Asymptotically Stable Interval Observers for Planar Systems with Complex Poles

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***Abstract*—In some parametric domains, the problem of designing an exponentially stable interval observer for an exponentially stable two dimensional time-invariant linear system is open. We show that, in some cases, no linear time-invariant change of coordinates can help to determine an exponentially stable interval observer. Next, we solve the problem by constructing interval observers of a new type, which have as key feature the property of being time-varying. This new design is applied to the chaotic Chua's system.**

Keywords. Interval observer, time-varying, exponential stability, robustness.

I. Introduction

State estimation when the model is uncertain is a key issue, especially in the biotechnological domain where models are rough approximations [1]. To tackle this problem, bounded error approaches assume that the uncertainties are bounded and they provide bounds for the state. Robust estimation approaches can rely on estimations through ellipsoidal sets [8] or be based on interval computation methods [5]. They were originally developed for discrete time systems and more recently extended to continuous time systems satisfying properties of monotone differential systems [4], [16]. The idea consists in bounding the real state by solutions of a joined dynamical system, and to ensure that the error dynamics satisfies some cooperativity conditions [17]. If the initial condition is sure to lay within a given interval, then the state is guaranteed to be bounded by some solutions of the joined dynamical systems. Some works have tried to combine both advantages of discrete and continuous approaches [6], [15], [7].

A key advantage of the interval approaches is that they address a crucial issue of the end user: the interval width provides at any time an hint on the observer accuracy. Indeed, the particularity of a usual observer is that it is designed to guarantee asymptotic convergence. However, in general, it is not possible to assess the convergence state, i.e. to know the residual error. In contrast to this, the interval observers have not only asymptotic properties, but they also provide intervals where the solutions of the system studied are sure to lay at any time, and thus they give a bound of the error at any time. In addition, when the only uncertainty is on the initial condition, it is even possible to assess the convergence of the observer. The consequence of this is that interval observers should not be understood as tools useful only when the classical observers cannot be applied: they should be regarded as techniques providing with complementary information about the solutions of a system.

Finally the interval approach has the key advantage to allow the comparison between several interval observers, since the best upper bound can be defined as the lowest of the upper bounds, and *vice versa* for the lower bound. This motivated the development of bundles of interval observers in [2], [12] where several interval observers are run in parallel and the best value provided by the observers set is taken at each time instant. More generally, in the bundles of interval observers, some estimates may even be unstable. Such bounds are called framers, in the sense that they simply bound the state without having any stability property.

The best context to derive an interval observer is when the system (or the error system) can be written under a cooperative form. If this desirable property is not satisfied by a system but if this system, of dimension denoted n, is monotone, Müller's Theorem [13], [3] shows how to design a framer by immersing the system into a cooperative system of dimension $2n$. Eventually, when the system is not monotone, it can also be immersed into a cooperative system of dimension $2n$, provided that some Lipschitz conditions are satisfied [10] and then again framers can be constructed. Unfortunately, through this classical approach, it is often difficult to obtain systems which are cooperative and for which stable framers can be constructed.

This paper is focused on a very simple class of systems, for which the design of exponentially stable interval observers is an open problem. For linear time-invariant systems, the dimension 2 is enough to understand and illustrate the complexity induced by the presence of poles with an imaginary part different from 0. A first part of our work is devoted to the problem of exhibiting conditions which guarantee that a time-invariant linear and exponentially stable interval observer can be constructed via standard approaches. Next, we show that when this condition is violated, one can construct again exponentially stable linear interval observers but these interval observers have the remarkable feature of being *time-varying*. To illustrate the power of our approach, we apply it to a chaotic system which is known to be highly sensitive to uncertainties in the initial conditions. Observe that, in the present paper, we do not consider exponentially stable systems with real poles: indeed, it is already known that for these systems, constructing exponentially stable observers is always possible because they can be transformed through a linear time-invariant change of coordinates into exponentially stable system in Jordan form [14, Section 1.8], which is always a cooperative system. Note also that the interest of our results is not limited to planar linear systems: indeed the problem of constructing exponentially stable interval observers for general linear systems cannot be understood if not even the case of the linear systems of dimension two is understood. The present paper gives a complete picture of the difficulties and of the solutions which can be given for systems of dimension two for which constructing exponentially stable interval observers is not a trivial problem. Therefore, by solving the open problem of constructing interval observers for all linear systems of dimension two, our work makes possible the study of general linear systems and one can already immediately deduce from our result some properties for more general families of systems. But this issue is beyond the scope of the present work. Finally, notice that our result can help when one wants to design interval observers for some nonlinear systems as illustrated by the example we give in Section IV.

