Generating Positive and Stable Solutions through Delayed State Feedback ⋆

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Abstract

For time-varying forward-complete nonlinear systems with delay in the input, a *new reduction model* approach is proposed. It presents three advantages. First, the corresponding control laws do not include distributed terms. Second, it yields closed-loop systems with positive solutions that can be easily derived. Finally, the stabilized systems possess some robustness properties that can be estimated.

Key words: nonlinear; asymptotic stabilization; delay; positive solution.

1 Introduction

The presence of time-delay in control systems dynamics is often accompanied with instabilities and poor performances for the corresponding closed-loop schemes, as pointed out by [Niculescu, 2001], [Gu *et al.*, 2003] and the references therein. In the case on input delays, the problem received considerable attention in the last few years, see, for instance, [Mazenc *et al.*, 2004], [Karafyllis, 2006], [Michiels and Niculescu, 2007, Chapt. 8,9], [Lin and Fang, 2007], [Mazenc *et al.*, 2008] and [Krstic, 2009] to cite only a few. One of the most popular approaches used to cope with delays in the input is the *reduction model* approach, sometimes called finite spectrum assignment technique (see, for instance [Fiagbedzi *et al.*, 1986], [Wang *et al.*, 1999] for further discussions). To the best of the authors' knowledge, the reduction model approach originates in the contributions [Olbrot, 1978], [Manitius and Olbrot, 1979], [Kwon and Pearson, 1980] and [Artstein, 1982]. Since our present work owes a great deal to this technique, we recall its basic ideas in a simple case. Consider the linear system:

$$\dot{x}(t) = Ax(t) + Bu(t-\tau) \ , \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, and where $\tau > 0$ is a pointwise (or discrete) delay. The classical reduction model approach starts by introducing an operator acting on the state vector and on some "piece" of trajectory on $[t-\tau, t]$ of the input:

$$z(t) = x(t) + \int_{t-\tau}^{t} e^{A(t-\ell-\tau)} Bu(\ell) d\ell \ . \tag{2}$$

Its derivative along the trajectories of (1) in closed loop with

$$u(t) = Mz(t) \ , \tag{3}$$

where $M \in \mathbb{R}^{p\times n}$ is a constant matrix, rewrites as the delay-free system

$$\dot{z}(t) = \left(A + e^{-A\tau} BM\right) z(t). \tag{4}$$

Such a system is exponentially stable if the pair $(A, e^{-A\tau}B)$ is stabilizable and M is appropriately chosen such that $A + e^{-A\tau}BM$ is Hurwitz. Next, it can be shown that the exponential stability of (4) implies the exponential stability of the original system (1) in closed-loop with u defined by (3). Since the operator (2) includes a distributed term, this term needs to be

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estimated on-line and the closed-loop system (4) may be not robust against any arbitrary small implementation errors in the corresponding integral term, as pointed out and discussed by [Engelborghs *et al.*, 2001], [Van Assche *et al.*, 2004].

Our approach relies on the introduction of an operator different from (2) that leads to control laws without distributed terms. To ease the understanding of the major differences between this new approach and the classical reduction technique, we briefly summarize our method by applying it to the system (1). Instead of considering the operator (2), we consider

$$s(t) = x(t) + \int_{t-\tau}^{t} e^{L(t-\ell-\tau)} Bu(\ell) d\ell \ , \qquad (5)$$

where $L \in \mathbb{R}^{n\times n}$ is a matrix to be appropriately selected. Immediate calculations yield

$$\dot{s}(t) \ = \ Ls(t) + (A-L)x(t) + e^{-L\tau} Bu(t) \ . \qquad (6)$$

Next, taking $u(t) = Mx(t)$, where $M \in \mathbb{R}^{p\times n}$ is a constant matrix, we deduce that (6) simplifies to

$$\dot{s}(t) = Ls(t) \qquad (7)$$

if the matrix equality

$$A - L + e^{-L\tau} BM = 0 \qquad (8)$$

is satisfied. Moreover, if L is Hurwitz, (7) implies that $s(t)$ converges exponentially to zero. However the exponential stability of (7) does *not imply* the exponential convergence of the solution $x(t)$: this property holds *if and only if* the trivial solution of the functional equation

$$x(t) = \int_{t-\tau}^{t} e^{L(t-\ell)} (A-L) x(\ell) d\ell \qquad (9)$$

is exponentially stable. Observe that the analysis of the stability of (9) is not a trivial task; in particular, in spite of what the intuition may suggest, it is not true that if for any $\tau > 0$ there are matrices L and M such that (8) holds, then, when the delay is sufficiently small, these matrices are such that (9) is stable (we will give details on this fact at the end of Section 3.1). Therefore a rigorous analysis of the stability of (9) is needed and we will perform it by constructing Lyapunov-Krasovskii functionals which share some features with those proposed in [Carvalho, 1996]. Observe that other techniques, like for instance the ones proposed by [Teel, 1998], [Mazenc *et al.*, 2008] or LMI-based approaches, can be used to determine other types of stabilizing control laws without distributed terms. However, since our main objective is to define design strategies guaranteeing simultaneously closed-loop stability and positivity of solutions under positive initial conditions, we will not present in this paper comparisons between the advantages of such methods and our technique with respect, for instance, to issues related to the largest delay margin for constant delay, or the extension of the approaches to the case of a time-varying delay.

We believe that an important advantage of the results we are proposing is that they allow to derive simple conditions ensuring that the resulting solutions of the closed-loop systems are positive. To the best of our knowledge, this represents a novelty in the control literature. Our wish to determine positive solutions for systems with delay has several motivations: when this objective is reached, one can easily solve more general problems: in Section 5, we will see that our main result can be used to generate solutions which respect more general constraints than the constraint of sign and solutions which can be compared between each other, which is useful when is available only an approximate knowledge of the initial condition and an estimation of the state variables at each instant is desirable. Furthermore, as explained for instance in [Anh *et al.*(2009)], [Wu *et al.*, 2009], [Haddad and Chellaboina, 2004] (and the references therein), many economic models and many processes in biological, ecological and medical sciences present inherently positive variables and/or feedbacks which must respect bounds. For instance in [Grognard and Gouzé, 2005], families of Lotka-Volterra systems describing interactions between preys and predators are stabilized in the positive orthant through non-negative and bounded feedbacks without delays. Therefore having a control technique ensuring that, in the presence of delays, the solutions with initial conditions appropriately chosen respect constraints, may be crucial. However, it is worth pointing out that the main drawback of the method we are proposing here is that, except in particular cases, it applies only for sufficiently small delays and two matrices L and M such that L is Hurwitz and (8) holds have to be found, which is not always a feasible task. However, when B is invertible, $M = B^{-1} e^{L\tau}(L-A)$ is a solution of (8) and the case when B is invertible is obviously not the only one where (8) admits a solution as we shall see in Section 2.2. Moreover, even when no general solution can be found, since our stability analysis relies on a *strict* Lyapunov functional, through a robustness analysis, the stability can be established when the delay is sufficiently small by taking advantage of a sufficiently accurate approximation of the solutions of (8). We will see in Section 4 how this result can be established using our Lyapunov functional approach.

This paper is organized as follows. Section 2 is devoted to the problem formulation and to various discussions on the assumptions we are considering. In Section 3, we state and prove the main result of our work. An extension and extra results are presented in Sections 4 and 5. An example in Section 6 illustrates the main result.

Concluding remarks are drawn in Section 7.

