Global Stabilization of Oscillators With Bounded Delayed Input

Frédéric Mazenc[*], Sabine Mondié[†] and Silviu-Iulian Niculescu[‡]

Abstract. The problem of globally asymptotically stabilizing by bounded feedback an oscillator with an arbitrary large delay in the input is solved. A first solution follows from a general result on the global stabilization of null controllable linear systems with delay in the input by bounded control laws with a distributed term. Next, it is shown through a Lyapunov analysis that the stabilization can be achieved as well when neglecting the distributed terms. It turns out that this main result is intimately related to the output feedback stabilization problem.
Key words. Global stability, oscillator, delay, Razumikhin theorem.

1 Introduction

The family of the linear systems described by

$$\dot{x}(t) = Ax(t) + Bu(t-\tau) \tag{1}$$

where $A \in \Re^{n\times n}, B \in \Re^{n\times m}$, τ is a delay, u is the input and the initial condition is

$$x(\theta) = \phi(\theta), \theta \in [-\tau, 0]$$

is one of the simplest class of models with delay. Nevertheless, they describe properly a number of phenomena commonly present in controlled processes such as transport of information or products, lengthy computations, information processing, delays inherent to sensors, etc... Well known approaches for the control of these systems include in an explicit or implicit manner a predictor of the state at time $t+\tau$. Some of the more widely used are the Smith predictor [17], [16], Process-Model Control schemes [20], and finite spectrum assignment techniques [6], [4]. A common drawback, linked to the internal instability of the prediction, is that they fail to stabilize unstable systems [10]. As shown in [11], it is possible to overcome this problem by introducing a periodic resetting of the predictor.

A common concern in practical problems is the use of bounded control laws. It is well known that for linear systems with poles in the open right half plane, only locally stabilizing control laws can be obtained if there is a bound on the input, and that global asymptotic stability can be achieved only for systems with poles in the closed left half plane, named null-controllable systems. As shown in [18], the problem can be decomposed into two fundamental subproblems, namely, the control of chains of integrators of arbitrary finite length, and that of oscillators of arbitrary finite

[*]INRIA-INRA, Projet MERE, UMR Analyse des Systèmes et Biométrie, INNRA, 2 pl. Viala, 34060 Montpellier, France, email: mazenc@helios.ensam.inra.fr

[†]Depto. de Control Automático, CINVESTAV-IPN. A.P. 14-740. 07360, México D.F. email: smondie@ctrl.cinvestav.mx

[‡]Heudiasyc, UTC, UMR CNRS 6599, BP 20529, 60205 Compiègne, France, email: Silviu.Niculescu@hds.utc.fr

multiplicity. Solutions to this problem in the framework of systems with no delay are given in [19], [18], [7].

From the above, it follows that the study of linear systems whose inputs are both delayed and bounded is useful in many applications.

In this paper, we focus our attention on the stabilization of a simple oscillator with arbitrarily large delay in the input described by

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & x_2(t), \\ \dot{x}_2(t) & = & -x_1(t) + u(t-\tau). \end{array} \right. \tag{2}$$

We restrict our analysis to the case where the frequency is one for the sake of simplicity, but all the results that are presented can be adapted easily to the system

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & \alpha x_2(t), \\ \dot{x}_2(t) & = & -\beta x_1(t) + u(t-\tau), \end{array} \right. \tag{3}$$

where α and β are strictly positive or strictly negative real numbers because this system can be transformed into the system (2) by a time rescaling and a change of coordinates.

To the best authors' knowledge, the issue of the stabilization of the oscillator was first discussed in [9] in the 1940s for a second-order (delayed) friction equation. Further comments and remarks on delayed oscillatory systems can also be found in [2, 3].

The nature of oscillators is significantly different from that of chains of integrators: for example, when the input is zero, the solutions of oscillators are trigonometric functions of the time while those of integrators are polynomial functions of the time. Not surprisingly, the technique of proof we use is significantly different from the approach of [8] for chains of integrators with bounded delayed inputs based on the use of saturated control laws introduced in [19] that only require the knowledge of an upper bound on the delay. In the case of oscillators, we take advantage of the properties of the explicit solutions to zero inputs to determine expressions of the control laws which depend explicitly on the exact value of the delay. The proof of the result relies on the celebrated Lyapunov-Razumikhin theorem. The key feature of our control design are that for arbitrarily large delays it allows us to determine a family of globally asymptotically stabilizing state feedbacks which contains elements arbitrarily small in norm. The reinterpretation of this result in the context of output feedback stabilization leads to the following result : for *any* linear output one can find an arbitrarily large delay for which the problem of globally asymptotically stabilizing the oscillator by bounded feedback is solvable.