The paper is structured as follows. In Section II we present the family of systems under study, and for a subfamily of systems, we show that the standard interval observer design leads to unstable framers. Then in Section III, we propose a time-varying exponentially stable interval observer for any system of the family we consider. In Section IV, we develop an interval observer for Chua's chaotic system and we show its efficiency. Finally, concluding remarks are given in Section V.

II. Time invariant framers for planar systems

This section is devoted to the problem of establishing whether time-invariant interval observers for exponentially stable two dimensional systems with complex poles can be constructed. We exhibit a condition which ensures that this problem can be solved by using classical framers. We also show that these framers are exponentially unstable when another condition is satisfied. We also prove that, for a subfamily of the systems we consider, no time-invariant linear change of coordinates yield systems for which classical interval observers are exponentially stable.

A. Notations, definitions and hypotheses

We consider the 2 dimensional system

$$\dot{\xi} \; = \; A\xi + \phi(t) \; , \; A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \in \mathbb{R}^{2\times 2} \; , \qquad (1)$$

where $\xi = (\xi_1 \; \xi_2)^\top$ and where $\phi(t) = (\phi_1(t) \; \phi_2(t))^\top$ is a continuous function of time.

To simplify the notations, we introduce two functions:

$$\mathcal{L}(s) = \max\{0, s\} \; , \; \mathcal{M}(s) = \min\{0, s\} \; . \qquad (2)$$

We introduce some constants and matrices derived from matrix A:

$$\begin{array}{rcl} \mathcal{A}_1 & = & \begin{bmatrix} a_{1,1} & \mathcal{L}(a_{1,2}) \\ \mathcal{L}(a_{2,1}) & a_{2,2} \end{bmatrix} \; , \; \mathcal{A}_2 = A - \mathcal{A}_1 \; , \\ A^\dagger & = & \mathcal{A}_1 - \mathcal{A}_2 \; , \; \bar{\bar{A}} = \begin{bmatrix} \mathcal{A}_1 & -\mathcal{A}_2 \\ -\mathcal{A}_2 & \mathcal{A}_1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \; . \end{array} \qquad (3)$$

The class of system under interest is defined through the following assumption.

Assumption H1. *The eigenvalues of matrix A defined in (3) have a negative real part and an imaginary part different from zero.*

Assumption H1 means that the system (1) with $\phi \equiv 0$ admits the origin as a stable focus. We denote by ω the rotating pulsation of the autonomous system, and κ, the damping ratio:

$$\begin{array}{rcl} \kappa & = & -\frac{1}{2}\mathrm{tr}(A) \; , \; \omega = \sqrt{\det(A) - \frac{1}{4}(\mathrm{tr}(A))^2} \; , \\ G & = & \begin{bmatrix} -\kappa & \omega \\ -\omega & -\kappa \end{bmatrix} \in \mathbb{R}^{2\times 2} \; . \end{array} \qquad (4)$$

From Hypothesis H1, $\det(A) - \frac{1}{4}\mathrm{tr}(A)^2 > 0$. Hence ω is well-defined and the eigenvalues of A are then $-\kappa - \omega j$, $-\kappa + \omega j$. Observe that the smaller is κ with respect to ω, the closer are the solutions to those of an oscillator. This phenomenon may give the intuition of the reason why the construction of stable framers is difficult when the stability of these systems is, roughly speaking, weak.

Next, we introduce an assumption on the function ϕ, which typically represents a poorly known disturbance.

Assumption H2. *There are two known functions $\phi^- : \mathbb{R} \to \mathbb{R}^2$ and $\phi^+ : \mathbb{R} \to \mathbb{R}^2$ Lipschitz continuous and such that $\phi^-(t) \leq \phi(t) \leq \phi^+(t)$ for all $t \geq 0$*[1].

We now introduce a key mapping for any matrix $A \in \mathbb{R}^{2\times 2}$

$$\mathcal{F}(A) \; = \; \mathrm{tr}(A)^2 - 2\det(A) \; . \qquad (5)$$

Note that the value $\mathcal{F}$ is the same for all similar matrices: if $A_* = P^{-1}AP$ where P is any invertible matrix then $\mathcal{F}(A) = \mathcal{F}(A_*)$. Based on $\mathcal{F}(A)$, we define the following properties **Pi** and **Qi**.

