TABLE III
Comparison of the Advanced aQPmR With the Standard QPmR and DDE-Biftool Used to Compute the Zeros of the Quasi-Polynomial Given by (1) and Table I

ω_{max}	β_{min}	N_D	QPmR	aQPmR		ρ_D	DDE-Biftool	
			T_C	T_C	T_P		T_C	Δt
10	-1.5	43	3.2				1.15	0.15
20	-2.1	82	7.2	11.7	3.9	1.8	7.1	0.075
40	-2.8	161	15.6	17.2	4.3	2.6	48.6	0.038
100	-4.5	401	47.7	33.2	6.3	4.0		
200	-5.7	797	125	61.7	9.8	5		
300	-6.5	1196	334	97	12	5.6		

T_C [sec.]—overall time of computation (Matlab 7.4, PC—Core Duo, 1.66 GHz, 1 GB RAM), T_P [sec.]—time of the preparatory stage of aQPmR (defining the suspect regions free of zeros, applying the argument principle). ρ_D—ratio between the mapped areas of QPmR and aQPmR. N_D—number of zeros in $\boldsymbol{D}$. Parameters of aQPmR (QPmR): $\boldsymbol{D} = [\beta_{min}\beta_{max}] \times [0 \ \omega_{max}]$, $\beta_{max} = 3$, $\omega_{map} = 10(\beta_{map} = -1.4)$, $\Delta_g = 0.0157(N_g = 8)$, $\delta_s = 0.38$, accuracy $\varepsilon_N = 1e - 6$. Parameters of DDE-Biftool: Milne-Simpson 4th order method, accuracy 1e-6, Δt—discretization step.

[3] K. Engelborghs and D. Roose, "Numerical computation of stability and detection of Hopf bifurcation of steady state solutions of delay differential equations," *Adv. Comput. Math.*, vol. 10, no. 3–4, pp. 271–289, 1999.

[4] K. Verheyden, T. Luzyanina, and D. Roose, "Efficient computation of characteristic roots of delay differential equations using LMS methods," *J. Comput. Appl. Math.*, vol. 214, no. 1, pp. 209–226, 2008.

[5] D. Breda, "Solution operator approximation for characteristic roots of delay differential equations," *Appl. Numer. Math.*, vol. 56, pp. 305–331, 2006.

[6] D. Breda, S. Maset, and R. Vermiglio, "Computing the characteristic roots for delay differential equations," *IMA J. Numer. Anal.*, vol. 24, no. 1, pp. 1–19, 2004.

[7] D. Breda, S. Maset, and R. Vermiglio, "Pseudospectral differencing methods for characteristic roots of delay differential equations," *SIAM J. Sci. Comput.*, vol. 27, no. 2, pp. 482–495, 2005.

[8] K. Engelborghs, T. Luzyanina, and D. Roose, "Numerical bifurcation analysis of delay differential equations using DDE-Biftool," *ACM Trans. Math. Software*, vol. 28, no. 1, pp. 1–21, 2002.

[9] D. Breda, S. Maset, and R. Vermiglio, "TRACE-DDE: A tool for robust analysis and characteristic equations for delay differential equations," in *Topics in Time-Delay Systems: Analysis, Algorithms and Control*, ser. LNCIS, J. J. Loiseau, W. Michiels, S. I. Niculescu, and R. Sipahi, Eds. Berlin, Germany: Springer, 2009.

[10] P. Kravanja and M. Van Barel, *Computing the Zeros of Analytic Functions*. Berlin, Germany: Springer, 2000, vol. 1727.

[11] A. S. Householder, *The Numerical Treatment of a Single Nonlinear Equation*. New York: McGraw-Hill, 1970.

[12] M. N. Vrahatis, "An efficient method for locating and computing periodic orbits of nonlinear mappings," *J. Comput. Phys.*, vol. 119, pp. 105–119, 1995.

[13] M. Dellnitz, O. Schütze, and Q. Zheng, "Locating all the zeros of an analytic function in one complex variable," *J. Comput. Appl. Math.*, vol. 138, pp. 325–333, 2002.

[14] L. M. Delves and J. N. Lyness, "A numerical method for locating the zeros of analytic functions," *Math. Comput.*, vol. 21, pp. 543–560, 1967.

[15] T. Vyhlídal and P. Zítek, "Quasipolynomial mapping based rootfinder for analysis of time delay systems," in *Proc 4th IFAC Workshop, Time Delay Syst. (TDS'03)*, Rocquencourt, France, 2003, pp. 227–232.

