Strict Lyapunov Functions for Time-varying Systems ⋆

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Abstract

Uniformly asymptotically stable periodic time-varying systems for which is known a Lyapunov function with a derivative along the trajectories non-positive and negative definite in the state variable on non-empty open intervals of the time are considered. For these systems, strict Lyapunov functions are constructed.

Key words: Lyapunov functions, nonlinear systems, time-varying.

1 Introduction

It is well-known that the knowledge of continuously differentiable strict Lyapunov functions [1] can be of great help. The potential benefits they offer are so multiple that they cannot be exhaustively enumerated. Observe in particular that the backstepping and the forwarding techniques require in some cases for subsystems the knowledge of strict Lyapunov functions, using a strict Lyapunov function a subset of the basin of attraction of a locally asymptotically stable system can be determine, when a control Lyapunov function satisfying the small control property is available, one can apply the universal formula proposed in (Sontag, 1989) to get an expression of an asymptotically stabilizing feedback which is optimal with respect to the control Lyapunov function as optimal value function (see (Sepulchre, Jankovic & Kokotovic, 1996, Section 3.5.3)), recent advances in the stabilization of nonlinear delay systems are based on the knowledge of continuously differentiable strict Lyapunov functions: see in particular (Teel, 1998; Jankovic, 1999, 2001; Mazenc & Niculescu, 2001). At last, observe that strict Lyapunov functions are known to be very efficient tools for robustness analysis.

Unfortunately, the problem of constructing strict Lyapunov functions for time-varying periodic systems is in general more difficult than that of constructing strict Lyapunov functions for time-invariant systems. One can easily understand why by observing that the derivative of $V(x) = x^2$ along the trajectories of the one dimensional system $\dot{x} = -x$ is negative definite: $\frac{\partial V}{\partial x}(x)(-x) = -2x^2$ whereas $V(x)$ is not a strict Lyapunov function for the one-dimensional globally uniformly asymptotically stable time-varying system $\dot{x} = -\sin^2(t)x$ because the function $\frac{\partial V}{\partial x}(x)(-\sin^2(t)x) = -2\sin^2(t)x^2$, is equal to zero when $t = k\pi$ where k is an integer.

So an important and very general question arises. In which cases is the knowledge of a Lyapunov function for a time-varying system which is not a strict Lyapunov function helpful ? Basically, in the literature, two answers can be found. In the periodic case, (and in slightly more general cases), when a Lyapunov function with a negative semi-definite derivative is available, the LaSalle Invariance Principle (see (Barbashin & Krasovskii, 1957; LaSalle, 1968)) can be applied to prove the uniform asymptotic stability of the null solution. For a nonperiodic and nonlinear time-varying system $\dot{x} = f(t,x)$, Narendra and Annaswamy have proved in (Narendra & Annaswamy, 1987) that if there exists a Lyapunov function $V(t,x)$ such that $\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x) \leq 0$ and there exists $T > 0$ such that, for all $t \geq 0$, $V(t+T, x(t+T)) - V(t, x(t)) \leq -\gamma(|x(t)|)$ where $\gamma(\cdot)$ is a strictly increasing continuous function on $\Re^+$ which is zero at zero, then this system is uniformly asymptotically stable. Extensions of this result to the case of

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[1] See the preliminaries at the end of the introduction for the definition of strict Lyapunov functions for time-varying systems we use.

Lyapunov functions which admit derivatives along the trajectories which are not negative semi-definite are proposed in (Aeyels & Peutman, 1998, 1997).

The objective of the present work is to give a new answer to the question mentioned above. Basically, we will show that if for a periodic time-varying nonlinear system

$$\dot{x} \;=\; f(t,x) \tag{1}$$

is known a Lyapunov function $V(t,x)$, a periodic function $q(t)$ and a nonnegative function $W(q(t),x)$ such that

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x) \;\leq\; -W(q(t),x) \tag{2}$$

where $W(q(t),x)$ is positive definite in x for all t in nonempty open intervals of the time *then one can construct a strict Lyapunov function for the system (1).*

The construction we propose is mainly motivated by two facts. On the one hand, as already explained, the knowledge of strict Lyapunov functions is beneficial in many circumstances. On the other hand, in many cases, for a time-varying system (1) a Lyapunov function $V(t,x)$ satisfying (2) is known. For example, by applying the main result of (Jiang & Nijmeijer, 1997) to the problem of tracking a periodic trajectory of a nonholonomic system in chained-form, such a Lyapunov function is obtained and the Lyapunov construction of (Mazenc & Praly, 2000) also yields frequently Lyapunov functions of this type.