Definition 1: A matrix $A \in \mathbb{R}^{2\times 2}$ satisfies Property **P1** if $\mathcal{F}(A) > (a_{1,1} - a_{2,2})^2$, it satisfies Property **P2** if $\mathcal{F}(A) = (a_{1,1} - a_{2,2})^2$, it satisfies Property **P3** if $\mathcal{F}(A) < (a_{1,1} - a_{2,2})^2$.

A matrix $A \in \mathbb{R}^{2\times 2}$ satisfies Property **Q1** if $\mathcal{F}(A) > 0$, it satisfies Property **Q2** if $\mathcal{F}(A) = 0$, it satisfies Property **Q3** if $\mathcal{F}(A) < 0$.

Remark: Since A is a constant matrix, Assumption H2 ensures that all the solutions of the system (1) are defined over $[0, +\infty)$. In addition, it is worth noting that if Assumption H1 is satisfied and if $||\phi(t)||$ converges to zero, then all the solutions of (1) converge to zero. One of the advantages of the main results we propose is that none of them relies on the restrictive assumption that the unknown mapping $\phi(t)$ vanishes, because it is very usefull in practice to have a technique enabling to cope with disturbances which do not vanish.

B. Design of standard interval observers

Framers given by the theorem below play a crucial role in the literature devoted to the design of interval observers. To the best of our knowledge, all the designs of framers available in the literature when applied to (1) lead to them.

Theorem 1: Assume that the system (1) satisfies Assumptions H1 and H2. Then the system

$$\left\{ \begin{array}{rcl} \dot{\xi}^+ & = & \mathcal{A}_1\xi^+ + \mathcal{A}_2\xi^- + \phi^+(t) \; , \\ \dot{\xi}^- & = & \mathcal{A}_1\xi^- + \mathcal{A}_2\xi^+ + \phi^-(t) \; , \end{array} \right. \qquad (6)$$

with $\mathcal{A}_i$, $i = 1, 2$ defined in (3), $\xi^+ = (\xi_1^+ \; \xi_2^+)^\top$, $\xi^- = (\xi_1^- \; \xi_2^-)^\top$, is a framer for system (1) *i.e.* if $\xi^-(0) \leq \xi(0) \leq$

[1] All the inequalities must be understood component by component.

$\xi^+(0)$, then $\xi^-(t) \le \xi(t) \le \xi^+(t)$ for all $t \ge 0$. Moreover, in the case of perfect knowledge of ϕ, i.e. $\phi^-(t) = \phi(t) = \phi^+(t)$, for all $t \ge 0$, the observation error is exponentially stable if and only if matrix A satisfies Property **P1** and it is exponentially unstable if and only if matrix A satisfies Property **P3**.

Proof. To show that the system (6) is a framer, we rewrite the system (1) as:

$$\dot{\xi} \;=\; \mathcal{A}_1\xi + \mathcal{A}_2\xi + \phi(t) \tag{7}$$

and defined the error vector by $e = (e_1\ e_2)^\top$, $e_1 = \xi^+ - \xi$, $e_2 = \xi - \xi^-$. From (6) and (7) we deduce easily that

$$\dot{e} = \bar{\bar{A}}e + \bar{\bar{\phi}}(t) \tag{8}$$

with $\bar{\bar{\phi}}(t) = (\phi^+(t) - \phi(t) \quad \phi(t) - \phi^-(t))^\top$ and $\bar{\bar{A}}$ defined in (3). The matrix $\bar{\bar{A}}$ is cooperative [17], and Assumption H2 ensures that $\bar{\bar{\phi}}(t) \ge 0$ for all $t \ge 0$. Consequently, when $e(0) \ge 0$, $e(t)$ is positive at any nonnegative time, and thus (6) is a framer for (1).