[16] A. D. Gouveia, "An aid to two-dimensional contouring using nonuniform orthogonal grids—A fortran algorithm," *Comput. Geosci.*, vol. 19, no. 8, pp. 1071–1076, 1992.

[17] W. Michiels and S. Niculescu, *Stability and Stabilization of Time-Delay Systems. An Eigenvalue Based Approach*, ser. Advances in Design and Control 12. Philadelphia, PA: SIAM Publications, 2007.

[18] T. Luzyanina and D. Roose, "Numerical stability analysis and computation of Hopf bifurcation points for delay differential equations," *J. Comput. Appl. Math.*, vol. 72, pp. 379–392, 1996.

[19] P. Deuflhard, *Newton Methods for Nonlinear Problems. Affine Invariance and Adaptive Algorithms*. Berlin, Germany: Springer, 2004, vol. 35.

A Simplified Design for Strict Lyapunov Functions Under Matrosov Conditions

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Abstract—We construct strict Lyapunov functions for broad classes of nonlinear systems satisfying Matrosov conditions. Our new constructions are simpler than the designs available in the literature. We illustrate the practical interest of our designs using a globally asymptotically stable biological model.

Index Terms—Biological systems, Lyapunov function, Matrosov theorem.

I. Introduction

Lyapunov functions play an essential role in modern nonlinear systems analysis and controller design. Oftentimes, *non*-strict Lyapunov functions are readily available. However, *strict* (i.e., *strong*) Lyapunov functions are preferable since they can be used to quantify the effects of disturbances; see the precise definitions below. Strict Lyapunov functions have been used in several biological contexts e.g. to quantify the effects of actuator noise and other uncertainty on the steady state concentrations of competing species in chemostats [13], but their explicit construction can be challenging. For some large classes of systems, there are mechanisms for transforming non-strict Lyapunov functions into the required strict Lyapunov functions e.g. [4], [11], [12], [14], [15].

For systems satisfying conditions of Matrosov's type [8], [10], strict Lyapunov functions were constructed in [15], under very general conditions. However, the generality of the assumptions in [15] makes its constructions complicated and therefore difficult to apply. Moreover, the Lyapunov functions provided by [15] are not locally bounded from below by positive definite quadratic functions, even for asymptotically stable linear systems, which admit a quadratic strict Lyapunov function. The shape of Lyapunov functions, their local properties and their simplicity matter when they are used to investigate robustness and construct feedbacks and gains.

In the present work, we revisit the problem of constructing strict Lyapunov functions under Matrosov's conditions. Our results have the

following desirable features. First, they lead to simplified constructions of strict Lyapunov functions; see Remark 3 below. For a large family of systems, the Lyapunov functions we construct have the added advantage of being locally bounded from below by positive definite quadratic functions, with time derivatives along the trajectories that are locally bounded from above by negative definite quadratic functions. Second, our work does not require a non-strict positive definite radially unbounded Lyapunov function. Rather, we only require a non-strict positive definite function whose derivative along the trajectories is non-positive.

One of our motivations is that one can frequently find non-strict Lyapunov-like functions which are not proper but which make it possible to establish global asymptotic stability of an equilibrium point. For instance, the celebrated Lyapunov function from [5] for a multi-species chemostat (also reported in [16]) is not proper. In such cases, the stability proof is often based on the fact that the models are derived from mass balance properties [1] leading to the boundedness of the trajectories. Our work yields robustness in the sense of input-to-state stability (ISS). The ISS notion is a fundamental paradigm of nonlinear control that makes it possible to quantify the effects of uncertainty [17], [18]. While our assumptions are more restrictive than those used in [8], [15], they are general in the sense that, to the best of our knowledge, they are satisfied by all examples whose stability can be established by the generalized Matrosov's theorem; specifically, see e.g. the examples in [15] whose auxiliary functions satisfy our Assumptions 1–2 below.

II. DEFINITIONS AND NOTATION

We omit the arguments of our functions when they are clear. We use the standard classes of comparison functions $\mathcal{K}_\infty$ and $\mathcal{KL}$; see [18] for their well known definitions. We always assume that $D \subseteq \mathbb{R}^n$ is an open set for which $0 \in D$. A function $V : D \times \mathbb{R} \to \mathbb{R}$ is *positive definite* on D provided $V(0,t) \equiv 0$ and $\inf_t V(x,t) > 0$ for all $x \in D \setminus \{0\}$. A function V is *negative definite* provided $-V$ is positive definite. Let $|\cdot|$ (resp., $|\cdot|_\infty$) denote the standard Euclidean norm (resp., essential supremum). We always assume that our functions are sufficiently smooth.