The paper is organized as follows. In Section 2 a general result on the stabilization of null-controllable systems is established via distributed control laws. The procedure is applied to the oscillator. Next, a stabilizing bounded state feedback is proposed. In Section 3 this last result is embedded in the output feedback context. Some concluding remarks are presented in Section 4. Razumikhin's Theorem is recalled in the appendix, and an illustrative example is given.

Preliminaries:

- For a real valued C^1 function $k(\cdot)$, we denote by $k'(\cdot)$ its first derivative.
- A function $\alpha : [0,+\infty) \longrightarrow [0,+\infty)$ is said to be of class $\mathcal{K}_\infty$ if it is zero at zero, strictly increasing and unbounded.
- We denote by $\sigma(\cdot)$ an odd nondecreasing smooth saturation such that

  - $\sigma(s) = s$ when $s \in [0,1]$,
  - $\sigma(s) = \frac{3}{2}$ when $s \geq 2$,
  - $-s \leq \sigma(s) \leq s$, $0 \leq \sigma'(s) \leq 1$.

2 Stabilization by bounded delayed control laws

We establish first a general result on the stabilization of null-controllable linear systems with delayed bounded inputs, using a control law that contains distributed elements. Next we particularize this result to the cases of the oscillator. Finally, we exploit this last result in order to determine state feedback laws which do not contain distributed elements.

2.1 Distributed control laws for null-controllable systems

Lemma 2.1 *Consider a controllable linear multivariable system with delay in the input described by*

$$\dot{x}(t) = Ax(t) + Bu(t-\tau) \tag{4}$$

where $A \in \Re^{n\times n}, B \in \Re^{n\times m}$ is such that (A, B) is null-controllable, and the delays in the input is $\tau \geq 0$, with initial conditions

$$x(\theta) = \phi(\theta), \theta \in [-\tau, 0]$$

where $\phi(\cdot)$ is a continuous function. Then the following problems are equivalent.
(i) The control law

$$u(t) = \xi(x) \tag{5}$$

globally asymptotically stabilizes the system (4) when $\tau = 0$.
(ii) The distributed control law

$$u(t-\tau) = \xi(e^{A\tau}x(t-\tau) + \int_{t-\tau}^{t} e^{(t-s)A}Bu(s-\tau)ds) \tag{6}$$

globally asymptotically stabilizes the system (4).

Proof. The result follows in a straightforward manner from the fact that

$$x(t) = e^{A\tau}x(t-\tau) + \int_{t-\tau}^{t} e^{A(t-s)}Bu(s-\tau)ds.$$

■

2.2 Distributed control laws for the oscillator

As a preliminary step to the main part of the work, we specialize this general result to the case of an oscillator. Recall first a very simple result whose proof is omitted.

Lemma 2.2 *Consider the system*

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & x_2(t), \\ \dot{x}_2(t) & = & -x_1(t) + u, \end{array} \right. \tag{7}$$

where u is the input. This system is globally asymptotically stabilized by the bounded control law

$$u(x_1(t), x_2(t)) = -\varepsilon\sigma(x_2(t)) \tag{8}$$

where ε is a strictly positive parameter and $\sigma(\cdot)$ is the bounded function defined in the preliminaries.

Using the Cauchy formula for the oscillator

$$\begin{aligned} x_1(t) &= \cos(\tau)x_1(t-\tau) + \sin(\tau)x_2(t-\tau) - \int_{t-\tau}^{t} \sin(s-t)u(s-\tau)ds, & (9) \\ x_2(t) &= -\sin(\tau)x_1(t-\tau) + \cos(\tau)x_2(t-\tau) - \int_{t-\tau}^{t} \cos(s-t)u(s-\tau)ds, & (10) \end{aligned}$$

and Lemma 2.1, it follows straightforwardly that the distributed delayed control law

$$u(t-\tau) = -\varepsilon\sigma(-\sin(\tau)x_1(t-\tau) + \cos(\tau)x_2(t-\tau) - \int_{t-\tau}^{t} \cos(s-t)u(s-\tau)ds) \tag{11}$$

globally asymptotically stabilizes the system (2).