Next, we analyze the stability property of (8) when $\bar{\bar{\phi}} \equiv 0$. The spectrum of $\bar{\bar{A}}$ is the union of the spectra of $A = \mathcal{A}_1 + \mathcal{A}_2$ and $A^\dagger = \mathcal{A}_1 - \mathcal{A}_2$. Indeed, it is easy to check that a vector $V = (v_1^\top v_2^\top)^\top$ with $v_1 \in \mathbb{R}^2$, $v_2 \in \mathbb{R}^2$ is an eigenvector of $\bar{\bar{A}}$ if and only if there exists $\lambda \in \mathbb{C}$ such that $(\mathcal{A}_1 + \mathcal{A}_2)(v_1 - v_2) = \lambda(v_1 - v_2)$, $(\mathcal{A}_1 - \mathcal{A}_2)(v_1 + v_2) = \lambda(v_1 + v_2)$. Since matrix A is Hurwitz, we deduce that the stability of $\bar{\bar{A}}$ depends only on the eigenvalues of $A^\dagger$. Since $\mathrm{tr}(A^\dagger) = -2\kappa < 0$, $A^\dagger$ is Hurwitz if and only if its determinant is positive. The value of this determinant is $a_{1,1}a_{2,2} - |a_{1,2}||a_{2,1}|$. Since the eigenvalues of A are complex, necessarily $a_{1,2}$ and $a_{2,1}$ have an opposite sign. Therefore $\det(A^\dagger) = \frac{1}{2}\left[\mathcal{F}(A) - (a_{1,1} - a_{2,2})^2\right]$. This allows us to conclude.

C. Obstacle to the design of time-invariant interval observers

Theorem 1 shows that the design of exponentially stable interval observers is straightforward when Property **P1** is satisfied. In contrast to this case, when Property **P2** or **P3** is satisfied, the only technique of construction of interval observers suggested by the literature consists in transforming the system through a time-invariant change of coordinates into another one for which Property **P1** is satisfied. But we prove below that this idea cannot always be successfully applied: it works when Property **Q1** is satisfied, but when Property **Q3** holds then no time-invariant linear change of coordinates transforms the matrix A into a new one which satisfies Property **P1**.

Theorem 2: Consider a matrix $A \in \mathbb{R}^{2\times 2}$ which satisfies Assumption H1 and Property **Q3**. Then, for all invertible matrix $P \in \mathbb{R}^{2\times 2}$, the matrix $A_* = PAP^{-1}$ satisfies Assumption H1 and Property **P3**. Assume that A satisfies Property **Q1** (resp. Property **Q2**). Then there exists an invertible matrix $H \in \mathbb{R}^{2\times 2}$ such that HAH^{-1} satisfies Assumption H1 and Property **P1** (resp. Property **P2**).

Proof. Consider a matrix $A \in \mathbb{R}^{2\times 2}$ which satisfies Assumption H1 and Property **Q3** and let $A_* = PAP^{-1}$ where P is any invertible matrix. Then the matrix A_* satisfies Assumption H1 and Property **Q3** as well: $\mathcal{F}(A_*) < 0$. Therefore $\mathcal{F}(A_*) < (a_{1,1*} - a_{2,2*})^2$. Thus A_* satisfies Property **P3**.

Next, assume that A satisfied Property **Q1** (resp. Property **Q2**). Then we deduce from Section 1.6 in [14] that there exists an invertible matrix $H \in \mathbb{R}^{2\times 2}$ such that $HAH^{-1} = G$ where G is the matrix defined in (4). Since A satisfied Property **Q1** (resp. Property **Q2**), necessarily G satisfied Property **Q1** (resp. Property **Q2**). For G, Property **Q1** (resp. Property **Q2**) is equivalent to Property **P1** (resp. Property **P2**). This concludes the proof.

Consequences of Theorem 2: (i) We deduce easily from Theorem 2 that if a system (1) satisfies Assumptions H1 and H2 with a matrix A which satisfies Property **Q1** (*i.e.* with $\mathcal{F}(A) > 0$), then one can construct an exponentially stable interval for this system by applying Theorem 1 to the system obtained after the change of coordinates $X = H\xi$ where $H \in \mathbb{R}^{2\times 2}$ is such that $HAH^{-1} = G$ where G is the matrix defined in (4). Indeed, in the new coordinates, the system (1) is $\dot{X} = GX + H\phi(t)$, where G satisfies Assumption H1 and Property **P1** and $H\phi$ satisfies Assumption H2.

(ii) We deduce easily from Theorem 2 that if a system (1) has a matrix A which satisfies Assumption H1 and Property **Q3** (*i.e.* with $\mathcal{F}(A) < 0$), there exists no linear time-invariant change of coordinates which results in a system for which the classical framer (6) is, in the absence of disturbances, exponentially stable.