Consider a general nonlinear system

$$\dot{x} = \mathcal{F}(x, t, \delta(t)) \qquad (1)$$

evolving on a forward invariant open set $\mathcal{G}$ that is diffeomorphic to $\mathbb{R}^n$, with disturbances δ in the set $\mathcal{L}_\infty(\mathcal{C})$ of all measurable essentially bounded functions valued in a given subset $\mathcal{C}$ of Euclidean space. Assume $0 \in \mathcal{G}$, $0 \in \mathcal{C}$, $\mathcal{F} \in C^1$ with $\mathcal{F}(0,t,0) \equiv 0$, where C^1 means continuously differentiable.

A C^1 function $V : D \times \mathbb{R} \to \mathbb{R}$ is a *Lyapunov-like function* for (1) with $\delta \equiv 0$ provided V is positive definite and $\dot{V}(x,t) := (\partial V/\partial t)(x,t) + (\partial V/\partial x)(x,t)\mathcal{F}(x,t,0) \leq 0$ for all $x \in D$ and $t \geq 0$. If in addition $\dot{V}(x,t)$ is negative definite, then V is a *strict Lyapunov-like function* for (1) with $\delta \equiv 0$. A function $W : D \to \mathbb{R}$ is *radially unbounded* (or *proper*) provided $\lim_{x \in D, |x| \to +\infty} W(x) = +\infty$. A (strict) Lyapunov-like function is a *(strict) Lyapunov function* provided it is also proper.

Let $t \mapsto \phi(t; t_o, x_o, \delta)$ denote the solution for (1) with arbitrary initial condition $x(t_0) = x_o$ and any $\delta \in \mathcal{L}_\infty(\mathcal{C})$, which we always assume to be uniquely defined on $[t_o, +\infty)$. Let $M(\mathcal{G})$ denote the set of all continuous functions $\mathcal{M} : \mathcal{G} \to [0, \infty)$ for which (A) $\mathcal{M}(0) = 0$ and (B) $\mathcal{M}(x) \to +\infty$ as $x \to \text{boundary}(\mathcal{G})$ or $|x| \to +\infty$ while remaining in $\mathcal{G}$. We say that (1) is *ISS on $\mathcal{G}$ with disturbances in $\mathcal{C}$* (or just ISS when $\mathcal{G}$ and $\mathcal{C}$ are clear) [17], [18] provided there exist $\beta \in \mathcal{KL}$, $\mathcal{M} \in M(\mathcal{G})$ and $\gamma \in \mathcal{K}_\infty$ such that $|\phi(t; t_o, x_o, \delta)| \leq \beta(\mathcal{M}(x_o), t - t_o) + \gamma(|\delta|_\infty)$ for all $t \geq t_o \geq 0$, $x_o \in \mathcal{G}$, and $\delta \in$

$\mathcal{L}_\infty(\mathcal{C})$. When $\mathcal{G} = \mathbb{R}^n$ and $\mathcal{M}(x) = |x|$, this becomes the usual ISS definition. The ISS property reduces to the standard (uniformly) globally asymptotically stable condition when $\delta \equiv 0$ but is far more general because it quantifies the effects of disturbances.

III. MAIN RESULT

A. Statement of Assumptions and Result

For simplicity, we first state our main result for time invariant systems $\dot{x} = f(x)$ evolving on D; see Section IV for generalizations to time-varying systems $\dot{x} = f(x,t)$. We assume:

Assumption 1: There exist an integer $j \geq 2$; functions $V_i : D \to \mathbb{R}$, $\mathcal{N}_i : D \to [0, +\infty)$, and $\phi_i : [0, +\infty) \to (0, +\infty)$; and constants $a_i \in (0, 1]$ such that (a) $V_i(0) = \mathcal{N}_i(0) = 0$ for all i, (b) $\nabla V_1(x) f(x) \leq -\mathcal{N}_1(x)$ and $\nabla V_i(x) f(x) \leq -\mathcal{N}_i(x) + \phi_i(V_1(x)) \sum_{l=1}^{i-1} \mathcal{N}_l^{a_i}(x) V_1^{1-a_i}(x)$ for $i = 2, \ldots, j$ for all $x \in D$, and (c) the function V_1 is positive definite on D.