2.3 Delayed state feedbacks for the oscillator

The control law (11) involves distributed elements. The implementation of such terms using numerical approximation is time consuming and, in some cases, may lead to instability [10]. One can observe that in (10) the distributed term is of the order of ε whereas the other term does not depend on ε, which implies that, roughly speaking, the control law (11) is equal to a term of order ε plus a distributed term of order ε^2. This remark leads us to conjecture that as ε decreases, the influence of the distributed term decreases as well. Then, the question that arises naturally, is wether or not stability is preserved if, when ε, the integral terms are neglected, that is to say, if instead of (6), the control law

$$u(t-\tau) = -\varepsilon\sigma(-\sin(\tau)x_1(t-\tau) + \cos(\tau)x_2(t-\tau)) \tag{12}$$

is used. The main result of the work, stated below, gives an answer to this question. In Appendix B, we perform simulations to validate this result.

Theorem 2.3 *Let τ be a positive number. The origin of the system (2) is globally asymptotically stabilized by the control law*

$$u(x_1, x_2) = -\varepsilon\sigma(-\sin(\tau)x_1 + \cos(\tau)x_2) \tag{13}$$

where

$$\varepsilon \in \left]0, \min\left\{\frac{1}{2}, \frac{1}{324\tau^2}, \frac{1}{40\tau}\right\}\right] \tag{14}$$

and $\sigma(\cdot)$ is the bounded function defined in the preliminaries.

Proof. The system (2) in closed loop with the feedback (13) rewrites

$$\left\{ \begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -x_1(t) - \varepsilon\sigma\left(x_2(t) + B_2(t)\right), \end{aligned} \right. \tag{15}$$

with

$$\begin{aligned} B_2(t) &= -\int_{t-\tau}^{t} \cos(s-t)u_s(x_1(s-\tau), x_2(s-\tau))ds \\ &= \varepsilon\int_{t-\tau}^{t} \cos(s-t)\sigma(-\sin(\tau)x_1(s-\tau) + \cos(\tau)x_2(s-\tau))ds. \end{aligned} \tag{16}$$

One can check readily that

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -x_1(t) - \varepsilon\sigma(x_2(t)) + \beta(t), \end{cases} \tag{17}$$

with

$$\beta(t) = \varepsilon B_2(t) \int_0^1 \sigma'\left(x_2(t) + lB_2(t)\right) dl. \tag{18}$$

The result is established through a Lyapunov-Razumikhin approach. Consider the function

$$U(x_1, x_2) = \lambda(x_1^2 + x_2^2) + x_1 x_2 \tag{19}$$

where $\lambda(\cdot)$ is a function of class $\mathcal{K}_\infty$ such that $\lambda(s) \geq s,\ s \geq 0$ to be specified later. With such a choice for $\lambda(\cdot)$, $U(x_1, x_2)$ is a positive definite and radially unbounded function.

Its time derivative along the trajectories of system (17) satisfies

$$\begin{aligned} \dot{U}(x_1, x_2) &= -2\varepsilon\lambda'(x_1^2 + x_2^2)x_2\sigma(x_2) + 2\lambda'(x_1^2 + x_2^2)x_2\beta(t) + x_2^2 - x_1^2 - \varepsilon x_1\sigma(x_2) + x_1\beta(t) \\ &\leq -\varepsilon\lambda'(x_1^2 + x_2^2)x_2\sigma(x_2) + \tfrac{2\lambda'(x_1^2+x_2^2)x_2}{\varepsilon\sigma(x_2)}\beta(t)^2 + x_2^2 - \tfrac{1}{2}x_1^2 + \varepsilon^2\sigma(x_2)^2 + \beta(t)^2. \end{aligned} \tag{20}$$

According to the properties of $\sigma(\cdot)$ and the fact that $\varepsilon \leq \frac{1}{2}$, it follows that

$$\dot{U}(x_1, x_2) \leq -\varepsilon\lambda'(x_1^2 + x_2^2)x_2\sigma(x_2) + \left[\tfrac{2\lambda'(x_1^2+x_2^2)x_2}{\varepsilon\sigma(x_2)} + 1\right]\beta(t)^2 + \tfrac{5}{4}x_2^2 - \tfrac{1}{2}x_1^2. \tag{21}$$