At last, observe that the construction we carry out is simple and can be easily performed in practice. This construction, which is the main result of the work, is presented in Section 2. In Section 3, we have chosen to illustrate on a simple example how the main result can be combined with the backstepping approach to construct strict Lyapunov functions for nonholonomic systems in closed-loop with feedbacks which globally asymptotically stabilize a periodic trajectory. Concluding remarks in Section 4 end the work.

1.1 Preliminaries

1) Throughout the paper we assume that the functions are sufficiently smooth.

2) A real-valued function $k(\cdot)$ is of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing and

$$\lim_{r\to+\infty} k(r) \;=\; +\infty \;.$$

3) A function $V(t,x)$ is a Lyapunov function if it is continuously differentiable and there exist two functions $\alpha_1(\cdot), \alpha_2(\cdot)$ of class $\mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \;\leq\; V(t,x) \;\leq\; \alpha_2(|x|) \;. \tag{3}$$

4) Consider the time-varying system

$$\dot{x} \;=\; f(t,x) \;. \tag{4}$$

A continuously differentiable function $V(t,x)$ is a strict Lyapunov function for the system (4) if it is a Lyapunov function and there exists a positive definite function $\alpha_3(\cdot)$ such that

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x) \;\leq\; -\alpha_3(|x|) \;. \tag{5}$$

2 Strict Lyapunov function

This part splits up into two parts. First, we construct a strict Lyapunov function in a simple case. All the key ideas of the approach take place in this first part. Second, for the sake of generality, we explain how the result of the first part can be straightforwardly extended to a more general case.

2.1 A simple case

Consider the time-varying system

$$\dot{x} \;=\; f(t,x) \tag{6}$$

with $x \in \Re^n$ and where $f(t,x)$ is a nonlinear function periodic in time of period $T > 0$. We introduce the following assumptions:

Assumption A1. A Lyapunov function $V(t,x)$, periodic in time and of period T, a positive definite function $W(x)$ and a non-negative function $p(t)$, periodic and of period T, such that

$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)f(t,x) \;\leq\; -p(t)W(x) \tag{7}$$

and two functions $\alpha_i(\cdot)$, $i = 1$ to 2 of class $\mathcal{K}_\infty$, such that

$$\alpha_1(|x|) \;\leq\; V(t,x) \;\leq\; \alpha_2(|x|) \tag{8}$$

are known.

Assumption A2. The constant $\displaystyle\int_0^T p(s)ds$ is strictly positive.

Theorem 1 *If Assumptions A1 and A2 are satisfied by the system (6), one can determine the explicit expressions of a continuously differentiable function $\Gamma(\cdot)$ of class $\mathcal{K}_\infty$ and of a positive definite function $\lambda(\cdot)$ continuously differentiable, with a positive first derivative, such that the function*

$$U(t,x) \;=\; \Gamma(V(t,x)) + P(t)\lambda(V(t,x)) \tag{9}$$

with

$$P(t) \;=\; -t\int_0^T p(s)ds + T\int_0^t p(s)ds \tag{10}$$

is a strict Lyapunov functions for the system (6).

Remark 2 *Theorem 1 is not just a result which allows us to prove the more general theorem given in Section 2.2. In many situations it can be used: the example in Section 3 makes understand that when the backstepping approach applies to a time-varying system then in general a Lyapunov function satisfying Assumptions A1 and A2 can be constructed. The same is true for a time-varying system which can be handled by the forwarding approach.*

Remark 3 *One can extend Theorem 1 to the case of systems which are not periodic but such that there exist $T > 0$ and $\delta > 0$ such that, for all $t \geq 0$, $\int_t^{t+T} p(s)ds \geq \delta$. The function $P(t)$ corresponding to that case is*

$$P(t) \;=\; \int_t^{t+T} (s-t-T)p(s)ds \tag{11}$$

whose derivative is

$$\dot{P}(t) \;=\; -\int_t^{t+T} p(s)ds + Tp(t) \;.$$

For the sake of clarity, we have chosen to prove our result with the function (10) instead of the function (11).

Proof.