III. Design of time-varying interval observers

The objective of this section is to overcome the obstacle to the construction of exponentially stable interval observers which occurs when Property **Q1** is not satisfied, *i.e.* $\mathcal{F}(A) \le 0$. We will consider the time-varying change of coordinates $z = \lambda(t)\xi$ ($\lambda(t) = (\lambda_{i,j}(t)) \in \mathbb{R}^{2\times 2}$) with

$$\begin{array}{l} \lambda_{1,1}(t) = \cos(\omega t) + \frac{a_{2,2}-a_{1,1}}{2\omega}\sin(\omega t)\ , \\ \lambda_{1,2}(t) = -\frac{a_{1,2}}{\omega}\sin(\omega t)\ , \\ \lambda_{2,1}(t) = -\frac{a_{2,1}}{\omega}\sin(\omega t)\ , \\ \lambda_{2,2}(t) = \cos(\omega t) - \frac{a_{2,2}-a_{1,1}}{2\omega}\sin(\omega t). \end{array} \tag{9}$$

Observe that, for all $t \in \mathbb{R}$, $\det(\lambda(t)) = 1$. By implicitly using the fact that λ transforms (1) into the diagonal system $\dot{z} = -\kappa z + \lambda(t)\phi(t)$, one can deduce exponentially stable interval observers for (1) from classical approaches of framer constructions.

To define the proposed interval observer, we define $\mu(t) = \lambda^{-1}(t)$ ($\mu(t) = (\mu_{i,j}(t)) \in \mathbb{R}^{2\times 2}$), which can be computed as follows:

$$\begin{array}{l} \mu_{1,1}(t) = \lambda_{2,2}(t)\ ,\ \mu_{1,2}(t) = -\lambda_{1,2}(t)\ , \\ \mu_{2,1}(t) = -\lambda_{2,1}(t)\ ,\ \mu_{2,2}(t) = \lambda_{1,1}(t)\ . \end{array} \tag{10}$$

Finally, we also define λ_p, λ_n, μ_p and μ_n:

$$\begin{array}{l} \lambda_p(t) = (\mathcal{L}(\lambda_{i,j}(t)))\ ,\ \mu_p(t) = (\mathcal{L}(\mu_{i,j}(t)))\ , \\ \lambda_n(t) = (\mathcal{M}(\lambda_{i,j}(t)))\ ,\ \mu_n(t) = (\mathcal{M}(\mu_{i,j}(t)))\ . \end{array} \tag{11}$$

Note that the functions λ, μ, λ_p, μ_p λ_n, μ_n are Lipschitz continuous and periodic. The main result of the section is given below

Theorem 3: The following system defines an interval observer for the system (1) under Assumptions H1 and H2:

$$\left\{\begin{array}{rcl} \dot{z}^+ &=& -\kappa z^+ + \lambda_p(t)\phi^+(t) + \lambda_n(t)\phi^-(t) \ , \\ \dot{z}^- &=& -\kappa z^- + \lambda_p(t)\phi^-(t) + \lambda_n(t)\phi^+(t) \ , \\ \xi^+ &=& \mu_p(t)z^+ + \mu_n(t)z^- \ , \\ \xi^- &=& \mu_p(t)z^- + \mu_n(t)z^+ \ , \end{array}\right. \tag{12}$$

where κ is the constant defined in (4) where the functions λ_p, λ_n, μ_p, μ_n are defined in (11). It means that, if the initial conditions of (1) and (12) are such that $\xi(0)^- \leq \xi(0) \leq \xi(0)^+$, then for any positive time t, we have $\xi(t)^- \leq \xi(t) \leq \xi(t)^+$. Moreover, if $\phi^+(t) = \phi^-(t)$ for all $t \geq 0$ then $||\xi(t)^+ - \xi(t)^-||$ converges exponentially to zero.

Remark: Note that, at initial time, $\mu(0)$ is the identity matrix. Therefore, the bounds $(z^+(0), z^-(0))$ on the initial condition of z are straightforwardly deduced from the (known) bounds on the initial condition for ξ: $z^-(0) = \xi(0)^-$ and $z^+(0) = \xi(0)^+$.

Proof. Consider solutions of (1) and (12) such that $\xi(0)^- \leq \xi(0) \leq \xi(0)^+$. Let $e^+ = z^+ - \lambda(t)\xi$, $e^- = \lambda(t)\xi - z^-$. Through direct calculations one can show that the function $\lambda(t)$ whose entries are defined in (9) satisfies for all $t \in \mathbb{R}$

$$\dot{\lambda}(t) + \lambda(t)A \quad = \quad -\kappa\lambda(t) \ . \tag{13}$$

It follows that, for all $t \geq 0$,

$$\left\{\begin{array}{rcl} \dot{e}^+(t) &=& -\kappa e^+(t) + \psi^+(t) \ , \\ \dot{e}^-(t) &=& -\kappa e^-(t) + \psi^-(t) \ , \end{array}\right. \tag{14}$$