Assumption 2: (i) There exists a function $\rho : [0, +\infty) \to (0, +\infty)$ such that $\sum_{l=1}^j \mathcal{N}_l(x) \geq \rho(V_1(x)) V_1(x)$ for all $x \in D$. (ii) There exist functions $p_2, \ldots, p_j : [0, +\infty) \to [0, +\infty)$ such that $|V_i(x)| \leq p_i(V_1(x)) V_1(x)$ for all $x \in D$ holds for $i = 2, 3, \ldots, j$.

Theorem 1: Assume that there exist $j \geq 2$ and functions satisfying Assumptions 1–2. Then one can build explicit functions $k_l, \Omega_l \in \mathcal{K}_\infty \cap C^1$ such that $S(x) = \sum_{l=1}^j \Omega_l(k_l(V_1(x)) + V_l(x))$ satisfies

$$S(x) \geq V_1(x) \text{ and } \nabla S(x) f(x) \leq -\frac{1}{4} \rho(V_1(x)) V_1(x) \qquad (2)$$

for all $x \in D$.

Remark 1: The proof of Theorem 1 uses the triangular structure of the inequalities in Assumption 1(b) in an essential way. The differences between Assumptions 1–2 and the assumptions from [15] are these. First, while Assumption 1 above ensures that V_1 is positive definite but not necessarily proper, [15] requires a radially unbounded non-strict Lyapunov function. Second, our Assumption 1 is a restrictive version of [15, Assumption 2] because we specify the local properties of the functions which correspond to the χ_i's of [15, Assumption 2]. Finally, our Assumption 2 imposes relations between the functions $\mathcal{N}_i$ and V_1, which are not required in [15]. Note that we do not require $V_2, \ldots, V_j$ to be nonnegative.

Remark 2: If $D = \mathbb{R}^n$ and V_1 is radially unbounded, then (2) implies that S is a strict Lyapunov function for $\dot{x} = f(x)$. If V_1 is not radially unbounded, then one cannot conclude from Lyapunov's theorem that the origin is globally asymptotically stable. However, in many cases, global asymptotic stability can be proved through a Lyapunov-like function and extra arguments. We illustrate this in Section V. If V_1 is bounded from below by a positive definite quadratic form in a neighborhood of 0, then we get positive definite quadratic lower bounds on S (by (2)) and $(1/4)\rho(V_1(x))V_1(x)$ near 0.

B. Proof of Theorem 1

Step 1: Construction of the k_i's and Ω_i's: Fix $j \geq 2$ and functions satisfying Assumptions 1–2. Fix $k_2, \ldots, k_j \in C^1 \cap \mathcal{K}_\infty$ such that $k_i(s) \geq s + p_i(s)s$ and $k_i'(s) \geq 1$ for all $s \geq 0$ for $i = 2, 3, \ldots, j$.

Lemma 1: The functions $U_1(x) = V_1(x)$ and $U_i(x) = k_i(V_1(x)) + V_i(x)$ satisfy $2k_i(V_1(x)) \geq U_i(x) \geq V_1(x)$ for all $i = 2, \ldots, j$ and all $x \in D$.

In fact, Assumption 2(ii) and our choices of the k_i's give $U_i(x) \geq V_1(x) + p_i(V_1(x))V_1(x) - p_i(V_1(x))V_1(x) = V_1(x)$ and $U_i(x) \leq k_i(V_1(x)) + p_i(V_1(x))V_1(x) \leq 2k_i(V_1(x))$ for $i \geq 2$, which proves Lemma 1. Set $k_1(s) \equiv s$.

Returning to the proof of the theorem, define the functions U_i according to Lemma 1. Let $\Omega_1, \ldots, \Omega_j \in \mathcal{K}_\infty \cap C^1$ be functions for which $\Omega_i'(s) \geq 1$ for all $s \geq 0$ and all i, and such that

$$\Omega_i'(U_i) \geq 2\Phi(V_1) \sum_{l=1+i}^{j} \Omega_l'(U_l)^{\frac{1}{a_l}}, \quad \text{where}$$

$$\Phi(V_1) = \max_{i=2,\ldots,j} \left\{ \phi_i(V_1)^{\frac{1}{a_i}} \left[\frac{4(j-1)(i-1)}{\rho(V_1)} \right]^{(1-a_i)/a_i} \right\} \quad (3)$$

for $i = 1, 2, \ldots, j$. Specifically, take $\Omega_i(p) = \int_0^p \mu_i(r) dr$ where the nondecreasing functions $\mu_i : [0, \infty) \to [1, \infty)$ are from Lemma A.1 in Appendix A below. In particular, we take $\Omega_j(p) \equiv p$.