1. The properties of $U(t,x)$.

The function $P(t)$ is periodic of period T:

$$\begin{aligned} P(t+T) \;&=\; -(t+T)\int_0^T p(s)ds \\ &\quad +T\int_0^{t+T} p(s)ds \\ &=\; -t\int_0^T p(s)ds + T\int_0^t p(s)ds \\ &\quad -T\int_0^T p(s)ds + T\int_t^{t+T} p(s)ds \\ &=\; P(t)\;. \end{aligned} \tag{12}$$

It follows that the function $U(t,x)$ defined in (9) is periodic of period T and that $P(t)$ is bounded in norm by a positive real number P_M. Since $\lambda(\cdot)$ is a non-negative function, the inequalities

$$\begin{array}{l} \Gamma(V(t,x)) - P_M\lambda(V(t,x)) \leq U(t,x) \\ U(t,x) \leq \Gamma(V(t,x)) + P_M\lambda(V(t,x)) \end{array} \tag{13}$$

are satisfied. One can choose $\Gamma(\cdot)$ of class $\mathcal{K}_\infty$ and $\lambda(\cdot)$, zero at zero, continuously differentiable, with a positive first derivative, such that, for all v,

$$\Gamma(v) \;\geq\; 2P_M\lambda(v) \tag{14}$$

which, in combination with (13), yields

$$\frac{1}{2}\Gamma(V(t,x)) \;\leq\; U(t,x) \;\leq\; \frac{3}{2}\Gamma(V(t,x))\;. \tag{15}$$

According to Assumption A1, $V(t,x)$ is a Lyapunov function. One can easily deduce from (15) that $U(t,x)$ is a Lyapunov function [2] as well.

2. The derivative of $U(t,x)$ along the trajectories of (6).

The derivative of $U(t,x)$ along the trajectories of (6) satisfies

$$\begin{aligned} \dot{U}(t,x) \;&=\; \Gamma'(V(t,x))\dot{V}(t,x) \\ &\quad + \left[-\int_0^T p(s)ds + Tp(t)\right]\lambda(V(t,x)) \\ &\quad + P(t)\lambda'(V(t,x))\dot{V}(t,x)\;. \end{aligned} \tag{16}$$

Choosing $\Gamma(\cdot)$ and $\lambda(\cdot)$ satisfying (14) and such that, for all v,

$$\frac{1}{2}\Gamma'(v) \;\geq\; P_M\lambda'(v)\;,$$

we obtain

$$\begin{aligned} \dot{U}(t,x) \;&\leq\; -\tfrac{1}{2}\Gamma'(V(t,x))p(t)W(x) \\ &\quad + \left[-\int_0^T p(s)ds + Tp(t)\right]\lambda(V(t,x))\;. \end{aligned} \tag{17}$$

Using the inequalities (8) in Assumption A1, (Mazenc & Praly, 1996, Lemma B2) and the technique of construction used to prove (Mazenc, 1998, Theorem 3.1), one can determine two functions $\Gamma(\cdot)$ and $\lambda(\cdot)$ such that, for all t and x,

$$\frac{1}{4}\Gamma'(V(t,x))W(x) \;\geq\; T\lambda(V(t,x))\;. \tag{18}$$

[2] See item 4 in the preliminaries.

When this inequality is satisfied, then

$$\begin{aligned}\dot{U}(t,x) \;\le\; & -\tfrac{1}{4}\Gamma'(V(t,x))p(t)W(x) \\ & -\left(\int_0^T p(s)ds\right)\lambda(V(t,x)) \;. \end{aligned} \tag{19}$$

Since the first derivative of $\lambda(\cdot)$ is positive, we deduce from Assumption A2 and (8) that

$$\begin{aligned}\dot{U}(t,x) \;\le\; & -\left(\int_0^T p(s)ds\right)\lambda(\alpha_1(|x|)) \\ & < 0 \;,\; \forall x \neq 0 \;. \end{aligned} \tag{20}$$

This concludes the proof.

Remark 4 *When $V(t,x)$ and $W(x)$ are lower bounded on a neighborhood of the origin by positive definite quadratic functions of x, the identity function is always a possible choice for $\lambda(\cdot)$.*

2.2 General case

We replace Assumptions A1 and A2 by two less restrictive assumptions.

Assumption B1. A Lyapunov function $V(t,x)$, periodic in time of period T, a non-negative function $\mathcal{W}(q(t),x)$ where $q(t)$ is periodic and of period T, such that

$$\dot{V}(t,x) \;\le\; -\mathcal{W}(q(t),x) \tag{21}$$

and two functions $\alpha_i(\cdot)$, $i = 1$ to 2, of class $\mathcal{K}_\infty$ such that

$$\alpha_1(|x|) \;\le\; V(t,x) \;\le\; \alpha_2(|x|) \tag{22}$$

are known.