with $\psi^+(t) = \lambda_p(t)[\phi^+(t) - \phi(t)] - \lambda_n(t)[\phi(t) - \phi^-(t)]$ and $\psi^-(t) = \lambda_p(t)[\phi(t) - \phi^-(t)] - \lambda_n(t)[\phi^+(t) - \phi(t)]$. Since λ_p and λ_n are Lipschitz continuous and periodic, ψ^+ and ψ^- satisfy Assumption H2. Since $e^+(0) = z^+(0) - \lambda(0)\xi(0) = z^+(0) - \xi(0) = \xi^+(0) - \xi(0) \geq 0$ and $e^-(0) = \lambda(0)\xi(0) - z^-(0) = \xi(0) - z^-(0) = \xi(0) - \xi^-(0) \geq 0$ and since Assumption H2 implies that $\psi^+(t) \geq 0$, $\psi^-(t) \geq 0$ for all $t \geq 0$, we deduce that, for all $t \geq 0$, $e^+(t) \geq 0$ and $e^-(t) \geq 0$. These inequalities imply that, for all $t \geq 0$, $z^-(t) \leq \lambda(t)\xi(t) \leq z^+(t)$. It follows that, for all $t \geq 0$,

$$\begin{array}{c} \mu_p(t)z^-(t) \leq \mu_p(t)\lambda(t)\xi(t) \leq \mu_p(t)z^+(t) \ , \\ \mu_n(t)z^-(t) \geq \mu_n(t)\lambda(t)\xi(t) \geq \mu_n(t)z^+(t) \ . \end{array} \tag{15}$$

By adding these inequalities and by observing that $\mu_p(t)\lambda(t) + \mu_n(t)\lambda(t)$ is equal to the identity matrix for all $t \geq 0$, we deduce that, for all $t \geq 0$,

$$\begin{array}{c} \mu_p(t)z^-(t) + \mu_n(t)z^+(t) \leq \xi(t) \ ; \\ \xi(t) \leq \mu_p(t)z^+(t) + \mu_n(t)z^-(t) \ . \end{array} \tag{16}$$

It follows that for all $t \geq 0$, we have $\xi(t)^- \leq \xi(t) \leq \xi(t)^+$. Finally, noticing that if $\phi^+ \equiv \phi^-$ then $||z^+(t) - z^-(t)||$ converges exponentially to zero and noticing that μ_p and μ_n are Lipschitz continuous and periodic functions, we deduce that if $\phi^+ \equiv \phi^-$ then $||\xi^+(t) - \xi^-(t)||$ converges exponentially to zero.

IV. Example of the chaotic Chua's system

In order to assess the efficiency of our approach, we have chosen to apply it to a system which belongs to the class of the chaotic systems, which are highly sensitive to the initial conditions and therefore are difficult to handle. We have selected Chua's model often considered as a paragon of the chaotic systems. The equations of Chua's system are:

$$\left\{\begin{array}{rcl} \dot{\xi}_1 &=& -\xi_1 + \xi_2 + \xi_3 \ , \\ \dot{\xi}_2 &=& -\beta\xi_1 - \gamma\xi_2 \ , \\ \dot{\xi}_3 &=& \alpha\xi_1 - \alpha[\xi_3(1+b) + g(\xi_3)] \ , \end{array}\right. \tag{17}$$

where $g(\xi_3) = \frac{1}{2}(a-b)(|\xi_3+1| - |\xi_3-1|)$. We assume that the variable ξ_3 can be measured, but with a measurement noise $r(t)$. We define $y(t) = \xi_3(t) - r(t)$ the measured signal. For the simulation purpose we take parameter values that are known to generate chaos: $\alpha = 11.85$, $\beta = 14.9$, $\gamma = 0.29$, $a = -1.14$ and $b = -0.71$. Observe that, for this system, one can easily construct a Luenberger observer [9] with an arbitrarily fast rate of convergence. However, the problem of constructing exponentially stable interval observers for this system is open. Since ξ_3 is measured, we focus on the (ξ_1, ξ_2)-subsystem of (17). We assume that there is a known nonnegative function $r_P(t)$ that bounds the noise, which is Lipschitz continuous and such that $-r_P(t) \leq r(t) \leq r_P(t)$ for all $t \geq 0$ i.e. r_P bound the norm of the noise. We define $\phi(t) = (\xi_3(t) \quad 0)^\top$. Next, observe that the system