Step 2: Stability Analysis: Since $\Omega_i'(s) \geq 1$ everywhere, $\Omega_1(U_1(x)) \geq U_1(x) = V_1(x)$ everywhere. Hence, $S(x) = \Omega_1(2V_1(x)) + \sum_{i=2}^{j} \Omega_i(U_i(x))$ satisfies the first requirement in (2). To check the decay estimate in (2), first note that Assumption 1(b) and our choices of the k_i's give

$$\nabla S(x) f(x) = 2\Omega_1'(2U_1)\dot{V}_1 + \sum_{i=2}^{j} \Omega_i'(U_i) \left[k_i'(V_1)\dot{V}_1 + \dot{V}_i \right]$$

$$\leq \sum_{i=1}^{j} \Omega_i'(U_i)\dot{V}_i$$

$$\leq - \sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i$$

$$+ \sum_{i=2}^{j} \Omega_i'(U_i) \left(\phi_i(V_1) \sum_{l=1}^{i-1} \mathcal{N}_l^{a_i} V_1^{1-a_i} \right) \quad (4)$$

along the trajectories of $\dot{x} = f(x)$. Define the positive functions $\Gamma_2, \ldots, \Gamma_j$ by $\Gamma_i(x) = [4(j-1)(i-1)\Omega_i'(U_i(x))\phi_i(V_1(x))]/\rho(V_1(x))$. For any $i \geq 2$ for which $0 < a_i < 1$, we can apply Young's Inequality $v_1 v_2 \leq v_1^p + v_2^q$ with $p = 1/a_i$, $q = 1/(1-a_i)$, $v_1 = \Gamma_i(x)^{1-a_i}\mathcal{N}_l(x)^{a_i}$, and $v_2 = \{V_1(x)/\Gamma_i(x)\}^{1-a_i}$ to get $\mathcal{N}_l(x)^{a_i}V_1(x)^{1-a_i} \leq \Gamma_i(x)^{(1-a_i)/a_i}\mathcal{N}_l(x) + V_1(x)/\Gamma_i(x)$ for all $x \in D$. The preceding inequality also holds when $a_i = 1$, so we can substitute it into (4) to get

$$\nabla S(x) f(x) \leq - \sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i$$

$$+ \sum_{i=2}^{j} \left(\Omega_i'(U_i)\phi_i(V_1)\Gamma_i^{\frac{1-a_i}{a_i}} \sum_{l=1}^{i-1} \mathcal{N}_l \right)$$

$$+ \left(\sum_{i=2}^{j} \Omega_i'(U_i) \frac{\phi_i(V_1)(i-1)}{\Gamma_i} \right) V_1$$

$$\leq - \sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i + \frac{1}{4}\rho(V_1)V_1$$

$$+ \sum_{i=2}^{j} \left(\Omega_i'(U_i)\phi_i(V_1) \right.$$

$$\times \left[\frac{4(j-1)(i-1)\Omega_i'(U_i)\phi_i(V_1)}{\rho(V_1)} \right]^{\frac{1-a_i}{a_i}}$$

$$\left. \times \sum_{l=1}^{i-1} \mathcal{N}_l \right)$$

$$\leq - \sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i + \frac{1}{4}\rho(V_1)V_1$$

$$+ \Phi(V_1) \sum_{i=2}^{j} \left(\Omega_i'(U_i)^{\frac{1}{a_i}} \sum_{l=1}^{i-1} \mathcal{N}_l \right) \quad (5)$$

by our choices of the Γ_i's and the formula for Φ in (3).

Since $\Omega_i' \geq 1$ for all i, Assumption 2(i) gives $\sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i \geq \rho(V_1)V_1$. Hence, (5) gives $\nabla S(x)f(x) \leq -(1/4)\rho(V_1)V_1 - (1/2)\sum_{i=1}^{j} \Omega_i'(U_i)\mathcal{N}_i + \Phi(V_1)\sum_{i=2}^{j}(\Omega_i'(U_i)^{1/a_i}\sum_{l=1}^{i-1}\mathcal{N}_l)$. It follows that $\nabla S(x)f(x) \leq -(1/4)\rho(V_1)V_1 + \sum_{i=1}^{j-1}[-(1/2)\Omega_i'(U_i)+\Phi(V_1)\sum_{l=1+i}^{j}\Omega_l'(U_l)^{1/a_l}]\mathcal{N}_i$, by an easy switching of the order of summation. Since the $\mathcal{N}_i$'s are nonnegative, (2) now readily follows from (3).