Notations and definitions. • The notations will be simplified whenever no confusion can arise from the context. • We denote by Id_n the identity matrix in $\mathbb{R}^{n\times n}$. • We let $|\cdot|$ denote the Euclidean norm of matrices and vectors of any dimension. • Given $\phi : \mathcal{I} \to \mathbb{R}^p$ defined on an interval $\mathcal{I}$, let $|\phi|_{\mathcal{I}}$ denote its (essential) supremum over $\mathcal{I}$. • We let $C_{\text{in}} = C([-\tau, 0], \mathbb{R}^n)$ denote the set of all continuous $\mathbb{R}^n$-valued functions defined on a given interval $[-\tau, 0]$. • For a vector $V = (v_1, ..., v_k)^\top \in \mathbb{R}^k$, we will write $V \geq 0$ (resp. $V > 0$) to indicate that $v_i \geq 0$ (resp. $v_i > 0$) for all $i = 1$ to k. In this case, we will say that V is non-negative (resp. positive). • A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times p}$ is non-negative (resp. positive) if $a_{ij} \geq 0$, $i = 1, ..., n$, $j = 1, ..., p$, (resp. $a_{ij} > 0$, $i = 1, ..., n$, $j = 1, ..., p$). In this case, we will write $A \geq 0$ (resp. $A > 0$). A square matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is essentially non-negative, or cooperative, or Metzler if $a_{ij} \geq 0$, $i, j = 1, ..., n$, $i \neq j$. • Let T be a positive real number or $+\infty$. A real function $u : [0, T) \to \mathbb{R}^k$ is a non-negative (resp. positive) function if $u(t) \geq 0$ (resp. $u(t) > 0$) on the interval $[0, T)$. In this case, we will write $u \geq 0$ (resp. $u > 0$). • For a function $x : [-\tau, +\infty) \to \mathbb{R}^k$, for all $t \geq 0$, the function x_t is defined by $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-\tau, 0]$.

2 Problem statement

In the next two sections, we consider the system

$$\dot{x}(t) = f(t, x(t)) + f_\tau(t-\tau, x(t-\tau))u(t-\tau), \tag{10}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, and $\tau > 0$ is a pointwise delay and where f and f_τ are Lipschitz continuous functions. Our control objective is the design of control laws which *globally exponentially stabilize* the origin and ensure that a wide family of *positive solutions* exists and can be determined through simple criteria. A particular attention will be paid to linear and time-invariant systems of the form (10).

2.1 Assumptions

We introduce some assumptions needed for our developments.

Assumption H1. There exists a matrix $L \in \mathbb{R}^{n\times n}$, Hurwitz and Metzler, such that the function $\overline{f}(t, x) = f(t, x) - Lx$ satisfies $\overline{f}(t, x) \geq 0$ when $x \geq 0$ and $t \geq -\tau$.

Assumption H2. There exists a Lipschitz continuous function $g(t, x)$ such that, for all $x \in \mathbb{R}^n$ and $t \geq -\tau$,

$$\overline{f}(t, x) = e^{-L\tau} f_\tau(t, x) g(t, x) \tag{11}$$

and the following inequality is satisfied

$$\sup_{t\geq-\tau} |\overline{f}(t, x)| \leq f_m|x| \ , \forall x \in \mathbb{R}^n \ , \tag{12}$$

where f_m is a known positive real number.

Assumption H3. There exist an invertible matrix $R \in \mathbb{R}^{n\times n}$ and a matrix $G \in \mathbb{R}^{n\times n}$ such that

$$0 < \tau \int_{-\tau}^{0} \left|e^{G\ell}\right|^2 d\ell \leq \frac{1}{4f_m^2|R|^2|R^{-1}|^2} \tag{13}$$

and

$$L = RGR^{-1} \ . \tag{14}$$

2.2 Discussions on Assumptions H1-H3

• Some of the requirements of Assumption H1 are present to make it possible to find positive solutions. If only stability is desired, then the assumptions that L is Metzler and $\overline{f}(t, \cdot)$ is non-negative can be dropped (these assumptions are neither used in the first part of the proof of Theorem 1 in Section 3.2 nor in Appendix A).

• Recently, some conditions ensuring that a matrix is Metzler and Hurwitz have been obtained in [Narendra and Shorten, 2009]. They can be used to find matrices L such that Assumptions H1 and H2 are satisfied. In addition, in the particular case where $f(t, x) = Ax$, where A is a constant matrix, the property $\overline{f}(t, x) \geq 0$ is equivalent to $A - L \geq 0$ and then following result sheds light on Assumption H1:

Lemma 1. *Let $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ be a constant matrix. Then there exists a constant matrix $L = (l_{i,j}) \in \mathbb{R}^{n\times n}$, Metzler and Hurwitz, such that $A - L \geq 0$ if and only if A is Metzler.*

The proof of this lemma relies on the fact that $A \geq L$ implies that for all i, j, $a_{i,j} \geq l_{i,j}$ and that if A is Metzler then $A \geq -qId_n$ with $q = 1 + \sup_i\{|a_{i,i}|\}$.

• Equality (11) rewrites $f(t, x) - Lx = e^{-L\tau} f_\tau(t, x)g(t, x)$. It does not always exist a matrix L and a function $g(t, x)$ such that this equality is satisfied or finding L and $g(t, x)$ may be difficult. However, when $f_\tau(t, x)$ is invertible everywhere, (11) is satisfied with $g(t, x) = f_\tau(t, x)^{-1} e^{L\tau} \overline{f}(t, x)$. Moreover, we will show (see Remark 1), how, in some cases where the delay τ is sufficiently small, the obstacle presented by the equality (11) can be overcome by using an approximation of this equality. In the particular case where $f(t, x) = Ax$, $f_\tau(t, x) = B$, where A and B are constant matrices, then (11) is satisfied if there exists a constant matrix M such that (8) is satisfied. If the pair $(A, e^{-A\tau}B)$ is stabilizable, i.e. there exists a constant matrix N such that $A + e^{-A\tau}BN$ is Hurwitz, and, if $ABN = BNA$, then one can check that (8) is satisfied with $L = A + e^{-A\tau}BN$ and $M = Ne^{e^{-A\tau}BN\tau}$ (see Lemma 3 in Appendix D). Unfortunately, this result does not hold if $ABN \neq BNA$. However, using the Brouwer's fixed point theorem, one can prove that if there exists a constant matrix M such that $A + BM$ is Hurwitz and if τ is sufficiently small, then there exists a constant matrix L such that (8) is satisfied and L is Hurwitz (see Lemma 4 in Appendix D).

• Frequently, it is convenient to simply choose in Assumption H3 the matrices $R = Id_n$ and $L = G$. Then (13) reduces to the simple inequality $\tau \int_{-\tau}^{0} \left|e^{L\ell}\right|^2 d\ell \leq \frac{1}{4f_m^2}$ which is satisfied if $\tau \int_{-\tau}^{0} e^{2|L|\tau} d\ell \leq \frac{1}{4f_m^2}$ and therefore is satisfied for instance if $0 \leq \tau \leq \min\left\{\frac{1}{2|L|}, \frac{1}{2\sqrt{e}f_m}\right\}$. However, we have chosen to keep Assumption H3 as defined, since choosing systematically $R = Id_n$ leads to conservative results.

3 Stabilization and positive solutions

We are ready to state and prove the main result.

Theorem 1. *Consider the system (10). Assume that it satisfies Assumptions H1 to H3. Then this system in closed-loop with the control law*

$$u(t-\tau, x(t-\tau)) = -g(t-\tau, x(t-\tau)) \tag{15}$$

admits the origin as a globally uniformly exponentially stable equilibrium point. Moreover, any solution $x(t)$ of the closed-loop system with an initial condition $\phi_x \in C_{\text{in}}$ such that $\phi_x \geq 0$ and

$$\phi_x(0) - \int_{-\tau}^{0} e^{-L\ell}\overline{f}(\ell, \phi_x(\ell))d\ell > 0 \tag{16}$$

satisfies, for all $t \geq 0$, the inequality $x(t) > 0$.