$$\dot{\xi} = A\xi + \phi(t) \quad \text{with} \quad A = \left[\begin{array}{cc} -1 & 1 \\ -\beta & -\gamma \end{array}\right] \tag{18}$$

and $\xi = (\xi_1, \xi_2)$ satisfies Assumption H2 with $\phi^+(t) = (y(t) + r_P(t) \ 0)^\top$, $\phi^-(t) = (y(t) - r_P(t) \ 0)^\top$ (since all the solutions of (17) are defined over $[0, +\infty)$) and Assumption H1 because A is Hurwitz with non-real eigenvalues. Next, observe that $\text{tr}(A)^2 = 1.29^2$, $\det(A) = 15.19$ and therefore A satisfies Property **Q3**. Next, by combining Theorem 2 and Theorem 1, we deduce that there exists no linear time-invariant change of coordinates that transforms (18) into a system for which the interval observer (6) is exponentially stable. This leads us to apply Theorem 3 to derive an exponentially stable interval observer for the system (18). This theorem leads us to propose for system (18) the following interval observer:

$$\left\{\begin{array}{rcl} \dot{z}_1^+ &=& -\kappa z_1^+ + \left[c(t) + \frac{1-\gamma}{2\omega}s(t)\right] y(t) \\ && + \left|c(t) + \frac{1-\gamma}{2\omega}s(t)\right| r_P(t) \ , \\ \dot{z}_2^+ &=& -\kappa z_2^+ - \frac{\beta}{\omega}s(t)y(t) + \frac{\beta}{\omega}\left|s(t)\right| r_P(t) \ , \\ \dot{z}_1^- &=& -\kappa z_1^- + \left[c(t) + \frac{1-\gamma}{2\omega}s(t)\right] y(t) \\ && - \left|c(t) + \frac{1-\gamma}{2\omega}s(t)\right| r_P(t) \ , \\ \dot{z}_2^- &=& -\kappa z_2^- - \frac{\beta}{\omega}s(t)y(t) - \frac{\beta}{\omega}\left|s(t)\right| r_P(t) \ , \\ \xi_1^+ &=& \mathcal{L}(c(t) - \frac{1-\gamma}{2\omega}s(t))z_1^+ + \mathcal{L}\left(\frac{1}{\omega}s(t)\right) z_2^+ \\ && + \mathcal{M}\left(c(t) - \frac{1+\gamma}{2\omega}s(t)\right) z_1^- + \mathcal{M}\left(\frac{1}{\omega}s(t)\right) z_2^- \ , \\ \xi_2^+ &=& \mathcal{L}\left(\frac{\beta}{\omega}s(t)\right) z_1^+ + \mathcal{L}\left(c(t) + \frac{1-\gamma}{2\omega}s(t)\right) z_2^+ \\ && + \mathcal{M}\left(\frac{\beta}{\omega}s(t)\right) z_1^- + \mathcal{M}\left(c(t) + \frac{1-\gamma}{2\omega}s(t)\right) z_2^- \ , \\ \xi_1^- &=& \mathcal{L}\left(c(t) - \frac{1-\gamma}{2\omega}s(t)\right) z_1^- + \mu_{p12}(t)z_2^- \\ && + \mathcal{M}\left(c(t) - \frac{1-\gamma}{2\omega}s(t)\right) z_1^+ + \mathcal{M}\left(\frac{1}{\omega}s(t)\right) z_2^+ \ , \\ \xi_2^- &=& \mathcal{L}\left(\frac{\beta}{\omega}s(t)\right) z_1^- + \mathcal{L}\left(c(t) + \frac{1-\gamma}{2\omega}s(t)\right) z_2^- \\ && + \mathcal{M}\left(\frac{\beta}{\omega}s(t)\right) z_1^+ + \mathcal{M}\left(c(t) + \frac{1-\gamma}{2\omega}s(t)\right) z_2^+ \ , \end{array}\right. \tag{19}$$

with $\kappa = \frac{1+\gamma}{2}$ and $\omega = \sqrt{\gamma + \beta - \frac{1}{4}(1+\gamma)^2}$ with $c(t) = \cos(\omega t)$, $s(t) = \sin(\omega t)$.