IV. EXTENSION TO TIME-VARYING SYSTEMS

One can prove an analog of Theorem 1 for $\dot{x} = f(x,t)$, as follows. We assume that there exists $R \in \mathcal{K}_\infty$ such that $|f(x,t)| \leq R(|x|)$ everywhere, and that time-varying analogs of Assumptions 1–2 hold. These analogs of Assumptions 1–2 are obtained by replacing their arguments x by (x,t), and $\nabla V_i(x)f(x)$ by $\dot{V}_i(x,t) = (\partial V_i/\partial t)(x,t) + (\partial V_i/\partial x)(x,t)f(x,t)$, assuming $\mathcal{N}_i(0,t) \equiv V_i(0,t) \equiv 0$.

More generally, assume that these time-varying versions of Assumptions 1–2 hold *except* that the lower bound on $\Sigma_i\mathcal{N}_i$ is replaced by a relation of the form

$$\mathcal{S}_j(x,t) := \sum_{l=1}^{j} \mathcal{N}_l(x,t) \geq \underline{p}(t)\rho(V_1(x,t))V_1(x,t). \quad (6)$$

Here ρ is again positive, and $\underline{p}(t)$ is assumed to be non-negative and admit constants $\bar{B}, T, p_m > 0$ such that $\int_t^{T+t} \underline{p}(s)ds > p_m$ and $\underline{p}(t) \leq \bar{B}$ for all t. This allows $\underline{p}(t) = 0$ for some t's (e.g., $\underline{p}(t) = \cos^2(t)$ and $T = \pi$), so $\mathcal{S}_j(x,t)$ is not necessarily positive definite. Nevertheless, we can prove an analog of Theorem 1 in this situation, as follows.

Set $V_{j+1}(x,t) = [\int_{t-T}^{t} \int_s^t \underline{p}(l)\,dl\,ds]V_1(x,t)$. Since $\dot{V}_1(x,t) \leq 0$ and $\underline{p}$ and V_1 are nonnegative

$$\dot{V}_{j+1} = - V_1(x,t) \int_{t-T}^{t} \underline{p}(l)dl + T\underline{p}(t)V_1(x,t)$$

$$+ \dot{V}_1 \int_{t-T}^{t} \int_s^t \underline{p}(l)dl\,ds$$

$$\leq - p_m V_1(x,t) + T\underline{p}(t)V_1(x,t)$$

$$\leq - p_m V_1(x,t) + T \frac{\mathcal{S}_j(x,t)}{\rho(V_1(x,t))} \quad (7)$$

along the trajectories of $\dot{x} = f(x,t)$, where the first inequality is by our choice of p_m. Therefore

$$\dot{V}_i \leq - \mathcal{N}_i(x,t) + \phi_i(V_1(x,t)) \sum_{l=1}^{i-1} \mathcal{N}_l(x,t)^{a_i} V_1(x,t)^{1-a_i}$$

for $2 \leq i \leq j$, and

$$\dot{V}_{j+1} \leq - \mathcal{N}_{j+1}(x,t) + \phi_{j+1}(V_1(x,t)) \sum_{l=1}^{j} \mathcal{N}_l(x,t) \quad (8)$$

with $\mathcal{N}_{j+1}(x,t) = p_m V_1(x,t)$ and $\phi_{j+1}(V_1(x,t)) = (T/\rho(V_1(x,t)))$. Also, $\sum_{l=1}^{j+1} \mathcal{N}_l(x,t) \geq p_m V_1(x,t)$, $\dot{V}_1 \leq -\mathcal{N}_1(x,t)$, and $|V_{j+1}(x,t)| \leq T^2\bar{B}V_1(x,t)$. Therefore, the properties required to apply the time-varying version of Theorem 1 are satisfied by $V_1, V_2, \ldots, V_{j+1}$.

