = -e^{\tau L} H(x, Z) + L e^{\tau L} Z \ , \tag{3.11}$$

which corresponds to the control (3.2), we obtain

$$\dot{\beta}(t) = L\beta(x_t, Z_t) \tag{3.12}$$

and

$$\beta(\phi_x, \phi_Z) = \phi_Z(0) + \int_{-\tau}^{0} e^{-Lm} [-H(\phi_x(m), \phi_Z(m)) + L\phi_Z(m)] dm \ . \tag{3.13}$$

We note that $\beta(x_t, Z_t)$ is a solution of an exponentially stable system since $L < 0$. We shall use later this property implicitly. For the time being, we observe that the equality

$$Z(t) = \beta(x_t, Z_t) - \int_{t-\tau}^{t} e^{L(t-m)} [-H(x(m), Z(m)) + LZ(m)] dm \tag{3.14}$$

and the negativity of L imply that, for all $t \geq 0$, the inequality

$$|Z(t)| \ \leq \ \int_{t-\tau}^{t} |H(x(m), Z(m)) - LZ(m)| dm + |\beta(x_t, Z_t)| \tag{3.15}$$

holds. From (2.5) in Assumption H2 and the definition of H in (2.4), we deduce that, for all $t \geq 0$,

$$|Z(t)| \leq \int_{t-\tau}^{t} [c_4|Z(m)| + G_1(x(m))]dm + |\beta(x_t, Z_t)| \ , \tag{3.16}$$

with $c_4 = c_1 - L$. This inequality together with the inequality (A.1), Cauchy-Schwartz inequality and the inequality (A.2) imply that, for all $t \geq 0$,

$$\begin{aligned} Z(t)^2 &\leq \tfrac{5}{4}\tau \int_{t-\tau}^{t} [c_4|Z(m)| + G_1(x(m))]^2 dm + 5\beta(x_t, Z_t)^2 \\ &\leq \tfrac{5}{2}\tau \int_{t-\tau}^{t} [c_4^2 Z(m)^2 + G_1(x(m))^2]dm + 5\beta(x_t, Z_t)^2 \ . \end{aligned} \tag{3.17}$$

Then, we introduce a new operator $\gamma : C_{in} \mapsto \Re$ defined by

$$\gamma(\phi_Z) = \int_{-\tau}^{0} \int_{\ell}^{0} \phi_Z(m)^2 dm d\ell \tag{3.18}$$

which satisfies, along the trajectories of (3.5), for all $t \geq 0$, $\gamma(Z_t) = \displaystyle\int_{t-\tau}^{t} \int_{\ell}^{t} Z(m)^2 dm d\ell$ and

$$\dot{\gamma}(t) = \tau Z(t)^2 - \int_{t-\tau}^{t} Z(m)^2 dm = -\tau Z(t)^2 - \int_{t-\tau}^{t} Z(m)^2 dm + 2\tau Z(t)^2 \ . \tag{3.19}$$

Combining this equality with (3.17), we obtain, for all $t \geq 0$, the inequality

$$\begin{aligned} \dot{\gamma}(t) &\leq -\tau Z(t)^2 - \int_{t-\tau}^{t} Z(m)^2 dm \\ &\quad + 2\tau \left[\tfrac{5}{2}\tau \int_{t-\tau}^{t} [c_4^2 Z(m)^2 + G_1(x(m))^2]dm + 5\beta(x_t, Z_t)^2 \right] \end{aligned} \tag{3.20}$$

which rewrites, by grouping the terms, as

$$\begin{aligned} \dot{\gamma}(t) &\leq -\tau Z(t)^2 + (5\tau^2 c_4^2 - 1) \int_{t-\tau}^{t} Z(m)^2 dm \\ &\quad + 5\tau^2 \int_{t-\tau}^{t} G_1(x(m))^2 dm + 10\tau\beta(x_t, Z_t)^2 \ . \end{aligned} \tag{3.21}$$