3.1 Discussion of Theorem 1.

• First of all, we wish to stress that a solution of the system (10) in closed-loop with the feedback (15) with a positive initial condition is not necessarily positive, even if Assumptions H1 to H3 are satisfied. To understand why, consider the simple system $\dot{x}(t) = u(t-\tau)$, with $x \in \mathbb{R}$ which satisfies Assumptions H1 to H3 if τ is sufficiently small. From Theorem 1, we deduce that for any real number $L < 0$, we can take (with the notations of Theorem 1) $\overline{f}(t,x) = -Lx$, $g(t,x) = -e^{L\tau}Lx$, $u(t-\tau) = e^{L\tau}Lx(t-\tau)$ and therefore the closed-loop system is $\dot{x}(t) = e^{L\tau}Lx(t-\tau)$ which admits solutions with a positive initial condition which take positive and negative values, no matter how small $\tau > 0$ is (see [Mazenc *et al.*, 2009]). Hence, in general, the closed-loop systems we obtain admit solutions with positive initial conditions that take negative values and, of course, they are not non-negative systems (see, for instance, [Haddad and Chellaboina, 2004] for a definition of non-negative systems). Consequently, the results of [Haddad and Chellaboina, 2004] are not helpful to solve the problem we address because they mainly concern the stability of non-negative systems. However, some of the results presented in [Haddad and Chellaboina, 2004] play an indirect, but crucial, role in our proof of positivity of some solutions of our closed-loop systems.

• We can give an insight into the reasons why the control laws we propose give closed-loop systems with a broad family of positive solutions although they are not non-negative. To do so, we focus our attention on system (1) in the particular case where $x \in \mathbb{R}$. In this case, if $A - L > 0$, then one can easily deduce from (7) and (9) that any solution $x(t)$ with a positive initial condition such that $s(0) > 0$ is positive for all $t \geq -\tau$.

• The assumption that f has a linear growth cannot be removed. If (12) is violated, the system (10) may be not stabilizable by a continuous state feedback $u(t-\tau, x(t-\tau))$. We prove this in Appendix C.

• It is well-known that, if $f(t,x) = Ax$ where A admits an eigenvalue with a positive real part and $f_\tau = B$, where B is constant, then the system (10) cannot be locally asymptotically stabilized by a feedback $u(x(t-\tau))$ (that is by a feedback with the pointwise delay τ only), when τ is larger than a specific constant. Therefore Assumption H3 cannot be removed without being replaced by something else.

• When, for all $\tau > 0$, there are a matrix L^τ and a function g^τ such that Assumptions H1 and H2 are satisfied, it does not follow that Assumption H3 is satisfied by L^τ when τ is in a sufficiently small neighborhood of the origin. To illustrate this phenomenon, consider the system (10) with $x \in \mathbb{R}$, $f(t,x) = x$, $f_\tau(t,x) = 1$ and take $L^\tau = -\frac{1}{2\tau}\ln(1 + \frac{1}{\tau})$. Then, for any $\tau > 0$, Assumptions H1 and H2 are satisfied with $L = L^\tau$, $g(t,x) = e^{L^\tau\tau}(1 - L^\tau)$, $f_m = (1 - L^\tau)$. Next, one can check readily that with $G = L^\tau$ and any $R \neq 0$,

$$\tau \int_{-\tau}^{0} (e^{G\ell})^2 d\ell = \tau \frac{1 - e^{-2L^\tau\tau}}{2L^\tau} = \frac{\tau}{\ln(1 + \frac{1}{\tau})}$$

and $4f_m^2|R|^2|R^{-1}|^2 = 4[1 + \frac{1}{2\tau}\ln(1 + \frac{1}{\tau})]^2$. We deduce easily that inequality (13) does not hold when $\tau \in (0, \frac{1}{9}]$.

3.2 Proof of Theorem 1.

The proof splits up into two parts. The first is devoted to the stabilization of the system (10). The second is devoted to the sign analysis of some solutions of the system (10) in closed-loop with (15). To begin with, we recall a result which will be used and is an immediate consequence of Proposition 3.1 in [Haddad and Chellaboina, 2004] and its proof.

Lemma 2. *A matrix $N \in \mathbb{R}^{n\times n}$ is essentially non-negative if and only if the matrix e^{Nt} is non-negative for all $t \geq 0$.*

1. Stabilization of the system (10).

Assumption H2 ensures that for any Lipschitz continuous feedback $u(t-\tau, x(t-\tau))$, the solutions of (10) are defined over $[-\tau, +\infty)$ i.e. the system (10) is forward-complete (see [Krstic, 2009] for the definition of forward-complete system).

Next, we consider a solution $x(t)$ of (10) with an initial condition $\phi_x \in C_{\text{in}}$ and rewrite (10) as follows:

$$\dot{x}(t) = Lx(t) + f_\tau(t-\tau, x(t-\tau))u(t-\tau) + \overline{f}(t, x(t)) \ . \tag{17}$$

The time derivative of

$$\alpha(t) = \int_{t-\tau}^{t} e^{L(t-\ell-\tau)} f_\tau(\ell, x(\ell))u(\ell)d\ell \tag{18}$$

is

$$\begin{aligned}\dot{\alpha}(t) = {} & L\alpha(t) + e^{-L\tau} f_\tau(t, x(t))u(t) \\ & -f_\tau(t-\tau, x(t-\tau))u(t-\tau) \ .\end{aligned} \tag{19}$$

Thus, the derivative of

$$s(t) = x(t) + \alpha(t) \tag{20}$$

along the trajectories of (17) satisfies

$$\begin{aligned}\dot{s}(t) &= Lx(t) + \overline{f}(t, x(t)) + L\alpha(t) \\ &\quad + e^{-L\tau} f_\tau(t, x(t))u(t) \\ &= Ls(t) + e^{-L\tau} f_\tau(t, x(t))u(t) + \overline{f}(t, x(t)) \ .\end{aligned} \tag{21}$$

From (11) in Assumption H2, we deduce that:

$$\begin{aligned}\dot{s}(t) &= Ls(t) + e^{-L\tau} f_\tau(t, x(t))g(t, x(t)) \\ &\quad + e^{-L\tau} f_\tau(t, x(t))u(t) \\ &= Ls(t) + e^{-L\tau} f_\tau(t, x(t))\left[g(t, x(t)) + u(t)\right]\end{aligned} \tag{22}$$

and therefore, choosing u defined in (15), we obtain

$$\dot{s}(t) = Ls(t) \tag{23}$$

and

$$\begin{aligned}s(t) &= x(t) - \int_{t-\tau}^{t} e^{L(t-\ell-\tau)} f_\tau(\ell, x(\ell))g(\ell, x(\ell))d\ell \\ &= x(t) - \int_{t-\tau}^{t} e^{L(t-\ell)} \overline{f}(\ell, x(\ell))d\ell \ .\end{aligned} \tag{24}$$

Next, we establish that the closed-loop system is globally exponentially stable by constructing a strict Lyapunov-Krasovskii functional. We perform in details this construction in Appendix A by using the representation (23)-(24).