Let now choose

$$\lambda(s) = 2k \int_0^s \sqrt{\frac{l}{\sigma(l)}}\, dl \tag{22}$$

where $k \geq \frac{1}{2}$. The properties satisfied by $\sigma(\cdot)$ ensure that this function is well defined, of class $\mathcal{K}_\infty$, and such that $\lambda''(s) \geq 0$. Then,

$$\begin{aligned} \dot{U}(x_1, x_2) &\leq -\varepsilon k\sqrt{\tfrac{x_1^2+x_2^2}{\sigma(x_1^2+x_2^2)}}x_2\sigma(x_2) - \tfrac{\varepsilon k}{\sqrt{\sigma(x_2^2)}}x_2^2|\sigma(x_2)| \\ &\quad + \left[\frac{4k\sqrt{\tfrac{x_1^2+x_2^2}{\sigma(x_1^2+x_2^2)}}x_2}{\varepsilon\sigma(x_2)} + 1\right]\beta(t)^2 + \tfrac{5}{4}x_2^2 - \tfrac{1}{2}x_1^2 \\ &\leq -\tfrac{1}{2}x_1^2 - \varepsilon k\sqrt{\tfrac{x_1^2+x_2^2}{\sigma(x_1^2+x_2^2)}}x_2\sigma(x_2) + \left[-\tfrac{2\varepsilon k}{3} + \tfrac{5}{4}\right]x_2^2 \\ &\quad + \left[\tfrac{4k}{\varepsilon\sigma(x_2)}\sqrt{\tfrac{x_1^2+x_2^2}{\sigma(x_1^2+x_2^2)}}x_2 + 1\right]\beta(t)^2. \end{aligned} \tag{23}$$

Choosing $k = \frac{4}{\varepsilon}$ (which, according to (14), is larger than $\frac{1}{2}$),

$$\dot{U}(x_1, x_2) \leq -\tfrac{1}{2}x_1^2 - 4\sqrt{\tfrac{x_1^2+x_2^2}{\sigma(x_1^2+x_2^2)}}x_2\sigma(x_2) - \tfrac{4}{3}x_2^2 + \left[\tfrac{16}{\varepsilon^2\sigma(x_2)}\sqrt{\tfrac{x_1^2+x_2^2}{\sigma(x_1^2+x_2^2)}}x_2 + 1\right]\beta(t)^2. \tag{24}$$

On the other hand, the use of the inequalities $0 \leq \sigma'(s) \leq 1$ and (16) leads to

$$\begin{aligned} \beta(t)^2 &= \varepsilon^2 B_2(t)^2 \left[\int_0^1 \sigma'\left(x_2(t) + lB_2(t)\right) dl\right]^2 \\ &\leq \varepsilon^4 \left[\int_{t-\tau}^t |\sigma(-\sin(\tau)x_1(s-\tau) + \cos(\tau)x_2(s-\tau))| ds\right]^2. \end{aligned} \tag{25}$$

We distinguish now between two cases.
First case: $x_1^2+x_2^2 \geq \frac{1}{4}$. Inequalities (24) and (25) imply that

$$\dot{U}(x_1,x_2) \;\leq\; -\tfrac{1}{2}x_1^2 - 4\sqrt{\tfrac{x_1^2+x_2^2}{\sigma(x_1^2+x_2^2)}}x_2\sigma(x_2) - \tfrac{3}{4}x_2^2 + \left(1+\tfrac{\varepsilon^2}{8}\right)\tfrac{16\varepsilon^2\tau^2}{\sigma(x_2)}\sqrt{\tfrac{x_1^2+x_2^2}{\sigma(x_1^2+x_2^2)}}x_2. \tag{26}$$

One can check readily that $\frac{s}{\sigma(s)} \leq 1+s$ for all $s \geq 0$. It follows that

$$\begin{aligned}\dot{U}(x_1,x_2) &\leq -\tfrac{1}{2}x_1^2 - 4\sqrt{\tfrac{x_1^2+x_2^2}{\sigma(x_1^2+x_2^2)}}x_2\sigma(x_2) - \tfrac{3}{4}x_2^2 + \left(1+\tfrac{\varepsilon^2}{8}\right)16\varepsilon^2\tau^2(1+|x_2|)\sqrt{1+x_1^2+x_2^2} \\ &\leq -\tfrac{1}{2}x_1^2 - 4\sqrt{\tfrac{x_1^2+x_2^2}{\sigma(x_1^2+x_2^2)}}x_2\sigma(x_2) - \tfrac{3}{4}x_2^2 + \left(1+\tfrac{\varepsilon^2}{8}\right)16\varepsilon^2\tau^2 10(x_1^2+x_2^2).\end{aligned} \tag{27}$$