The inequality (3.1) implies that $5\tau^2 c_4^2 - 1 \leq -3\tau^2 c_4^2$. It follows that, for all $t \geq 0$,

$$\begin{aligned} \dot{\gamma}(t) &\leq -\tau Z(t)^2 - 3c_4^2\tau^2 \int_{t-\tau}^{t} Z(m)^2 dm \\ &\quad + 5\tau^2 \int_{t-\tau}^{t} G_1(x(m))^2 dm + 10\tau\beta(x_t, Z_t)^2 \ . \end{aligned} \tag{3.22}$$

We continue our Lyapunov construction by considering the functional $U_1 : C_{in} \mapsto \Re$ defined by

$$U_1(\phi_x, \phi_Z) = V(\phi_x(0)) + c_5\gamma(\phi_Z) \ , \tag{3.23}$$

where V is the function given by Assumption H1 and where $c_5 > 0$ is a constant to be chosen later. Then, from Assumption H1 and (3.22), we deduce that, for all $t \geq 0$,

$$\begin{array}{rcl} \dot{U}_1(t) & \leq & -W(x(t)) + \frac{\partial V}{\partial x}(x(t))g(x(t))Z(t) - c_5\tau Z(t)^2 \\ & & -c_6\tau^2 \int_{t-\tau}^{t} Z(m)^2 dm + 5c_5\tau^2 \int_{t-\tau}^{t} G_1(x(m))^2 dm \\ & & +10c_5\tau\beta(x_t, Z_t)^2 \ , \end{array} \tag{3.24}$$

with $c_6 = 3c_5c_4^2$. From the inequality (2.6) in Assumption H2, we deduce that, for all $t \geq 0$,

$$\begin{array}{rcl} \dot{U}_1(t) & \leq & -W(x(t)) + \sqrt{c_2 W(x(t))}|Z(t)| - c_5\tau Z(t)^2 \\ & & -c_6\tau^2 \int_{t-\tau}^{t} Z(m)^2 dm + 5c_5c_3\tau^2 \int_{t-\tau}^{t} W(x(m))dm \\ & & +10c_5\tau\beta(x_t, Z_t)^2 \ . \end{array} \tag{3.25}$$

This inequality leads us to consider the functional $U_2 : C_{in} \mapsto \Re$ defined by

$$\begin{array}{rcl} U_2(\phi_x, \phi_Z) & = & U_1(\phi_x, \phi_Z) + 6c_5c_3\tau^2 \int_{-\tau}^{0} \int_{\ell}^{0} W(\phi_x(m))dmd\ell \\ & & +k\beta(\phi_x, \phi_Z)^2 \end{array} \tag{3.26}$$

with $k = \frac{1+10c_5\tau}{-2L}$ (since $L < 0$, $k > 0$). Bearing in mind (3.12), through elementary calculations, we obtain that for all $t \geq 0$, the following inequality holds:

$$\begin{array}{rcl} \dot{U}_2(t) & \leq & (6c_5c_3\tau^3 - 1)W(x(t)) + \sqrt{c_2 W(x(t))}|Z(t)| - c_5\tau Z(t)^2 \\ & & -\beta(x_t, Z_t)^2 - c_6\tau^2 \int_{t-\tau}^{t} Z(m)^2 dm \\ & & -c_5c_3\tau^2 \int_{t-\tau}^{t} W(x(m))dm \ . \end{array} \tag{3.27}$$

Therefore there exists a constant $c_7 > 0$ such that, for all $t \geq 0$,

$$\begin{array}{rcl} \dot{U}_2(t) & \leq & -c_7 W(x(t)) - c_7 Z(t)^2 - \beta(x_t, Z_t)^2 - c_6\tau^2 \int_{t-\tau}^{t} Z(m)^2 dm \\ & & -c_5c_3\tau^2 \int_{t-\tau}^{t} W(x(m))dm \end{array} \tag{3.28}$$