V. APPLICATION TO A REAL BIOTECHNOLOGICAL SYSTEM

Several adaptive control problems for bioreactors have been solved [1], [6], [9]. The proofs in these works construct *non-strict* Lyapunov functions. Here we use Theorem 1 to construct a *strict* Lyapunov-like function for the system and corresponding adaptive controller in [9].

We then use our strict Lyapunov-like function to quantify the robustness of the controller to uncertainty.

Consider an experimental anaerobic digester used to treat waste water [3], [9], [19]. This process degrades a polluting organic substrate s with the anaerobic bacteria x and produces a methane flow rate y_1. The methane and substrate can generally be measured, so the system is

$$\dot{s} = u(s_{in} - s) - kr(s,x), \quad \dot{x} = r(s,x) - \alpha ux, \quad y = (\lambda r(s,x), s) \quad (9)$$

where the biomass growth rate r is any nonnegative C^1 function that admits positive functions $\underline{\Delta}$ and $\bar{\Delta}$ such that

$$s\bar{\Delta}(s,x) \geq r(s,x) \geq xs\underline{\Delta}(s,x) \quad (10)$$

u is the nonnegative input (i.e. dilution rate); α is a known positive real number representing the fraction of the biomass in the liquid phase; and λ, k, and s_{in} are positive constants representing methane production and substrate consumption yields and the influent substrate concentration, respectively. Hence, $y_1 = \lambda r(s,x)$. We wish to regulate the variable s to a prescribed positive real number $s_* \in (0, s_{in})$. We assume that there are known constants $\gamma_M > \gamma_m > 0$ such that

$$\gamma_* := k/[\lambda(s_{in} - s_*)] \in (\gamma_m, \gamma_M) \text{ and } k/[\lambda s_{in}] < \gamma_m. \quad (11)$$

We introduce the notation $v_* = s_{in} - s_*$ and $x_* = (v_*/k\alpha)$.

The work [9] leads to a non-strict Lyapunov-like function and an adaptive controller for an error dynamics associated with (9). We next review these earlier results. In Section V-A, we use them to build a strict Lyapunov-like function for the error dynamics, and in Section V-B we use our Lyapunov construction in a robustness analysis. We introduce the dynamics $\dot{\gamma} = y_1(\gamma - \gamma_m)(\gamma_M - \gamma)\nu$ evolving on (γ_m, γ_M), where ν is a function to be selected that is independent of x. With $u = \gamma y_1$, the system (9) with its dynamic extension becomes

$$\dot{s} = y_1[\gamma(s_{in} - s) - k/\lambda], \quad \dot{x} = y_1 \alpha[1/(\alpha\lambda) - \gamma x],$$
$$\dot{\gamma} = y_1(\gamma - \gamma_m)(\gamma_M - \gamma)\nu \quad (12)$$

by the definition of y_1, with the same output y as before. The dynamics (12) evolves on the invariant domain $E = (0, +\infty) \times (0, +\infty) \times (\gamma_m, \gamma_M)$. The following is easily checked:

Lemma 2: For each initial value $(s(t_0), x(t_0), \gamma(t_0)) \in E$, there is a compact set $K_o \subseteq (0, \infty)^2$ so that the corresponding solution of (12) is such that $(s(t), x(t)) \in K_o$ for all $t \geq t_0$.

It follows from Lemma 2 and (10) that we can re-parameterize (12) in terms of $\tau = \int_{t_o}^t y_1(l)dl$. Doing so and setting $\tilde{x} = x - x_*$, $\tilde{s} = s - s_*$, and $\tilde{\gamma} = \gamma - \gamma_*$ yields the error dynamics

$$\dot{\tilde{s}} = -\gamma\tilde{s} + \tilde{\gamma}v_*, \quad \dot{\tilde{x}} = \alpha[-\gamma\tilde{x} - \tilde{\gamma}x_*], \quad \dot{\tilde{\gamma}} = (\gamma - \gamma_m)(\gamma_M - \gamma)\nu \quad (13)$$

for $t \mapsto (\tilde{s}, \tilde{x}, \tilde{\gamma})(\tau^{-1}(t))$. The state space of (13) is $D = (-s_*, +\infty) \times (-x_*, +\infty) \times (\gamma_m - \gamma_*, \gamma_M - \gamma_*)$. The system (13) has an uncoupled triangular structure; i.e., its $(\tilde{s}, \tilde{\gamma})$-subsystem does not depend on $\tilde{x}$, and the $\tilde{x}$-subsystem is globally input-to-state stable with respect to $\tilde{\gamma}$ with the ISS Lyapunov function $\tilde{x}^2$ [18]. Therefore (13) is globally asymptotically stable to 0 if and only if the system

$$\dot{\tilde{s}} = -\gamma\tilde{s} + \tilde{\gamma}v_*, \quad \dot{\tilde{\gamma}} = (\gamma - \gamma_m)(\gamma_M - \gamma)\nu \quad (14)$$

with state space $F = (-s_*, +\infty) \times (\gamma_m - \gamma_*, \gamma_M - \gamma_*)$ is globally asymptotically stable to 0. Therefore, we may limit our analysis to (14) in the sequel.