2. Sign of the solutions.

In this part, we consider a solution of (10) in closed-loop with (15) with an initial condition $\phi_x \geq 0$ such that (16) holds. From the expression of s in (24) and (16), we deduce that $s(0) > 0$. Since the matrix L is Metzler (see [Smith, 1995]), it follows that, for all $t \geq 0$, $s(t) > 0$ or, equivalently,

$$x(t) > \int_{t-\tau}^{t} e^{L(t-\ell)} \overline{f}(\ell, x(\ell))d\ell \ . \tag{25}$$

We deduce easily that $x(0) > 0$. Next, let us prove by contradiction that $x(t) > 0$ for all $t \in [0, +\infty)$. Let $x(t) = (x_1(t)....x_n(t))^\top \in \mathbb{R}^n$. Assume there exist $t_c > 0$ and $j \in \{1, ..., n\}$ such that $x_j(t_c) = 0$ and $x(t) > 0$ for all $t \in [0, t_c)$. Since Assumption H1 ensures that the matrix L is Metzler, Lemma 2 ensures that $e^{L(t-\ell)} \geq 0$ for all $\ell \leq t$. Moreover, we deduce from Assumption H1 and our assumption $x(t) > 0$ for all $t \in [0, t_c)$ that, for all $\ell \in [t_c - \tau, t_c]$, $e^{L(t_c-\ell)}\overline{f}(\ell, x(\ell)) \geq 0$. Consequently, $\int_{t_c-\tau}^{t_c} e^{L(t_c-\ell)}\overline{f}(\ell, x(\ell))d\ell \geq 0$. This inequality, in combination with (25), yields $x(t_c) > 0$, which contradicts the equality $x_j(t_c) = 0$. This allows us to conclude.

4 Robustness result

In this section, we extend Theorem 1 to systems

$$\dot{x}(t) = f(t, x(t)) + f_\tau(t-\tau, x(t-\tau))u(t-\tau) + h(t, x(t)) \tag{26}$$

with $x \in \mathbb{R}^n$, with the input $u \in \mathbb{R}^p$, where $\tau > 0$ is a pointwise delay, where f, f_τ and h are Lipschitz-continuous functions and where h is unknown, non-negative and *"small"* in a sense to be clarified below. Before introducing an extra assumption, we notice that Assumption H1 implies that there is a matrix $P \in \mathbb{R}^{n\times n}$, symmetric and positive definite, such that

$$PL + L^\top P \leq -R^\top R \ . \tag{27}$$

Assumption H4. There exists a real number h_m such that

$$0 \leq h_m \leq \frac{1}{8|P||R^{-1}|^2} \tag{28}$$

and for all $x \in \mathbb{R}^n$

$$\sup_{t \geq -\tau} |h(t,x)| \leq h_m|x| \ . \tag{29}$$

Moreover if $x \geq 0$, then for all $t \geq -\tau$, $h(t,x) \geq 0$.
Theorem 2. *Consider the system (26). Assume that it satisfies Assumptions H1 to H4. Then the system (26) in closed-loop with the control law*

$$u(t-\tau, x(t-\tau)) \ = \ -g(t-\tau, x(t-\tau)) \tag{30}$$

admits the origin as a globally uniformly exponentially stable equilibrium point. Moreover, any solution $x(t)$ of this closed-loop system with an initial condition $\phi_x \in C_{\text{in}}$ such that $\phi_x \geq 0$ and (16) is satisfied is such that, for all $t \geq 0$, $x(t) > 0$.

Proof. The proof, which is based on simple modifications of the proof of Theorem 1, is given in Appendix B.
Remark 1. *It is worth pointing out that Theorem 2 may be useful even for systems without uncertain term. Let us explain why. Consider the system*

$$\dot{x} = f_\alpha(t,x) + f_\tau(t-\tau, x(t-\tau))u(t-\tau) \tag{31}$$

with $x \in \mathbb{R}^n$, with $u \in \mathbb{R}^p$ the input, with $f_\tau(t-\tau, x(t-\tau)) = B$ where $B \in \mathbb{R}^{n\times p}$ is a constant matrix, where $\tau > 0$ is a pointwise delay and where f_α is a known Lipschitz-continuous function. Assume, instead of Assumptions H1 and H2, that there exist a matrix L, Hurwitz and Metzler, and a Lipschitz continuous function $g(t,x)$ such that $f_\alpha(t,x) - Lx = Bg(t,x)$, $e^{-L\tau}Bg(t,x) \geq 0$ for all $t \geq -\tau$ and $x \geq 0$ and there exists a constant $g_m > 0$ such that $|g(t,x)| \leq g_m|x|$ for all $t \geq 0$, $x \in \mathbb{R}^n$. Then (31) rewrites

$$\dot{x} = Bu(t-\tau) + f(t,x) + h(t,x) \tag{32}$$

with $f(t,x) = Lx + e^{-L\tau}Bg(t,x)$ and $h(t,x) = f_\alpha(t,x) - f(t,x)$ and the inequalities

$$|h(t,x)| \leq \left|Bg(t,x) - e^{-L\tau}Bg(t,x)\right| \leq h_m|x| \tag{33}$$

with $h_m = |B|g_m|Id_n - e^{-L\tau}|$ are satisfied everywhere. We deduce that the system $\dot{x} = Bu(t-\tau) + f(t,x)$ satisfies Assumptions H1 to H3, provided that τ is sufficiently small. If in addition $h(t,x) = [e^{L\tau} - Id_n][f(t,x) - Lx]$ is non-negative, then positive solutions can be found by using Theorem 2.

5 Applications of Theorem 1

In this section, we show how, using our technique, we succeed in finding controls which generate solutions which respect more general constraints than the mere positiveness. The results we give below are valid for linear systems only but nonlinear extensions are expected.

5.1 Comparisons between solutions

A consequence of Theorem 1 is that it provides with control laws which make it possible to compare the solutions of the closed-loop system between themselves. This is desirable when initial conditions are only approximately known and when one wants to estimate the solutions at each time instant, as is done for example in [Mazenc and Bernard, 2010] and in [Mazenc *et al.*, 2009] in different contexts.
Corollary 1. *Consider the system (10) in the particular case where $f(t,x) = Ax$, $f_\tau(t,x) = B$ where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ are constant matrices. Assume that this system satisfies Assumptions H1 to H3 with a matrix L and a function $g(t,x) = -Mx$ where $M \in \mathbb{R}^{p\times n}$ is a constant matrix. Then if $x_1(t)$, $x_2(t)$ are two solutions of the system (10) in closed-loop with $u(t-\tau) = Mx(t-\tau)$ with initial conditions $\phi_{x_1} \in C_{\text{in}}$ and $\phi_{x_2} \in C_{\text{in}}$ such that $\phi_{x_1} < \phi_{x_2}$ and*

$$\begin{aligned} &\phi_{x_1}(0) + \int_{-\tau}^0 e^{-L\ell}(A-L)\phi_{x_1}(\ell)d\ell < \\ &\phi_{x_2}(0) + \int_{-\tau}^0 e^{-L\ell}(A-L)\phi_{x_2}(\ell)d\ell \end{aligned} \tag{34}$$

then these solutions satisfy $x_1(t) < x_2(t)$ for all $t \geq -\tau$.