When $\varepsilon \leq \frac{1}{40\tau}$, the inequality

$$\left(1+\frac{\varepsilon^2}{8}\right)16\varepsilon^2\tau^2 10 \;\leq\; \frac{1}{4} \tag{28}$$

holds, and

$$\dot{U}(x_1,x_2) \;\leq\; -\tfrac{1}{4}x_1^2 - 4\sqrt{\tfrac{x_1^2+x_2^2}{\sigma(x_1^2+x_2^2)}}x_2\sigma(x_2) - \tfrac{1}{4}x_2^2. \tag{29}$$

Second case: $x_1^2+x_2^2 \leq \frac{1}{4}$. In this case $|x_1| \leq \frac{1}{2}$,$|x_2| \leq \frac{1}{2}$, $\sigma(x_2)=x_2$, $\sigma(x_1^2+x_2^2)=x_1^2+x_2^2$. Then it follows from (24) that

$$\dot{U}(x_1,x_2) \;\leq\; -\tfrac{1}{2}x_1^2 - \tfrac{4}{3}x_2^2 + \left[\tfrac{16}{\varepsilon^2}+1\right]\beta(t)^2 \tag{30}$$

and

$$\begin{aligned}\beta(t)^2 &= \varepsilon^2 B_2(t)^2\left[\int_0^1 \sigma'\left(x_2(t)+lB_2(t)\right)dl\right]^2 \\ &\leq \varepsilon^4\left[\int_{t-\tau}^{t}[|x_1(s-\tau)|+|x_2(s-\tau)|]ds\right]^2.\end{aligned} \tag{31}$$

Combining (30) and (31) leads to

$$\dot{U}(x_1,x_2) \;\leq\; -\frac{1}{2}x_1^2 - \frac{4}{3}x_2^2 + \left[16\varepsilon^2+\varepsilon^4\right]\left[\int_{t-\tau}^{t}[|x_1(s-\tau)|+|x_2(s-\tau)|]ds\right]^2 \tag{32}$$

$$\leq\; -\frac{1}{2}x_1^2 - \frac{4}{3}x_2^2 + 17\varepsilon^2 2\tau^2\left[\sup_{s\in[t-\tau,t]}[x_1(s-\tau)^2+x_2(s-\tau)^2]\right]. \tag{33}$$

Next, we determine the values of ε for which the feedback (13) globally asymptotically stabilizes the system (2) with the help of Razumikhin Theorem (see Appendix A). To do so we prove by exploiting (29) and (33) that when the inequality

$$U(x_1(t+\theta),x_2(t+\theta)) < 2U(x_1(t),x_2(t)) \tag{34}$$

holds for all $\theta \in [-2\tau,0]$, there exists a continuous, nondecreasing functions $w(\cdot)$ such that

$$\dot{U}(x_1,x_2) \leq -w(\|x\|)$$

with $x=(x_1,x_2)^T$. Recall that $U(\cdot)$ is positive definite and radially unbounded.
1. When $x_1(t)^2+x_2(t)^2 \geq \frac{1}{4}$, it follows from (29) that $\dot{U}(x_1(t),x_2(t)) \leq -\frac{1}{4}\|x(t)\|^2$.

2. When $x_1(t)^2 + x_2(t)^2 \leq \frac{1}{4}$, the saturations act in their linear regions hence,

$$U(x_1(t), x_2(t)) = \frac{8}{\varepsilon}(x_1(t)^2 + x_2(t)^2) + x_1(t)x_2(t). \tag{35}$$

One can check readily that

$$U(x_1(t), x_2(t)) \leq \frac{9}{\varepsilon}(x_1(t)^2 + x_2(t)^2). \tag{36}$$

On the other hand,

$$U(x_1, x_2) = \frac{8}{\varepsilon}\int_0^{{x_1}^2+{x_2}^2} \sqrt{\frac{l}{\sigma(l)}}dl + x_1x_2.$$