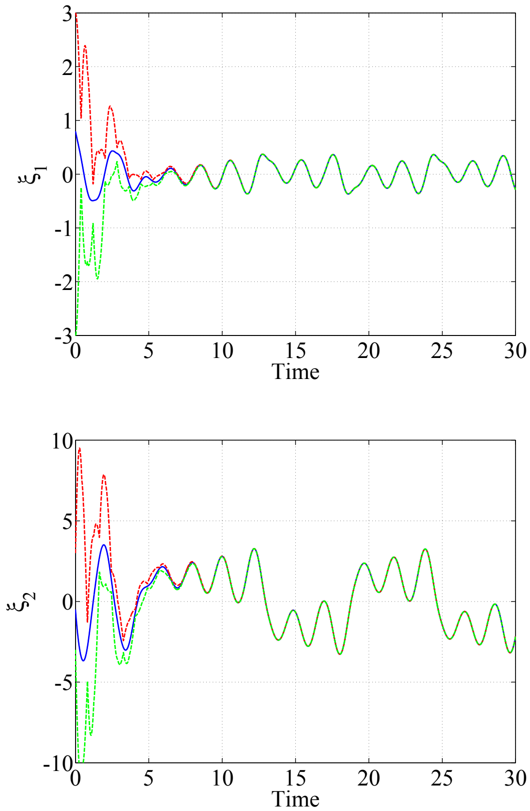

Fig. 1. Interval observer for Chua's system (- -) compared to real state (–), when $\xi_3(t)$ is measured without error.

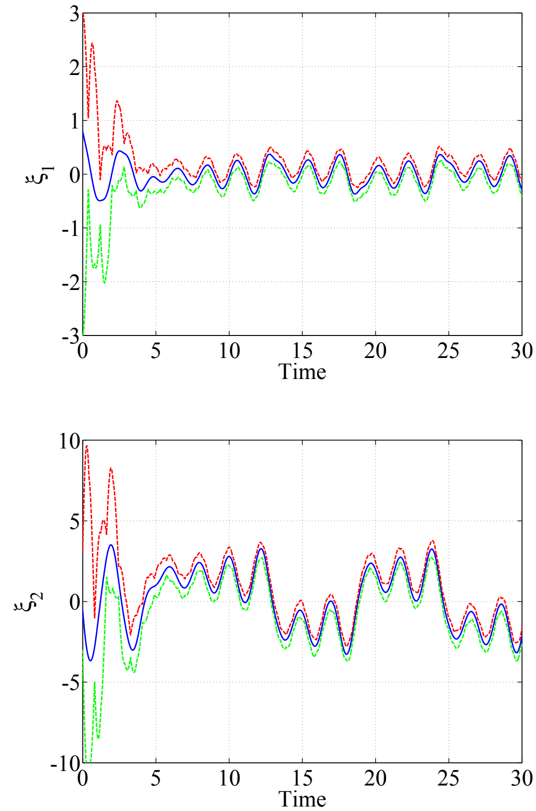

Fig. 2. Interval observer for Chua's system (- -) compared to real state (–), when $\xi_3(t)$ is measured with a disturbance noise smaller everywhere than 0.1.

Figure 1 shows the simulation results in the case of perfect knowledge. It is worth noting that the observer converges despite the chaotic regime. This can have applications in signal encrypting and secure communication [18].

The observer was then run assuming an error $r(t)$, on the measurements of $\xi_3(t)$, bounded between -0.1 and 0.1. The results are presented on Figure 2. The observer converges toward a narrow interval, and the proposed interval stays accurate enough to provide a useful state estimate.

This example can be compared to [11], where an observer was developed for Chua's system. However, in [11], Chua system is endowed with the output ξ_1, so that the problem of constructing a stable interval observer can be solved after introducing an error correction term that transforms the error dynamics into a cooperative and stable system.

V. Conclusion

In this paper we have shown how to construct an exponentially stable linear interval observer for a two dimensional exponentially stable linear system with complex poles having a negative real part. The main idea consists in performing a time-varying change of coordinates which yields a simple dynamics for which one can construct exponentially stable interval observers.

The example illustrates the efficiency of the approach with the non-trivial case of a chaotic system. The main advantage for encryption application is that the observer convergence can be assessed by the difference between the upper and lower bounds $||\xi^+ - \xi^-||$.

Much remains to be done. In further works, we shall extend our results to more general families of systems. Also we shall investigate whether, in the situations where interval observers of the form (6) are exponentially stable, the interval observers resulting from our time-varying change of coordinates are (at least in some cases) more efficient than the time-invariant interval observers or if, together, they can lead to a bundle of observers. The next step consists in considering uncertainties in the linear part of the model and, as in [11], in deriving the associated interval observer.

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