Proof. With the notations of Theorem 1, we have $\overline{f}(t,x) = (A-L)x$ and $A - L \geq 0$. Moreover, Assumption H2 gives $A - L + e^{-L\tau}BM = 0$. Theorem 1 ensures that (10) in closed-loop with $u(x) = Mx$, i.e. the system

$$\dot{x}(t) = Ax(t) + BMx(t-\tau) \ , \tag{35}$$

is such that if $x(t)$ is one of its solutions such that $\phi_x > 0$ and $\phi_x(0) > -\int_{-\tau}^0 e^{-L\ell}(A-L)\phi_x(\ell)d\ell$ then, for all $t \geq -\tau$, the inequality $x(t) > 0$ is satisfied. Let $\overline{x}(t) = x_2(t) - x_1(t)$ and let $\phi_{\overline{x}}$ denote the initial condition of $\overline{x}$. Then $\overline{x}$ is a solution of (35), $\phi_{\overline{x}} > 0$ and, according to (34), $\phi_{\overline{x}}(0) > -\int_{-\tau}^0 e^{-\overline{A}\ell}(A-L)\phi_{\overline{x}}(\ell)d\ell$. It follows that for all $t \geq -\tau$, the inequality $\overline{x}(t) > 0$ is satisfied. This allows us to conclude.

5.2 General bounds for the solutions

The following result shows that, for linear systems, more general constant constraints than the mere positivity can be taken into account using a control design which relies on Theorem 1.
Corollary 2. *Consider the system (10) in the particular case where $f(t,x) = Ax$, $f_\tau(t,x) = B$ where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ are constant matrices. Assume that this*

system satisfies Assumptions H1 to H3 with a matrix L and a function $g(t,x) = -Mx$ where $M \in \mathbb{R}^{p\times n}$ is a constant matrix. Then let $V \in \mathbb{R}^n$ and $W \in \mathbb{R}^n$ be vectors such that

$$V - \int_{-\tau}^{0} e^{-L\ell}(A-L)V d\ell \geq W \ . \qquad (36)$$

Then if $x(t)$ is a solution of (10) in closed-loop with $u(t-\tau) = Mx(t-\tau)$ and with an initial conditions $\phi_x \in C_{\text{in}}$ such that $V > \phi_x(\ell)$ (resp. $-V < \phi_x(\ell)$) for all $\ell \in [-\tau, 0]$ and $W > e^{Lt}s_0$ (resp. $-W < e^{Lt}s_0$) for all $t \geq 0$ with $s_0 = \phi_x(0) + \int_{-\tau}^{0} e^{-L\ell}L\phi_x(\ell)d\ell$, then $V > x(t)$ (resp. $-V < x(t)$) for all $t \geq -\tau$.

Remark 2. *Finding vectors $V > 0$, $W > 0$ such that (36) is satisfied is not a difficult task when τ is sufficiently small. In particular one can prove easily that for all real number $d > 0$ there exists a real number $v > 0$ such that $V = v(1....1)^\top$ satisfies $V - \int_{-\tau}^{0} e^{-L\ell}(A-L)V d\ell \geq \frac{d}{v}V$ when τ is sufficiently small. In addition, since L is Metzler and Hurwitz, it follows that for any vector $W > 0$, there exists $s_0 > 0$ (resp. $s_0 < 0$) such that $W \geq e^{Lt}s_0$ (resp $-W \leq e^{Lt}s_0$) for all $t \geq 0$.*

Proof. As in the proof of Corollary 1, we deduce that $\overline{f}(t,x) = (A-L)x$. From this equality, the definition of s_0 and Theorem 1, we deduce that for all $t \geq 0$,

$$x(t) - \int_{t-\tau}^{t} e^{L(t-\ell)}(A-L)x(\ell)d\ell = e^{Lt}s_0 \ . \qquad (37)$$

Let $\overline{x} = V - x$. Then, for all $t \geq 0$,

$$\overline{x}(t) + \int_{t-\tau}^{t} e^{L(t-\ell)}(A-L)[V - \overline{x}(\ell)]d\ell = V - e^{Lt}s_0 \ . \quad (38)$$

Combining (38) and (36), we deduce that, for all $t \geq 0$,

$$\overline{x}(t) - \int_{t-\tau}^{t} e^{L(t-\ell)}(A-L)\overline{x}(\ell)d\ell \geq W - e^{Lt}s_0 \ . \quad (39)$$

Since $W > e^{Lt}s_0$ for all $t \geq 0$, it follows that, for all $t \geq 0$,

$$\overline{x}(t) > \int_{t-\tau}^{t} e^{L(t-\ell)}(A-L)\overline{x}(\ell)d\ell \ . \qquad (40)$$

This allows us to conclude by arguing as we did in the second part of Section 3.2. Due to space limitation, we do not prove how lower bounds can be obtained (this proof is similar to that just given).

6 Illustrative Example

In this section, we illustrate Theorem 1 with the scalar nonlinear time-varying system:

$$\dot{x}(t) = u(t-\tau) - \sin(t)\tfrac{x^3(t)}{1+x^2(t)} \ , \qquad (41)$$

with $x \in \mathbb{R}$, where $u \in \mathbb{R}$ is the input and τ a constant delay such that

$$\tau \in \left(0, \frac{2}{25}\right] \ . \qquad (42)$$

Even for this simple system, finding controls which stabilize it and, at the same time, lead to a system for which one can find a large family of positive solutions, is not an easy task. For instance, if we choose a linear delayed-state feedback $u(t-\tau) = -kx(t-\tau)$, where k is a positive real number, we get

$$\dot{x}(t) = -kx(t-\tau) - \sin(t)\tfrac{x^3(t)}{1+x^2(t)} \qquad (43)$$

and analyzing the sign of the solutions of this system is not easy. Adopting the classical reduction model approach does not lead to a solution too. Indeed, if we take $z(t) = x(t) + \int_{t-\tau}^{t} u(\ell)d\ell$, we obtain

$$\dot{z}(t) = u(t) - \sin(t)\tfrac{x^3(t)}{1+x^2(t)} \qquad (44)$$

and we choose $u(t) = -kz(t) + \sin(t)\frac{x^3(t)}{1+x^2(t)}$, where k is a positive real number. Then the system $\dot{z}(t) = -kz(t)$ is obtained and $z(0) > 0$ implies that $z(t) > 0$ for all $t \geq 0$. Therefore $x(t) > -\int_{t-\tau}^{t} u(\ell)d\ell$ for all $t \geq 0$. However, this last inequality does not make it possible to analyze the sign of $x(t)$ because the expression of the control is $u(t) = -k\left[x(t) + \int_{t-\tau}^{t} u(\ell)d\ell\right] + \sin(t)\frac{x^3(t)}{1+x^2(t)}$ and therefore $u(t)$ can take positive and negative values. Likewise, the techniques of Chapter 11 in [Krstic, 2009] and [Karafyllis, 2010] apply, but does not lead to closed-loop systems with solutions for which the sign can be analyzed.