if there exists $c_5 > 0$ such that the inequality

$$c_2 + 4c_5\tau(6c_5c_3\tau^3 - 1) < 0 \tag{3.29}$$

holds. Since Assumption H3 ensures that $\tau < \frac{1}{\sqrt{6c_2c_3}}$, the constant $c_5 = \frac{1}{12c_3\tau^3}$ is such that (3.29) is satisfied because this value for c_5 leads to $c_2 + 4c_5\tau(6c_5c_3\tau^3 - 1) = c_2 - \frac{1}{6c_3\tau^2} < 0$. Then (3.28) holds with for instance $c_7 = \min\left\{\frac{1-6c_2c_3\tau^2}{4}, \frac{1-6c_2c_3\tau^2}{12c_3\tau^2(1+6c_2c_3\tau^2)}\right\}$. However, although the right hand side of (3.28) is nonpositive, we cannot apply the Lyapunov-Krasovskii Theorem yet (see for instance [6], [16]) because we do not know if there is a function λ of class $\mathcal{K}_\infty$ such that, for all functions $(\phi_x, \phi_Z) \in C_{\text{in}}$ the inequality

$$\lambda(|\phi_x(0)|^2 + \phi_Z(0)^2) \leq U_2(\phi_x, \phi_Z) \tag{3.30}$$

is satisfied. To overcome this obstacle, we replace U_2 by the functional $U_3 : C_{in} \mapsto \Re$ defined by

$$U_3(\phi_x, \phi_Z) = U_2(\phi_x, \phi_Z) + \frac{c_7}{2}\int_{-\tau}^{0} W(\phi_x(m))dm + \frac{c_7}{2}\int_{-\tau}^{0} \phi_Z(m)^2 dm \ . \tag{3.31}$$

We first observe that the functional U_2 satisfies, for all functions $(\phi_x, \phi_Z) \in C_{\text{in}}$

$$\begin{aligned} U_2(\phi_x, \phi_Z) &= V(\phi_x(0)) + c_5\int_{-\tau}^{0}\int_{\ell}^{0} \phi_Z(m)^2 dm d\ell \\ &+ 6c_5c_3\tau^2 \int_{-\tau}^{0}\int_{\ell}^{0} W(\phi_x(m)) dm d\ell \\ &+ k\left[\phi_Z(0) + \int_{-\tau}^{0} e^{-Lm}[-H(\phi_x(m), \phi_Z(m)) + L\phi_Z(m)]dm\right]^2 \ . \end{aligned} \tag{3.32}$$

Next, using successively the inequality (A.3), Cauchy-Schwartz inequality and the inequality (A.2), we deduce that for all $\rho \in (0, 1)$ and for all functions $(\phi_x, \phi_Z) \in C_{\text{in}}$,

$$\begin{aligned} U_2(\phi_x, \phi_Z) &\geq V(\phi_x(0)) + k\rho\phi_Z(0)^2 \\ &- \tfrac{k\rho}{1-\rho}\left[\int_{-\tau}^{0} e^{-Lm}[-H(\phi_x(m), \phi_Z(m)) + L\phi_Z(m)]dm\right]^2 \\ &\geq V(\phi_x(0)) + k\rho\phi_Z(0)^2 \\ &- \tfrac{2k\tau\rho}{1-\rho}\int_{-\tau}^{0} e^{-2Lm}[H(\phi_x(m), \phi_Z(m))^2 + L^2\phi_Z(m)^2]dm \ . \end{aligned} \tag{3.33}$$

From the inequality (2.5) in Assumption H2, we deduce that

$$\begin{aligned} &U_2(\phi_x, \phi_Z) \geq V(\phi_x(0)) + k\rho\phi_Z(0)^2 \\ &-\tfrac{2k\tau\rho}{1-\rho}\int_{-\tau}^{0} e^{-2Lm}[(c_1|\phi_Z(m)| + G_1(\phi_x(m)))^2 + L^2\phi_Z(m)^2]dm \\ &\geq V(\phi_x(0)) + k\rho\phi_Z(0)^2 \\ &-\tfrac{2k\tau\rho}{1-\rho}e^{-2L\tau}\int_{-\tau}^{0}[(2c_1^2 + L^2)\phi_Z(m)^2 + 2G_1(\phi_x(m))^2]dm \ . \end{aligned} \tag{3.34}$$