Assumption B2. Two values $\tau_1 \in [0,T[$ and $\tau_2 \in]\tau_1,T]$ and a positive definite function $W(x)$ such that, for all $t \in [\tau_1,\tau_2]$ and for all x, $\mathcal{W}(q(t),x) \ge W(x)$ are a known .

Theorem 5 *If the system (6) satisfies the Assumptions B1 and B2, one can determine explicit expressions of strict Lyapunov functions for the system (6).*

This theorem can be proved by combining Theorem 1 and the following lemma:

Lemma 6 *Let $q(t)$ be a periodic function of period T. Let $\mathcal{W}(q(t),x)$ be a non-negative function such that there exist $\tau_1 \in [0,T[, \tau_2 \in]\tau_1,T]$ such that, for all $t \in [\tau_1,\tau_2]$, $\mathcal{W}(q(t),x) \ge W(x)$, where $W(x)$ is a positive definite function. Then*

$$\mathcal{W}(q(t),x) \;\ge\; p(t)W(x) \tag{23}$$

where $p(t)$ is a periodic function of period T such that $\int_0^T p(s)ds > 0$ and

$$\begin{aligned} p(t) \;=\; 0 \quad & when \quad t \notin [\tau_1,\tau_2] \\ and \quad p(t) \in]0,1] \quad & when \quad t \in [\tau_1,\tau_2] \;. \end{aligned} \tag{24}$$

Proof. The proof is trivial.

3 Example

In (Jiang & Nijmeijer, 1997) the backstepping approach is used to design control laws which globally exponentially stabilize trajectories of nonholonomic systems in chained-form. However, no strict Lyapunov function is constructed in this work. We show, in a particular case, how the strategy of construction of strict Lyapunov function proposed in the present work allows us to construct a strict Lyapunov function in the context of (Jiang & Nijmeijer, 1997). A similar construction can be carried out for any nonholonomic systems in chained-form. But for the sake of simplicity, we have chosen to restrict our attention to a three dimensional system in chained-form.

1. Problem statement.

Consider the system

$$\begin{cases} \dot{x}_3 \;=\; x_2 u_1 \;, \\ \dot{x}_2 \;=\; u_2 \;, \\ \dot{x}_1 \;=\; u_1 \;, \end{cases} \tag{25}$$

where u_1 and u_2 are the inputs. Our objective is to globally asymptotically and locally exponentially stabilize the reference state trajectory $(0,0,-\cos(t))$ and to construct a strict Lyapunov function for the closed-loop system.

2. Stabilizing feedbacks.

In a first step, we determine stabilizing feedbacks for the error equation corresponding to the reference state trajectory $(0,0,-\cos(t))$. Choosing

$$u_1 \;=\; \sin(t) - x_{1e} \tag{26}$$

the error equation becomes

$$\begin{cases} \dot{x}_{3e} = x_{2e}(\sin(t) - x_{1e}) , \\ \dot{x}_{2e} = u_2 , \\ \dot{x}_{1e} = -x_{1e} . \end{cases} \quad (27)$$

We focus our attention on the system

$$\begin{cases} \dot{x}_{3e} = x_{2e}\sin(t) , \\ \dot{x}_{2e} = u_2 , \end{cases} \quad (28)$$

which is the (x_{2e}, x_{3e})-subsystem of (27) with $x_{1e} = 0$. The backstepping approach leads us to perform the time-varying change of coordinates (see (Jiang & Nijmeijer, 1997))

$$x_{2e} = -\sin^3(t)x_{3e} + z_{2e} \quad (29)$$

which yields

$$\begin{cases} \dot{x}_{3e} = -\sin^4(t)x_{3e} + z_{2e}\sin(t) , \\ \dot{z}_{2e} = u_2 + 3\sin^2(t)\cos(t)x_{3e} \\ \qquad + \sin^3(t)(-\sin^4(t)x_{3e} + z_{2e}\sin(t)) . \end{cases} \quad (30)$$

The derivative of the function

$$V(t, x_{3e}, z_{2e}) = \frac{1}{2}\left[x_{3e}^2 + z_{2e}^2\right] \quad (31)$$

along the trajectories of (30) in closed-loop with

$$\begin{aligned} u_2 = & -3\sin^2(t)\cos(t)x_{3e} \\ & -\sin^3(t)(-\sin^4(t)x_{3e} + z_{2e}\sin(t)) \\ & -z_{2e} - \sin(t)x_{3e} \end{aligned} \quad (32)$$

satisfies:

$$\begin{aligned} \dot{V}(t, x_{3e}, z_{2e}) &= -\sin^4(t)x_{3e}^2 - z_{2e}^2 \\ &\leq -2\sin^4(t)V(t, x_{3e}, z_{2e}) . \end{aligned} \quad (33)$$