Next, we check that Theorem 1 applies to (41) and obtain that way control laws which stabilize (41) and generate positive solutions. With the notations of Section 3, we have $f_\tau(t,x) = 1$, $f(t,x) = -\sin(t)\frac{x^3}{1+x^2}$. Assumption H1 is satisfied with, for instance, $L = -4$. Indeed, $\overline{f}(t,x) = 4x - \sin(t)\frac{x^3}{1+x^2}$ and therefore $\overline{f}(t,x) > 0$ if $x > 0$. Assumption H2 is satisfied with $g(t,x) = e^{-4\tau}\left[4x - \sin(t)\frac{x^3}{1+x^2}\right]$ and $f_m = 5$. Finally, we choose $R = 1$. Then $G = -4$. The property (42) ensures that τ is such that

$$0 < \tau \int_{-\tau}^{0} e^{-8\ell}d\ell \leq \tfrac{1}{100} \ , \qquad (45)$$

which implies that Assumption H3 is satisfied. From Theorem 1, we deduce that the system (41) in closed-loop with the feedback

$$u(t,x) = -e^{-4\tau}\left[4x - \sin(t)\frac{x^3}{1+x^2}\right]$$

admits the origin as a globally uniformly exponentially stable equilibrium point. Moreover, if a solution $x(t)$ has an initial condition $\phi_x \in C_{\text{in}}$ such that $\phi_x > 0$ and

$$s(0) = \phi_x(0) - a_0 > 0 \tag{46}$$

with $a_0 = \int_{-\tau}^{0} e^{4\ell}\left[4\phi_x(\ell) - \sin(\ell)\frac{\phi_x(\ell)^3}{1+\phi_x(\ell)^2}\right] d\ell$ then $x(t) > 0$ for all $t \geq -\tau$. The condition (46) allows us to determine positive solutions with, roughly speaking, simple initial conditions. For example, one can check that any solution with a positive and constant initial condition c_0 is positive everywhere as follows. Let $\phi_x(\ell) = c_0 > 0$ for all $\ell \in [-\tau, 0]$. Then

$$\begin{aligned} \tfrac{s(0)}{c_0} &= 1 - \int_{-\tau}^{0} e^{4\ell}\left[4 - \sin(\ell)\frac{c_0^2}{1+c_0^2}\right] d\ell \\ &= e^{-4\tau} + \tfrac{c_0^2}{1+c_0^2}\int_{-\tau}^{0} e^{4\ell}\sin(\ell)d\ell\ . \end{aligned} \tag{47}$$

Since $\tau \in (0, \frac{1}{25}]$, we deduce that

$$\tfrac{s(0)}{c_0} \geq e^{-4\tau} + \int_{-\tau}^{0} e^{4\ell}\sin(\ell)d\ell\ . \tag{48}$$

Using again $\tau \in (0, \frac{1}{25}]$, one can prove that $s(0) > 0$. We end this section with numerical simulations of the closed-loop system we obtained. This system rewrites

$$\begin{aligned} \dot{x}(t) &= e^{-4\tau}\left[-4x(t-\tau) + \sin(t-\tau)\tfrac{x(t-\tau)^3}{1+x(t-\tau)^2}\right] \\ &\quad - \sin(t)\tfrac{x^3(t)}{1+x^2(t)}, \end{aligned} \tag{49}$$

and the time-evolution of its solution for a positive initial condition is depicted in Fig.1.

7 Conclusions

For a family of nonlinear systems with a pointwise delay in the input, we have presented a control design based on a reduction model approach of a new type. A distinctive feature of our control laws is that they do *not incorporate distributed terms* and their novelty lies in the fact that they generate *positive solutions* which can be easily found by performing a simple analysis of the initial conditions. We have proved the robustness of the stabilization with respect to additive uncertain terms. Much remains to be done: for instance we surmise that robustness results with respect to uncertainties on the delay τ can be established. Also, we conjecture that our results extend to the case of multiple delays in the input.

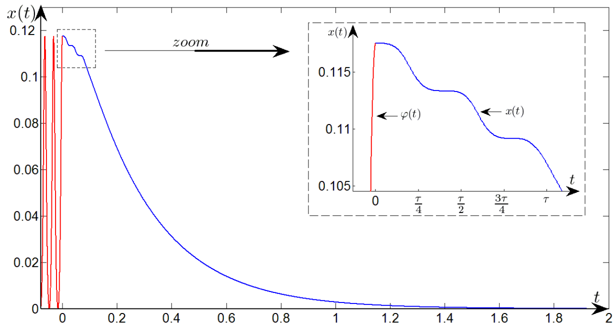

Fig. 1. The time-evolution of the system (49) under the initial condition $\varphi(t) := 1/20\left(\exp(\cos(5\pi t/\tau)) - e^{-1}\right)$, for all $t \in [-\tau, 0]$, with $\tau = 2/25$.

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A Lyapunov-Krasovskii construction

In this section, we construct a strict Lyapunov-Krasovskii functional for the system (23), (24). The derivative of the positive definite Lyapunov function

$$V_1(s) = s^\top P s \ , \tag{A.1}$$

where $P \in \mathbb{R}^{n\times n}$ is a matrix satisfying (27), along the trajectories of (23) satisfies, for all $t \geq 0$,

$$\dot{V}_1(t) \leq -|Rs(t)|^2 \ . \tag{A.2}$$

Next, from (14) in Assumption H3 and (24), it follows that, for all $m \in \mathbb{R}$, $Re^{Lm} = e^{Gm}R$ and therefore, for all $t \geq 0$,

$$Rx(t) = Rs(t) + \int_{t-\tau}^{t} e^{G(t-\ell)} R\overline{f}(\ell, x(\ell))d\ell \ .$$

This equality, in combination with Assumption H2, gives, for all $t \geq 0$,

$$|Rx(t)| \leq |Rs(t)| + f_m|R|\beta(t, x_t) \ ,$$

with $\beta(t,x_t) = \int_{t-\tau}^{t} \left|e^{G(t-\ell)}\right| |x(\ell)|d\ell$. Let $\gamma(x_t) = \int_{t-\tau}^{t} |x(\ell)|^2 d\ell$. From Cauchy-Schwarz's inequality, we deduce that

$$\beta(t,x_t) \leq \sqrt{\int_{t-\tau}^{t} \left|e^{G(\ell-t)}\right|^2 d\ell}\sqrt{\gamma(x_t)} \ . \tag{A.3}$$

We deduce that, for all $t \geq 0$,

$$\begin{aligned} |Rx(t)| \leq\ & |Rs(t)| \\ & + f_m|R|\sqrt{\int_{-\tau}^{0} \left|e^{G\ell}\right|^2 d\ell}\sqrt{\gamma(x_t)} \ . \end{aligned} \tag{A.4}$$

Next, we consider the quadratic function

$$Q(x) = |Rx|^2 \tag{A.5}$$

which is positive definite because R is invertible. We deduce from (A.4) that, for all $t \geq 0$,

$$Q(x(t)) \leq 2|Rs(t)|^2 + 2f_m^2|R|^2 \left(\int_{-\tau}^{0} \left|e^{G\ell}\right|^2 d\ell\right) \gamma(x_t) \ . \tag{A.6}$$

It follows that

$$Q(x(t)) \leq 2|Rs(t)|^2 + m(\tau)\delta(x_t) \tag{A.7}$$

with $m(\tau) = 2f_m^2|R|^2|R^{-1}|^2 \int_{-\tau}^{0} \left|e^{G\ell}\right|^2 d\ell$ and $\delta(x_t) = \int_{t-\tau}^{t} Q(x(\ell))d\ell$. Let

$$V_2(\phi) = \frac{1}{\tau}\int_{-\tau}^{0}\left(\int_{m}^{0} Q(\phi(\ell))d\ell\right)dm \ . \tag{A.8}$$

Then, for all $t \geq 0$, $V_2(x_t) = \frac{1}{\tau}\int_{t-\tau}^{t}\left(\int_{m}^{t} Q(x(\ell))d\ell\right)dm$ and

$$\dot{V}_2(t) = Q(x(t)) - \tfrac{1}{\tau}\delta(x_t) \ . \tag{A.9}$$

From (A.7), we deduce that, for all $t \geq 0$,

$$\dot{V}_2(t) \leq 2|Rs(t)|^2 + \tfrac{\tau m(\tau)-1}{\tau}\delta(x_t) \ . \tag{A.10}$$