Since $\sigma(l) \leq l,\ l \geq 0$, it follows that

$$U(x_1, x_2) \geq \frac{8}{\varepsilon}({x_1}^2 + {x_2}^2) + x_1x_2 \geq (\frac{8}{\varepsilon} - \frac{1}{2})({x_1}^2 + {x_2}^2).$$

In particular, this inequality implies that

$$14(x_1(t+\theta)^2 + x_2(t+\theta)^2) \leq U(x_1(t+\theta), x_2(t+\theta)). \tag{37}$$

Combining (36), (34) and (37),

$$14(x_1(t+\theta)^2 + x_2(t+\theta)^2) \ \leq \ \frac{18}{\varepsilon}(x_1(t)^2 + x_2(t)^2). \tag{38}$$

It follows from (38) and (33) that

$$\begin{array}{rcl} \dot{U}(x_1, x_2) & \leq & -\frac{1}{2}x_1^2 - \frac{4}{3}x_2^2 + 34\varepsilon^2\tau^2\frac{9}{7\varepsilon}[x_1(t)^2 + x_2(t)^2] \\ & \leq & -\frac{1}{2}x_1^2 - \frac{4}{3}x_2^2 + 162\varepsilon\tau^2[x_1(t)^2 + x_2(t)^2]. \end{array} \tag{39}$$

Choosing $\varepsilon \in]0, \min\left\{\frac{1}{2}, \frac{1}{40\tau}, \frac{1}{648\tau^2}\right\}]$, implies that

$$\dot{U}(x_1, x_2) \quad \leq \quad -\frac{1}{4}\left\|x\right\|^2. \tag{40}$$

This concludes the proof. ■

3 Output feedback stabilization

The above analysis shows that when the delay is τ, feedbacks depending only on

$$-\sin(\tau)x_1(t-\tau) + \cos(\tau)x_2(t-\tau)$$

stabilize the oscillator. This leads straightforwardly to the following interpretation of Theorem 2.3 in an output feedback framework.

Proposition 3.1 *Consider the system*

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) & = & x_2(t), \\ \dot{x}_2(t) & = & -x_1(t) + u(t), \end{array} \right. \tag{41}$$

with the output

$$y = ax_1 + bx_2 \tag{42}$$

where a and b are arbitrary real numbers such that $a^2 + b^2 > 0$. Then the system is globally asymptotically stabilized by a delayed feedback

$$u = -\varepsilon \sigma \left(\frac{y(t-\tau)}{\sqrt{a^2+b^2}} \right) \tag{43}$$

where $\sigma(\cdot)$ is the function defined in Theorem 2.3, and where τ is a strictly positive number such that

$$\begin{cases} a = -\sqrt{a^2+b^2}\sin(\tau), \\ b = \sqrt{a^2+b^2}\cos(\tau), \end{cases}$$

and $\varepsilon \in \left]0, \min\left\{\frac{1}{2}, \frac{1}{40\tau}, \frac{1}{648\tau^2}\right\}\right]$.

An interesting and perhaps surprising particular case is when the output is x_1. In this case, although the system is not stabilizable when the delay is zero, it is for suitably chosen delays, as shown in Corollary 3.1. The explanation of this fact is that, for a delay τ appropriately chosen, the past value of the output $x_1(t-\tau)$ gives information on the value of $x_2(t)$.

An analysis of the problem of output feedback stabilization by linear feedback, based on the study of the corresponding characteristic equation [14], or on the Nyquist criterion [1] can be carried out. It leads to the following result that provides additional informations on our problem.

Proposition 3.2 *[14], [1]The linear system*

$$\begin{cases} \dot{x}_1(t) &=& x_2(t), \\ \dot{x}_2(t) &=& -x_1(t) + u(t), \end{cases} \tag{44}$$

with the output

$$y(t) = x_1(t), \tag{45}$$

can be stabilized by delayed output feedback

$$u(t) = -ky(t-\tau) \tag{46}$$

for all the pairs (k, τ) satisfying simultaneously:

i) the gain $k \in (0,1)$

ii) the delay $\tau \in (\underline{\tau}_i(k), \overline{\tau}_i(k))$ where:

$$\begin{cases} \underline{\tau}_i(k) &=& \dfrac{(2i-1)\pi}{\sqrt{1-k}} \\ \overline{\tau}_i(k) &=& \dfrac{2i\pi}{\sqrt{1+k}} \end{cases} \tag{47}$$

for $i = 1, 2, \ldots$.