From the inequality (2.7) in Assumption H2, we deduce that

$$\begin{aligned} &U_3(\phi_x, \phi_Z) \geq V(\phi_x(0)) + k\rho\phi_Z(0)^2 \\ &-\tfrac{2k\tau\rho}{1-\rho}e^{-2L\tau}\int_{-\tau}^{0}[(2c_1^2 + L^2)\phi_Z(m)^2 + 2c_3W(\phi_x(m))]dm \\ &+\tfrac{c_7}{2}\int_{-\tau}^{0} W(\phi_x(m))dm + \frac{c_7}{2}\int_{-\tau}^{0} \phi_Z(m)^2 dm \ . \end{aligned} \tag{3.35}$$

Therefore, selecting ρ sufficiently small, we obtain

$$U_3(\phi_x, \phi_Z) \geq V(\phi_x(0)) + k\rho\phi_Z(0)^2 \ . \tag{3.36}$$

Since V is positive definite and radially unbounded, there is a function κ_1 of class $\mathcal{K}_\infty$ such that, for all $x \in \Re^n$, $V(x) \geq \kappa_1(|x|)$ and next, from the inequality (3.36) and Lemma B.1 in Appendix B, we deduce that there exists a function κ_s of class $\mathcal{K}_\infty$ such that, for all functions $(\phi_x, \phi_Z) \in C_{\text{in}}$, the inequality

$$\kappa_s\left(\sqrt{|\phi_x(0)|^2 + \phi_Z(0)^2}\right) \leq U_3(\phi_x, \phi_Z) \tag{3.37}$$

is satisfied. Moreover, one can easily verify that there exists a function κ_l of class $\mathcal{K}_\infty$ such that, for all functions $(\phi_x, \phi_Z) \in C_{\text{in}}$, the inequality

$$U_3(\phi_x, \phi_Z) \le \kappa_l \left(|(\phi_x, \phi_Z)|_{[-\tau,0]} \right) \tag{3.38}$$

is satisfied. Finally, the inequality (3.28) implies that, for all $t \ge 0$, the inequality

$$\begin{array}{rcl} \dot{U}_3(t) & \le & -\frac{c_7}{2} W(x(t)) - \frac{c_7}{2} Z(t)^2 - \beta(x_t, Z_t)^2 - \tau^2 c_6 \displaystyle\int_{t-\tau}^{t} Z(m)^2 dm \\ & & -c_5 c_3 \tau^2 \displaystyle\int_{t-\tau}^{t} W(x(m)) dm \end{array} \tag{3.39}$$

is satisfied. Since the function $-\frac{c_7}{2} W(x) - \frac{c_7}{2} Z^2$ is negative definite with respect to (x, Z), the conditions of the Lyapunov-Krasovskii theorem are satisfied (see for instance [6], [16]). We deduce that the origin of the system (3.5) in closed-loop with the control (3.11) is globally asymptotically stable. Since Assumption H1 ensures that $z_s(0) = 0$, it follows that the origin of the system (2.1) in closed-loop with the control (3.2) is globally asymptotically stable.

4. ISS closed-loop systems. In this section, we use the Lyapunov-Krasovskii functional constructed in the previous section to establish that, under extra assumptions, the control (3.2) gives an ISS property to the system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))z(t) + w_1(t) \ , \\ \dot{z}(t) & = & u(t-\tau) + h_1(x(t), z(t)) + h_2(x(t-\tau), z(t-\tau)) + w_2(t) \ , \end{array} \right. \tag{4.1}$$

with $x \in \Re^n$, $z \in \Re$, where $u \in \Re$ is the input, under appropriate initial conditions, where $\tau > 0$ is a constant, and f, g, h_1, h_2 are functions of class C^1 and where w_1 and w_2 are disturbances. More precisely, we establish the following result:

THEOREM 4.1. *Consider the system (4.1). Assume that it satisfies Assumptions H1 to H3 with a function W that is radially unbounded. Assume that there exists a constant $c_8 > 0$ such that, for all $x \in \Re^n$,*

$$\left| \frac{\partial z_s}{\partial x}(x) \right| \le c_8 \ . \tag{4.2}$$

Then for all $L \in \Re$ such that (3.1) holds, the system (4.1) in closed-loop with the control (3.2) is ISS with respect to (w_1, w_2).