From Assumption H3, we deduce that, for all $t \geq 0$,

$$4\dot{V}_2(t) \leq 8|Rs(t)|^2 - \tfrac{2}{\tau}\delta(x_t) \ . \tag{A.11}$$

On the other hand, (A.9) gives

$$-\dot{V}_2(t) = -Q(x(t)) + \tfrac{1}{\tau}\delta(x_t) \ . \tag{A.12}$$

Adding (A.11) and (A.12) and dividing by 3, we obtain, for all $t \geq 0$,

$$\dot{V}_2(t) \leq \lambda(s_t, x_t) \ , \tag{A.13}$$

with $\lambda(s_t, x_t) = \frac{1}{3}\left(8|Rs(t)|^2 - Q(x(t)) - \frac{1}{\tau}\delta(x_t)\right)$. Let

$$V_3(\psi,\phi) = \frac{8}{3}V_1(\psi) + V_2(\phi) + \frac{1}{6}\delta(\phi) \ . \tag{A.14}$$

Then (A.2) and (A.13) imply that, for all $t \geq 0$,

$$\begin{aligned}\dot{V}_3(t) &\leq -\tfrac{8}{3}|Rs(t)|^2 + \lambda(s_t,x_t) \\ &\quad + \tfrac{1}{6}[Q(x(t)) - Q(x(t-\tau))] \\ &\leq -\tfrac{1}{6}Q(x(t)) - \tfrac{1}{3\tau}\delta(x_t) \ .\end{aligned} \tag{A.15}$$

Finally, let

$$V_4(t,\phi) = V_3\left(\varsigma(t,\phi),\phi\right) \tag{A.16}$$

with $\varsigma(t,\phi) = \phi(0) - \int_{-\tau}^{0} e^{-L\ell}\overline{f}(t+\ell,\phi(\ell))d\ell$. We have

$$\begin{aligned}V_4(t,\phi) &= \tfrac{8}{3}V_1\left(\phi(0) - \int_{-\tau}^{0} e^{-L\ell}\overline{f}(t+\ell,\phi(\ell))d\ell\right) \\ &\quad + V_2(\phi) + \tfrac{1}{6}\delta(\phi) \ .\end{aligned} \tag{A.17}$$

Since P is symmetric and positive definite, we deduce that for all constant $c_1 \in (0,1)$ and for all vectors $v_1 \in \mathbb{R}^n$, $v_2 \in \mathbb{R}^n$ the inequality $(v_1+v_2)^\top P(v_1+v_2) \geq (1-c_1)v_1^\top P v_1 + \left(1 - \frac{1}{c_1}\right)v_2^\top P v_2$ is satisfied. Since V_2 is non-negative, we deduce that for all $c_1 \in (0,1)$, the inequality

$$\begin{aligned}V_4(t,\phi) &\geq \tfrac{8}{3}(1-c_1)\phi(0)^\top P\phi(0) \\ &\quad + \tfrac{8}{3}\left(1 - \tfrac{1}{c_1}\right)\varphi(t,\phi) + \tfrac{1}{6}\delta(\phi) \ ,\end{aligned} \tag{A.18}$$

with $\varphi(t,\phi) = \int_{-\tau}^{0} e^{-L\ell}\overline{f}(t+\ell,\phi(\ell))d\ell^\top P \int_{-\tau}^{0} e^{-L\ell}\overline{f}(t+\ell,\phi(\ell))d\ell$, holds. From Assumption H2 and the fact that Q is a positive definite quadratic function, we deduce that there exists a constant $c_2 > 0$ such that $\varphi(t,\phi) \leq c_2\left(\int_{-\tau}^{0}\sqrt{Q(\phi(\ell))}d\ell\right)^2$. From Jensen's inequality, we deduce that $\varphi(t,\phi) \leq c_3\delta(\phi)$ with $c_3 = \tau c_2$. It follows that

$$V_4(t,\phi) \geq \tfrac{8}{3}(1-c_1)\phi(0)^\top P\phi(0) + c_4\delta(\phi) \tag{A.19}$$

with $c_4 = \frac{1}{6} + \frac{8}{3}\left(1 - \frac{1}{c_1}\right)c_3$. Choosing $c_1 = \frac{16c_3}{1+16c_3} \in (0,1)$, we deduce that

$$V_4(t,\phi) \geq c_5\phi(0)^\top P\phi(0) \tag{A.20}$$

with $c_5 = \frac{8}{3(1+16c_3)}$. Similarly, through simple but lengthy calculations, one can determine a constant $c_6 > 0$ such that

$$V_4(t,\phi) \leq c_6|\phi|^2_{[-\tau,0]} \ . \tag{A.21}$$

It follows that V_4 is a candidate Lyapunov-Krassovskii functional (see for instance [Malisoff and Mazenc, 2009] for the definition of Lyapunov-Krasovskii functional). Moreover, it follows straightforwardly from (A.15) that, for all $t \geq 0$,

$$\dot{V}_4(t) \leq -\tfrac{1}{6}Q(x(t)) - \tfrac{1}{2\tau}\int_{t-\tau}^{t} Q(x(\ell))d\ell \ . \tag{A.22}$$

Since Q is positive definite, the Lyapunov-Krasovskii Theorem ensures that the origin of (17) in closed-loop with (15) is globally uniformly asymptotically stable. Furthermore, using the inequality $\int_{t-\tau}^{t}\int_{\ell}^{t} Q(x(m))dmd\ell \leq \tau\int_{t-\tau}^{t} Q(x(\ell))d\ell$, one can prove through some tedious but simple calculations that there exists a constant $c_7 > 0$ such that, for all $t \geq 0$,

$$\dot{V}_4(t) \leq -c_7 V_4(t, x_t) \ . \tag{A.23}$$

Finally, using this inequality and (A.20), (A.21), (A.23), one can prove that the closed-loop system is globally uniformly exponentially stable.

B Proof of Theorem 2

Since this proof is based on simple modifications of the proof of Theorem 1, we only present the steps where there are explicit differences between the two proofs.

1. Stabilization of the system (26).

With $s(t)$ defined in (24), we have, instead of (23),

$$\dot{s}(t) = Ls(t) + h(t, x(t)) \ . \tag{B.1}$$

It follows that

$$\dot{V}_1(t) \ \leq \ -|Rs(t)|^2 + 2|P||R^{-1}Rs(t)||h(t, x(t))| \ . \tag{B.2}$$

From Assumption H4, we deduce that

$$\dot{V}_1(t) \ \leq \ -|Rs(t)|^2 + 2|P||R^{-1}|h_m|Rs(t)||x(t)| \ . \tag{B.3}$$

Using the triangle inequality and the inequality $|x|^2 \leq |R^{-1}|^2 Q(x)$, we obtain

$$\dot{V}_1(t) \ \leq \ -\tfrac{1}{2}|Rs(t)|^2 + c_8 h_m^2 Q(x(t)) \ , \tag{B.4}$$

with $c_8 = 2|P|^2|R^{-1}|^4$. The inequality (A.13) is not modified by the presence of h. We deduce that

$$\dot{V}_3(t) \ \leq \ -\tfrac{1}{6}Q(x(t)) - \tfrac{1}{3\tau}\delta(x_t) + \tfrac{8}{3}c_8 h_m^2 Q(x(t)) \ . \tag{B.5}$$

Assumption H4 ensures that $\frac{8}{3}c_8 h_m^2 \leq \frac{1}{12}$. It follows that that

$$\dot{V}_3(t) \ \leq \ -\tfrac{1}{12}Q(x(t)) - \tfrac{1}{3\tau}\delta(x_t) \ . \tag{B.6}$$

Next, arguing as we did at the end of Section A, we can establish again that the closed-loop system is globally uniformly exponentially stable.