Furthermore, if $\tau = \underline{\tau}_i(k)$ or $\tau = \overline{\tau}_i(k)$, the corresponding characteristic equation in closed-loop has at least one eigenvalue on the imaginary axis.

The regions of stabilizing k shrink as the delay τ gets larger, and furthermore for each k there exists a value $\tau^*(k)$, such that for any $\tau > \tau^*(k)$ the closed-loop system is unstable. Besides, the robustness margin of the closed-loop system decreases as τ gets larger.

Remark 3.3 *Observe that in the case when $y = x_1$, the values of a and b in equation (42) in Corollary 3.1 are respectively* 1 *and* 0. *This implies that $\tau_i = -\frac{\pi}{2} + 2i\pi$, where i is an integer. Not surprisingly, these values are such that $\tau_i = -\frac{\pi}{2} + 2i\pi \in]\underline{\tau}_i(k), \overline{\tau}_i(k)[$ when k is sufficiently small: indeed, the feedback (43) is linear in a neighborhood of the origin which implies that the system (41) in closed-loop with $-\varepsilon \frac{y(t-\tau)}{\sqrt{a^2+b^2}}$ is globally asymptotically stable when $\tau = \tau_i$.*

Remark 3.4 *This observation gives a new way to establish that the delayed output feedback (46) satisfying i) and ii) are stabilizing. Indeed, boundaries of the regions of the plane k-τ described by i) and ii) correspond to crossing of the imaginary axis of the roots of the closed loop quasipolynomial. It follows from a continuity argument that in each region, the quasipolynomial has a fixed number of roots in the right-half plane, hence it is either stable or unstable in the whole region. A consequence of Remark 3.3 is that we are able to exhibit a stabilizing element (k_i, τ_i) in each of the regions described by i) and ii), hence all the control laws all the pairs (k, τ) satisfying simultaneously i) and ii) are stabilizing as well.*

4 Concluding remarks

The problem of globally asymptotically stabilizing by bounded feedback an oscillator with an arbitrary large delay in the input is solved. A first solution follows from a general result on the global stabilization of null-controllable linear systems with delay in the input by bounded control laws with a distributed term. Next, it is shown through a Lyapunov analysis that the stabilization can be achieved as well when neglecting the distributed terms. It turns out that this main result is intimately related to the output feedback stabilization problem. The robustness problems due to the need for the exact knowledge of the delay can be investigated with the help of the Lyapunov function we have constructed.

References

[1] Abdallah C., Dorato P., Benitez-Read and Byrne R., *Delayed positive feedback can stabilize oscillatory systems.* Proc. American Contr. Conf. (1993) 3106-3107.

[2] Ansoff H.I., *Stability of linear oscillating systems with constant time lag.* J. Appl. Mech. **16** (1949) 158-164.

[3] Ansoff H.I. and Krumhansl J. A., *A general stability criterion for linear oscillating systems with constant time lag.* Quart. Appl. Math. **6** (1948) 337-341.

[4] Artstein S., *Linear systems with delayed controls: a reduction.* IEEE Trans. Aut. Contr., Vol. AC-27, No. 4, 869-879, 1982.

[5] Hale J.K. and Verduyn Lunel S.M., *Introduction to functional differential equations.* Applied Math, Sciences, 99, Springer-Verlag, New-York, 1993.

[6] Manitius A.Z. and Olbrot A.W., *Finite spectrum assignment problem for systems with delays.* IEEE Trans. Aut. Contr., Vol. AC-24, No. 4, 541-553, 1979.

[7] Mazenc F. and Praly L., *Adding an integration and global asymptotic stabilization of feedforward systems.* IEEE Trans. Aut. Contr., Vol. 41, No.11, pp.1559-1578, 1996.

[8] Mazenc F., Mondié S. and Niculescu S., *Global asymptotic stabilization for chains of integrators with a delay in the input.* IEEE Trans. Aut. Contr., Vol. 48, No. 1, pp. 57 -63, 2003.

[9] Minorsky N.: *Self-excited oscillations in dynamical systems possessing retarded actions.* J. Applied Mech. **9** (1942) A65-A71.

[10] Mondié S., Dambrine M. and Santos O., *Approximation of control laws with distributed delays: a necessary condition for stability.* IFAC Symposium on Systems, Structure and Control, Prague, Czek Republic, August 2001.