Proof. We consider the system (4.1) in closed-loop with the control (3.2) after the coordinate transformation $Z = z - z_s(x)$. We infer from the proof of Theorem 3.1 that the derivative of the Lyapunov functional U_3 defined in (3.31) along the trajectories of this system satisfies, for all $t \ge 0$, the inequality:

$$\begin{array}{rcl} \dot{U}_3(t) & \le & -\frac{c_7}{2} W(x(t)) - \frac{c_7}{2} Z(t)^2 - \beta(x_t, Z_t)^2 - \tau^2 c_6 \displaystyle\int_{t-\tau}^{t} Z(m)^2 dm \\ & - & c_9 \displaystyle\int_{t-\tau}^{t} W(x(m)) dm \\ & + & \frac{\partial V}{\partial x}(x(t)) g(x(t)) w_1(t) + 2k\beta(x_t, Z_t) \left[w_2(t) - \frac{\partial z_s}{\partial x}(x) w_1(t) \right] \ , \end{array} \tag{4.3}$$

where β is the functional given in (3.13) and $c_9 = \tau^2 c_5 c_3$. From (2.6) in Assumption

H2 and (4.2), it follows that, for all $t \geq 0$,

$$\begin{array}{rcl} \dot{U}_3(t) & \leq & -\frac{c_7}{2}W(x(t)) - \frac{c_7}{2}Z(t)^2 - \beta(x_t, Z_t)^2 - \tau^2 c_6 \int_{t-\tau}^{t} Z(m)^2 dm \\ & & -c_9 \int_{t-\tau}^{t} W(x(m))dm \\ & & +\sqrt{c_2 W(x(t))}|w_1(t)| + 2k|\beta(x_t, Z_t)| \left[|w_2(t)| + c_8|w_1(t)|\right] . \end{array} \tag{4.4}$$

Using Young's inequality, we deduce that, for all $t \geq 0$,

$$\begin{array}{rcl} \dot{U}_3(t) & \leq & -\frac{c_7}{4}W(x(t)) - \frac{c_7}{2}Z(t)^2 - \frac{1}{2}\beta(x_t, Z_t)^2 - \tau^2 c_6 \int_{t-\tau}^{t} Z(m)^2 dm \\ & & -c_9 \int_{t-\tau}^{t} W(x(m))dm + \dfrac{c_2}{c_7}|w_1(t)|^2 \\ & & +2k^2 \left[|w_2(t)| + c_8|w_1(t)|\right]^2 . \end{array} \tag{4.5}$$

Since both V and W are positive definite and radially unbounded, there exists a $\mathcal{K}_\infty$ function μ and of class C^1 such that for all $x \in \Re^n$,

$$\frac{c_7}{4}W(x) \geq \mu(V(x)) . \tag{4.6}$$

From Lemma B.1 in Appendix B, we deduce that

$$\begin{array}{c} \frac{c_7}{4}W(x(t)) + \frac{c_7}{2}Z(t)^2 + \frac{1}{2}\beta(x_t, Z_t)^2 + \tau^2 c_6 \int_{t-\tau}^{t} Z(m)^2 dm + c_9 \int_{t-\tau}^{t} W(x(m))dm \\ \geq \mu_s\left(\psi(x_t, Z_t)\right) \end{array} \tag{4.7}$$

with the function $\mu_s(m) = \min\{\mu(\frac{m}{2}), \frac{m}{2}\}$, which is of class $\mathcal{K}_\infty$ and

$$\psi(x_t, Z_t) = V(x(t)) + \frac{c_7}{2}Z(t)^2 + \frac{1}{2}\beta(x_t, Z_t)^2 + \tau^2 c_6 \int_{t-\tau}^{t} Z(m)^2 dm + c_9 \int_{t-\tau}^{t} W(x(m))dm$$