2. Sign of the solutions.

Consider a solution $x(t)$ of the system (26) in closed-loop with (30) and with an initial condition $\phi_x \geq 0$ such that the condition (16) is satisfied. We have

$$\begin{aligned} \dot{s}(t) \ &= \ Ls(t) + h(t, x(t)) \ , \\ s(t) \ &= \ x(t) - \int_{t-\tau}^{t} e^{L(t-\ell)}\overline{f}(\ell, x(\ell))d\ell \ , \end{aligned} \tag{B.7}$$

$s(0) > 0$ and $x(0) > 0$. Let us show by contradiction that $x(t) > 0$ for all $t \geq 0$. Assume that there exists $t_c > 0$ and $j \in \{1, ..., n\}$ such that $x_j(t_c) = 0$ and $x(t) > 0$ for all $t \in [0, t_c)$. We can prove as we did in the second part of the proof of Theorem 1 that $\int_{t_c-\tau}^{t_c} e^{L(\ell-t_c)}\overline{f}(\ell, x(\ell))d\ell \geq 0$. It follows from the second equality in (24) that $x(t_c) \geq s(t_c)$. Assumption H4 implies that $h(t, x(t)) \geq 0$ for all $t \in [-\tau, t_c)$ and since $s(0) > 0$ and L is Metzler, we deduce from (B.1) that $s(t_c) > 0$. Hence, $x(t_c) \geq s(t_c)$ implies that $x(t_c) > 0$. This is in contradiction the equality $x_j(t_c) = 0$. This allows us to conclude.

C Finite escape time phenomenon

In this appendix, we adapt the result of the appendix of [Mazenc and Bliman, 2006] to prove that, for any delay $\tau > 0$ and for any continuous time-varying state feedback $u(t, x)$ the system

$$\dot{x}(t) = u(t - \tau, x(t - \tau)) + x(t)^4 + x(t) \tag{C.1}$$

with $x \in \mathbb{R}$ admits solutions which go to the infinity in finite time.

Let $u(t, x)$ be a continuous feedback and let $u_m = \sup_{\ell \in [-\tau, -\frac{2\tau}{3}]} |u(\ell, 0)|$. Consider a solution $x(t)$ of (C.1) in closed-loop with $u(t-\tau, x(t-\tau))$ with an initial condition $\phi_x \in C_{\text{in}}$ such that $\phi_x(\ell) = 0$ for all $\ell \in \left[-\tau, -\frac{2\tau}{3}\right]$ and $\phi_x(\ell) = \phi_0$ for all $\ell \in \left[-\frac{\tau}{3}, 0\right]$ where $\phi_0 = u_m + \left(\frac{2}{\tau}\right)^{\frac{1}{3}}$. Then, for all $t \in \left[0, \frac{\tau}{3}\right]$, we have

$$\dot{x}(t) = u(t - \tau, 0) + x(t)^4 + x(t) \ . \tag{C.2}$$

Next, we prove by contradiction that, $x(t) > u_m$, for all $t \in \left[0, \frac{\tau}{3}\right]$. Assume there exists $t_a \in \left(0, \frac{\tau}{3}\right]$ such that $x(t) > u_m$, for all $t \in [0, t_a)$ and $x(t_a) = u_m$. Then, we deduce from (C.2) and the definition of u_m that for all $t \in [0, t_a)$, $\dot{x}(t) \geq x(t)^4 \geq 0$. It follows that $x(t_a) \geq \phi_0 > u_m$. This yields a contradiction with the definition of t_a. Therefore $x(t) > u_m$, for all $t \in \left[0, \frac{\tau}{3}\right]$. We deduce that, for all $t \in \left[0, \frac{\tau}{3}\right]$,

$$\dot{x}(t) > x(t)^4 \ . \tag{C.3}$$

By integrating this inequality, we deduce that, for all $t \in \left[0, \min\{t_b, \frac{\tau}{3}\}\right)$, where $[0, t_b)$ is the largest interval of definition of $x(t)$, we have $x(t) \geq \frac{\phi_0}{\left[1-3\phi_0^3 t\right]^{\frac{1}{3}}}$. From the expression of ϕ_0, we deduce that, for all $t \in \left[0, \min\{t_b, \frac{\tau}{3}\}\right)$, $x(t) \geq \frac{\phi_0}{\left[1-\frac{6}{\tau}t\right]^{\frac{1}{3}}}$. It follows that $t_b \in \left(0, \frac{\tau}{6}\right]$ and therefore the finite escape time phenomenon occurs.

D Technical results

Lemma 3. *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $N \in \mathbb{R}^{p\times n}$ be constant matrices such that $ABN = BNA$. Then the matrices $L = A + e^{-\tau A}BN$ and $M = Ne^{e^{-\tau A}BN\tau}$ satisfy the equality*

$$A - L + e^{-L\tau}BM = 0 \ . \tag{D.1}$$

Proof. Let us introduce the simplifying notation $E = A - L + e^{-L\tau}BM$. We have

$$\begin{aligned} E &= A - L + e^{-L\tau}BNe^{e^{-\tau A}BN\tau} \\ &= A - L + e^{-L\tau}BNe^{(L-A)\tau} \\ &= [(A-L)e^{(-L+A)\tau} + e^{-L\tau}BN]e^{(L-A)\tau} \\ &= [e^{(-L+A)\tau}(A-L) + e^{-L\tau}BN]e^{(L-A)\tau} \ . \end{aligned} \tag{D.2}$$

Since $ABN = BNA$, we have $LA = AL$ and therefore $e^{(-L+A)\tau} = e^{-L\tau}e^{A\tau}$. We deduce that

$$E = e^{-L\tau}\left[e^{A\tau}(A-L) + BN\right]e^{(L-A)\tau} = 0 \ . \tag{D.3}$$

Lemma 4. *Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $M \in \mathbb{R}^{p\times n}$ be constant matrices. If*

$$\tau \in \left[0, \frac{1}{2e(|A| + |BM| + 1)}\right] \tag{D.4}$$

then there exists $L \in \mathbb{R}^{n\times n}$ such that

$$A - L + e^{-L\tau}BM = 0 \ . \tag{D.5}$$

Moreover, if $A + BM$ is Hurwitz, and if τ is sufficiently small, then L is Hurwitz.

Proof. Let

$$\mathcal{S} = \{X \in \mathbb{R}^{n\times n} : |X| \leq 2e(|A| + |BM| + 1)\} \ . \tag{D.6}$$

Let $\Psi \in C(\mathcal{S}, \mathbb{R}^{n\times n})$ be the function defined by

$$\Psi(X) = A + e^{-X\tau}BM \ . \tag{D.7}$$

Then, for all $X \in \mathcal{S}$,

$$\begin{aligned} |\Psi(X)| &\leq |A| + e^{|X|\tau}|BM| \\ &\leq (|A| + |BM|)\left(1 + e^{|X|\tau}\right) \\ &\leq [|A| + |BM|]\left[1 + e^{2e(|A|+|BM|+1)\tau}\right] \ . \end{aligned} \tag{D.8}$$

Since $0 \leq \tau \leq \frac{1}{2e(|A|+|BM|+1)}$, we deduce that

$$|\Psi(X)| \leq (|A| + |BM|)(1 + e) \ . \tag{D.9}$$

It follows that $\Psi(X) \in \mathcal{S}$. Since Ψ is continuous, we deduce from Brouwer's fixed point theorem that there exists $L \in \mathcal{S}$ such that $\Psi(L) = L$. This allows us to conclude that (D.5) admits a solution. Finally, since $\mathcal{S}$ is a compact set, we deduce that L converges to $A + BM$ as τ goes to zero. We deduce that L is Huwitz when τ is sufficiently small.