3. Strict Lyapunov function.

We apply Theorem 1 to construct a strict Lyapunov function for the system (30) in closed-loop with the feedback (32). Consider the function

$$\begin{aligned} U_1(t, x_{3e}, z_{2e}) = & \Gamma(V(t, x_{3e}, z_{2e})) \\ & +P(t)\lambda(V(t, x_{3e}, z_{2e})) \end{aligned} \quad (34)$$

where $\lambda(\cdot)$ is the identity, $\Gamma(v) = \pi v$ and

$$\begin{aligned} P(t) &= -t\int_0^{\pi}\sin^4(s)ds + \pi\int_0^t\sin^4(s)ds \\ &= -\frac{\pi}{4}\sin(2t) + \frac{\pi}{32}\sin(4t) . \end{aligned} \quad (35)$$

Using (33) and introducing the simplifying notations $\rho_1(t) = \pi - \frac{\pi}{4}\sin(2t) + \frac{\pi}{32}\sin(4t)$, $\rho_2(t) = -\frac{\pi}{2}\cos(2t) + \frac{\pi}{8}\cos(4t)$, $\rho_3(t) = -\frac{3\pi}{8} + \pi\sin^4(t)$, we obtain

$$\begin{aligned} \dot{U}_1(t, x_{3e}, z_{2e}) &= \rho_1(t)\dot{V}(t, x_{3e}, z_{2e}) \\ &\quad +\rho_2(t)V(t, x_{3e}, z_{2e}) \\ &\leq -2\sin^4(t)\rho_1(t)V(t, x_{3e}, z_{2e}) \\ &\quad +\rho_3(t)V(t, x_{3e}, z_{2e}) \\ &\leq -\frac{3\pi}{8}V(t, x_{3e}, z_{2e}) . \end{aligned} \quad (36)$$

Moreover, one can check easily that $U_1(t, x_{3e}, z_{2e})$ belongs to the family of the functions satisfying inequalities of the form (3). So $U_1(t, x_{3e}, z_{2e})$ a strict Lyapunov function for the system (30) in closed-loop with the feedback (32).

We are ready to construct a strict Lyapunov function for the system (27) in closed-loop with with the feedback (32). The derivative of $U_1(t, x_{3e}, z_{2e})$ along the trajectories of this system in closed-loop satisfies

$$\begin{aligned} \dot{U}_1(t, x_{3e}, z_{2e}) \leq & -\frac{3\pi}{8}V(t, x_{3e}, z_{2e}) \\ & +\frac{\pi}{2}\left[|x_{3e}| + |z_{2e}|\right]|x_{2e}||x_{1e}| . \end{aligned} \quad (37)$$

Using the triangular inequality, the inequality

$$\begin{aligned} \dot{U}_1(t, x_{3e}, z_{2e}) \leq & -\frac{\pi}{4}V(t, x_{3e}, z_{2e}) \\ & +8\pi V(t, x_{3e}, z_{2e})x_{1e}^2 \end{aligned} \quad (38)$$

is obtained. It follows that the derivative of

$$U_2(t, x_{3e}, z_{2e}, x_{1e}) = \ln(1 + U_1(t, x_{3e}, z_{2e})) + 7x_{1e}^2 \quad (39)$$

satisfies

$$\dot{U}_2(t, x_{3e}, z_{2e}, x_{1e}) \leq -\frac{\pi}{4}\frac{V(t, x_{3e}, z_{2e})}{1+U_1(t, x_{3e}, z_{2e})} - 2x_{1e}^2 . \quad (40)$$

Moreover, one can check easily that $U_2(t, x_{3e}, z_{2e}, x_{1e})$ belong to the family of the functions satisfying inequalities of the form (3). It follows that this Lyapunov function is a strict Lyapunov function for the system (27) in closed-loop with with the feedback (32). Moreover, on a neighborhood of the origin, the right-hand-side of (40) is upper bounded by a negative definite quadratic function. It follows that the feedbacks (26)(32) globally uniformly and locally exponentially stabilize the reference state trajectory.

4 Conclusion

The construction of strict Lyapunov functions we have developed is new, simple and, for all the reasons mentioned in the introduction, can be useful in many cases. In particular, in a future work, we will exploit this construction to prove for nonholonomic systems in closed-loop with control laws which globally uniformly asymptotically stabilize the origin some ISS properties.

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