We deduce that, for all $t \geq 0$,

$$\dot{U}_3(t) \leq -\mu_s\left(\psi(x_t, Z_t)\right) + c_{10}(|w_1(t)|^2 + w_2(t)^2) \tag{4.8}$$

with $c_{10} = \max\left\{\frac{c_2}{c_7} + 4k^2 c_8^2, 4k^2\right\}$. We show in Appendix C that there is a constant $c_{11} > 0$ such that, for all $t \geq 0$,

$$\dot{U}_3(t) \quad \leq \quad -\mu_s\left(c_{11}U_3(x_t, z_t)\right) + c_{10}(|w_1(t)|^2 + w_2(t)^2) . \tag{4.9}$$

It follows that U_3 is a ISS Lyapunov-Krasovskii functional (see [7], [25]). Using [7] or [25], we can complete the proof.

5. Illustrative example. In this section, we illustrate Theorem 4.1. We consider the pendulum equations:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & z(t) + w_1(t) , \\ \dot{z}(t) & = & u(t-\tau) + a\sin(x(t)) - bz(t) + w_2(t) , \end{array} \right. \tag{5.1}$$

where a is a constant different from 0, b and τ are positive constants, $x \in \Re$, $z \in \Re$, where $u \in \Re$ is the input and where w_1, w_2 are disturbances. Due to the presence of the term $a\sin(x) - bz$, the technique of [17] does not apply to (5.1). For this system, with the notations of Section 4, $f(x) = 0$, $g(x) = 1$, $h_1(x,z) = a\sin(x) - bz$, $h_2(x,z) = 0$. We apply Theorem 4.1 with the following functions $z_s : \Re \mapsto \Re$

$$z_s(x) = -rx \tag{5.2}$$

where r is a positive parameter different from b and $V : \Re \mapsto \Re$

$$V(x) = \frac{1}{2}x^2 \ . \tag{5.3}$$

Then, with the notation of Section 3, $W : \Re \mapsto \Re$, $F : \Re \mapsto \Re$

$$W(x) = rx^2 \ , \ F(x) = -rx \ , \ \frac{\partial V}{\partial x}(x)g(x) = x \tag{5.4}$$

and $H : \Re \times \Re \mapsto \Re$

$$H(x,Z) = r[-rx + Z] + a\sin(x) - b(Z - rx) \ . \tag{5.5}$$

It follows that, for all $(x,Z) \in \Re^2$,

$$\begin{array}{rcl} |H(x,Z)| & = & |(-r+b)rx + a\sin(x) + (r-b)Z| \\ & \leq & c_1|Z| + G_1(x) \ , \end{array} \tag{5.6}$$

with $c_1 = |b - r|$, $G_1(x) = |(b-r)rx + a\sin(x)|$. Finally, noticing that W is positive definite and radially unbounded, $\frac{\partial z_s}{\partial x}(x) = -r$,

$$\left(\frac{\partial V}{\partial x}(x)g(x)\right)^2 = x^2 = c_2 W(x) \tag{5.7}$$

with $c_2 = \frac{1}{r}$ and

$$G_1(x)^2 = |(b-r)rx + a\sin(x)|^2 \leq [r^2(b-r)^2 + a^2]x^2 = c_3 W(x) \ , \tag{5.8}$$

with $c_3 = \frac{r^2(b-r)^2+a^2}{r}$, we deduce from Theorem 4.1 that if

$$\tau < \min\left\{\frac{1}{2\sqrt{2}|b-r|}, \frac{1}{\sqrt{6}}\frac{r}{\sqrt{r^2(b-r)^2+a^2}}\right\} \tag{5.9}$$

then the system (5.1) in closed-loop with the control law

$$u_r(x,z) = Le^{\tau L}(z + rx) + e^{\tau L}\left[-rz - a\sin(x) + bz\right] \ , \tag{5.10}$$

with

$$L \in \left[|b-r| - \frac{1}{2\sqrt{2}\tau}, 0\right) \tag{5.11}$$

is globally ISS with respect to (w_1, w_2). Observe that, rewriting u_r as

$$u_r(x,z) = e^{\tau L}\left[Lrx - a\sin(x) + (L + b - r)z\right] \tag{5.12}$$

and choosing $r < b$ and $L = r - b$, we obtain the feedback

$$u_r(x,z) = e^{\tau(r-b)}\left[(r-b)rx - a\sin(x)\right] \tag{5.13}$$

which is independent from z. This feature may be of interest in cases where the variable of velocity z cannot be measured. Of course, ISS for the system (5.1) can be also achieved by linear output feedbacks