[11] Mondié S., Lozano R. and Collado, J., *Resetting Process-Model Control for Unstable Systems with Delay.* Submitted to IEEE Trans. Aut. Contr.

[12] Mufti I.H., *A note on the stability of an equation of third order with time lag.* IEEE Trans. Aut. Contr., Vol. AC-9, pp. 190-191, 1964.

[13] Neimark J. (1949). *$\mathcal{D}$-subdivisions and spaces of quasipolynomials. Prickl. Math. Mech.*, 13, 349-380.

[14] Niculescu S.-I.: *Delay effects on stability. A robust control approach.* Springer-Verlag: Heidelberg, LNCIS, May 2001.

[15] Niculescu S.-I. and Abdallah C.T., *Delay effects on static output feedback stabilization.* Proc. 39th IEEE Conf. Dec. Contr., Sydney, Australia, 2000.

[16] Palmor Z.J., *Time delay compensation-Smith predictor and its modifications*, in The Control Handbook, (W.S. Levine, Eds), CRSC Press, 224-237, 1996.

[17] Smith O.J.M., *Closer Control of loops with dead time.* Chem. Eng. Prog.,53, 217-219, 1959.

[18] Sussmann H.J., Sontag E.D. and Yang Y., *A general result on the stabilization of linear systems using bounded controls.* IEEE Trans. Autom. Contr., Vol. AC-39, 2411-2425, 1994.

[19] Teel A., *Global stabilization and restricted tracking for multiple integrators with bounded controls.* Systems and Control Letters, 18, 165-171, 1992.

[20] Watanabe K. and Ito M., *A process model control for linear systems with delay.* IEEE Trans. Aut. Contr., Vol. AC-26, No. 6, pp. 1261-1268, 1981.

A Razumikhin's Theorem

Theorem A.1 *[5] Consider the functional differential equation*

$$\dot{x}(t) = f(t, x_t), t \geq 0, \tag{48}$$

$$x_{t_0}(\theta) = \phi(\theta), \forall \theta \in [-\tau, 0] \tag{49}$$

(Note: $x_t(\theta) = x(t+\theta), \forall \theta \in [-\tau, 0]$). The function $f: R \times \mathcal{C}_{n,\tau}$ is such that the image by f of $R\times$(a bounded subset of $\mathcal{C}_{n,\tau}$) is a bounded subset of R^n and the functions $u, v, w: R^+ \longrightarrow R^+$ are continuous, nondecreasing, $u(s), v(s)$ positive for all $s > 0, u(0) = v(0) = 0$ and v is strictly increasing.
If there exists a function $V: R \times R^n \longrightarrow R$ such that
(a) $u(\|x\|) \leq V(t, x) \leq v(\|x\|), t \in R,\ x \in R^n$

(b) $\dot{V}(t,x) \leq -w(\|x\|)$ *if* $V(t+\theta, x(t+\theta)) \leq V(t, x(t)), \forall\theta \in [-\tau, 0]$
then the trivial solution of (48, 49) is uniformly stable.

Moreover, if $w(s) > 0$ *when* $s > 0$, *and there exists a function* $p : R^+ \to R^+$, $p(s) > s$ *when* $s > 0$ *such that:*
(a) $u(\|x\|) \leq V(t,x) \leq v(\|x\|), t \in R,\ x \in R^n$
(b) $\dot{V}(t,x) \leq -w(\|x\|)$ *if*

$$V(t+\theta, x(t+\theta)) < p(V(t, x(t))), \forall\theta \in [-\tau, 0] \tag{50}$$

then the trivial solution of (48, 49) is uniformly asymptotically stable.

B Illustrative example

In this section, we carry out some computer simulations to validate the result presented in Section 2.3. In Figure 1, one can observe the behaviour of the oscillator when the delay is $\frac{\pi}{4}$ and when the control law is $u = -0.1\sigma\left(-\sin\left(\frac{\pi}{4}\right)x_1 + \cos\left(\frac{\pi}{4}\right)x_2\right)$, which results from Theorem 2.3. Figures 2 and 3 show that the same system in closed-loop with the same feedback is unstable when the delay is not $\frac{\pi}{4}$ but the larger delay $\frac{3\pi}{4}$.

Figure 1: Closed-loop states ($\tau = \pi/4$)

Figure 2: Closed-loop states ($\tau = 3\pi/4$)

Figure 3: Delayed input ($\tau = 3\pi/4$)