$$u_l(x) = -sx \ , \ s \in \Re \ , \tag{5.14}$$

when the delay τ is sufficiently small. However, the families of stabilizing feedbacks (5.14) and (5.13) do not have the same features. Since, for all $(x, z) \in \Re^2$,

$$|u_r(x, z)| \leq e^{\tau(r-b)} \left[(b - r)r|x| + |a|\right] \tag{5.15}$$

and since r can be chosen arbitrary close to b (independently from τ), for any $\epsilon > 0$, the family (5.13) contains elements which satisfy, for all $(x, z) \in \Re^2$,

$$|u_r(x, z)| \leq \epsilon|x| + |a| \ . \tag{5.16}$$

(For instance, a possible choice for obtaining this is $r = b - \min\left\{\frac{\varepsilon}{b}, \frac{b}{2}\right\}$.) By contrast, one can prove that if a feedback $u_l(x) = -sx$ stabilizes the system (5.1), then there exists $x_l \in \Re$ such that $|u_l(x_l)| > \varepsilon_0|x_l| + 2|a|$ with $\varepsilon_0 = \frac{|a|}{9}$. To prove this result, we proceed by contradiction. Assume that there exists $s \in \Re$ such that the origin of (5.1) in closed-loop with $u_l(x) = -sx$ is globally asymptotically stable in the absence of (w_1, w_2) and, for all $x \in \Re$,

$$|u_l(x)| \leq \frac{|a|}{9}|x| + 2|a| \ . \tag{5.17}$$

Then necessarily, for all $x \neq 0$, $-sx + a\sin(x) \neq 0$ (otherwise, the closed-loop system would admit an equilibrium point different from the origin). Therefore, for all $x \neq 0$, $s \neq a\frac{\sin(x)}{x}$. It follows that $|s| \notin \left(0, \frac{2|a|}{3\pi}\right)$. Since $s = 0$ is not a possible case, it follows that $|s| \geq \frac{2|a|}{3\pi}$ so that, for all $x \in \Re$,

$$|u_l(x)| \geq \frac{2|a|}{3\pi}|x| \ . \tag{5.18}$$

Combining (5.17) and (5.18), we deduce that, for all $x \in \Re$,

$$\frac{2|a|}{3\pi}|x| \leq \frac{|a|}{9}|x| + |a| \ . \tag{5.19}$$

This implies that the inequality $\frac{2|a|}{3\pi} \leq \frac{|a|}{9}$ is satisfied. Hence we have a contradiction because $a \neq 0$ implies that $\frac{2|a|}{3\pi} > \frac{|a|}{9}$.

We conclude that the family of stabilizing control laws u_r given in (5.13) has an advantage over the family of the stabilizing linear control laws u_l given in (5.14), relative to the size of their elements: roughly speaking, outside a compact set, no stabilizing linear feedback will be smaller than some of the feedbacks provided by Theorem 4.1 .

6. Conclusion. We have developed a new backstepping method for a new family of systems with delay in the input. We have obtained state feedbacks of class C^1 whose analytic expressions include delay information, and explicit expressions of strict Lyapunov-Krasovskii functionals for the closed-loop systems. Much remains to be done. We conjecture that extensions to time-varying systems and semi-global versions of our main result relative to the size of the delay for systems with arbitrary nonlinearities can be done. As a future work, we also plan to establish robustness results to cope with the case where there are uncertainties on the delay.

Appendix A. Useful inequalities.
For all real numbers A, B the inequalities

$$(A+B)^2 \leq \frac{5}{4}A^2 + 5B^2 \ , \tag{A.1}$$

$$(A+B)^2 \leq 2A^2 + 2B^2 \ , \tag{A.2}$$

are satisfied. For all real numbers A, B and for all $\rho \in (0,1)$, the inequality

$$(A+B)^2 \geq \rho A^2 - \frac{\rho}{1-\rho}B^2 \tag{A.3}$$

is satisfied.

Appendix B. Technical lemma.
LEMMA B.1. *Let κ be a function of class $\mathcal{K}_\infty$. Then, for all $A \geq 0$, $B \geq 0$,*

$$\kappa(A) + B \geq \kappa_s(A+B) \tag{B.1}$$

where

$$\kappa_s(m) = \min\left\{\kappa\left(\frac{m}{2}\right), \frac{m}{2}\right\} \ . \tag{B.2}$$

Moreover, κ_s is a function of class $\mathcal{K}_\infty$.
Proof. Consider the case of constants $A \geq 0$ and $B \geq 0$ such that $A > B$. Then

$$\kappa(A) + B \geq \kappa(A) \geq \kappa\left(\frac{1}{2}A + \frac{1}{2}B\right) \geq \kappa_s(A+B) \ . \tag{B.3}$$

Consider the case of the constants $A \geq 0$ and $B \geq 0$ such that $A \leq B$. Then

$$\kappa(A) + B \geq B \geq \frac{1}{2}A + \frac{1}{2}B \geq \kappa_s(A+B) \ . \tag{B.4}$$

Now, $\kappa_s(0) = 0$ and $\lim_{m\to+\infty} \kappa_s(m) = +\infty$. Finally, we prove that κ_s is increasing. Let $m_1 \geq 0$ and $m_2 \geq 0$ be such that $m_1 > m_2$. Then $\kappa(\frac{m_1}{2}) > \kappa(\frac{m_2}{2})$. Therefore $\kappa(\frac{m_1}{2}) > \min\left\{\kappa(\frac{m_2}{2}), \frac{m_2}{2}\right\}$. Moreover, $\frac{m_1}{2} > \frac{m_2}{2} \geq \min\left\{\kappa(\frac{m_2}{2}), \frac{m_2}{2}\right\}$. We deduce that $\kappa_s(m_1) > \kappa_s(m_2)$.

Appendix C. Technical result.
The analytic expression of U_3 is

$$\begin{aligned} U_3(\phi_x, \phi_Z) \ &= \ V(\phi_x(0)) + c_5 \int_{-\tau}^{0}\int_{\ell}^{0} \phi_Z(m)^2 dm d\ell \\ &\quad +6c_9 \int_{-\tau}^{0}\int_{\ell}^{0} W(\phi_x(m)) dm d\ell \\ &\quad +k\beta(\phi_x, \phi_Z)^2 + \tfrac{c_7}{2}\int_{-\tau}^{0} W(\phi_x(m)) dm \\ &\quad +\tfrac{c_7}{2}\int_{-\tau}^{0} \phi_Z(m)^2 dm \end{aligned} \tag{C.1}$$

(see (3.31), (3.32)). It follows that, for all functions $(\phi_x, \phi_Z) \in C_{\text{in}}$,

$$
\begin{aligned}
U_3(\phi_x, \phi_Z) \quad \leq \quad & V(\phi_x(0)) + \left(c_5\tau + \tfrac{c_7}{2}\right) \int_{-\tau}^{0} \phi_Z(m)^2 dm \\
& + \left(6c_9\tau + \tfrac{c_7}{2}\right) \int_{-\tau}^{0} W(\phi_x(m)) dm + k\beta(\phi_x, \phi_Z)^2 \ .
\end{aligned} \tag{C.2}
$$

This allows us to conclude that there exists $c_{11} > 0$ such that, for all functions $(\phi_x, \phi_Z) \in C_{\text{in}}$, the inequality

$$
\begin{aligned}
c_{11} U_3(\phi_x, \phi_Z) \quad \leq \quad & V(\phi_x(0)) + \tfrac{c_7}{2}\phi_Z(0)^2 + \tfrac{1}{2}\beta(\phi_x, \phi_Z)^2 \\
& + \tau^2 c_6 \int_{-\tau}^{0} \phi_Z(m)^2 dm + c_9 \int_{-\tau}^{0} W(\phi_x(m)) dm
\end{aligned} \tag{C.3}
$$

is satisfied.

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