For a given a tuning parameter $K > 0$, [9] uses the Lyapunov-like function

$$V_1(\tilde{s}, \tilde{\gamma}) = \frac{1}{2\gamma_m}\tilde{s}^2 + \frac{v_*}{K\gamma_m}\int_0^{\tilde{\gamma}} \frac{l}{(l + \gamma_* - \gamma_m)(\gamma_M - \gamma_* - l)}dl \quad (15)$$

for (14), which is positive definite on $D := (-s_*, +\infty) \times (\gamma_m - \gamma_*, \gamma_M - \gamma_*)$. Also, $\dot{V}_1 = (1/\gamma_m)[-\gamma\tilde{s}^2 + \tilde{s}\tilde{\gamma}v_*] + (v_*/K\gamma_m)\tilde{\gamma}\nu$ along the trajectories of (14). Choosing $\nu(\tilde{s}) = -K\tilde{s}$ gives $\dot{V}_1 = -(\gamma/\gamma_m)\tilde{s}^2 \leq -\mathcal{N}_1(\tilde{s})$, where $\mathcal{N}_1(\tilde{s}) = \tilde{s}^2$ (because $\gamma(t) \in (\gamma_m, \gamma_M)$ for all t). Using the LaSalle Invariance Principle [7], [9] shows that (14) is globally asymptotically stable to 0 when $\nu(\tilde{s}) = -K\tilde{s}$.

A. Construction of a Strict Lyapunov-Like Function for System (14)

Set $V_2(\tilde{s}, \tilde{\gamma}) = -\tilde{s}\tilde{\gamma}$. Along the trajectories of (14), in closed loop with $\nu(\tilde{s}) = -K\tilde{s}$, simple calculations yield $\dot{V}_2 = \gamma\tilde{s}\tilde{\gamma} - \tilde{\gamma}^2 v_* + (\gamma - \gamma_m)(\gamma_M - \gamma)K\tilde{s}^2$. From the relation $\gamma\tilde{s}\tilde{\gamma} \leq v_*\tilde{\gamma}^2/2 + \gamma^2\tilde{s}^2/(2v_*)$ and the fact that the maximum value of $(\gamma - \gamma_m)(\gamma_M - \gamma)$ over $\gamma \in [\gamma_m, \gamma_M]$ is $(\gamma_M - \gamma_m)^2/4$, we get $\dot{V}_2 \leq -\mathcal{N}_2(\tilde{\gamma}) + [(\gamma_M^2/2v_\star) + (K(\gamma_M - \gamma_m)^2/4)]\mathcal{N}_1(\tilde{s})$, where $\mathcal{N}_2(\tilde{\gamma}) = (v_*/2)\tilde{\gamma}^2$. Moreover, since V_1 is bounded from above by a positive definite quadratic function near 0, we can find a positive function ρ so that $\rho(V_1)V_1 \leq \min\{1, (v_*/2)\}(\tilde{s}^2 + \tilde{\gamma}^2)$ on D. (In fact, we can choose ρ so that outside a neighborhood of zero, $\rho(v) = c/(1 + v)$ for a suitable constant c.) Thus, Assumptions 1 and 2(i) are satisfied. Also

$$V_1(\tilde{s}, \tilde{\gamma}) \geq \frac{1}{2\gamma_m}\tilde{s}^2 + \frac{2v_*}{K\gamma_m(\gamma_M - \gamma_m)^2}\tilde{\gamma}^2 \quad (16)$$

holds on D. This gives $V_2(\tilde{s}, \tilde{\gamma}) \leq \bar{c}V_1(\tilde{s}, \tilde{\gamma})$ for some constant $\bar{c} > 0$. (Our choice of V_2 was motivated by our desire to have the preceding estimate.) Hence, Assumption (ii) is satisfied as well, so Theorem 1 applies with $p_2(V_1) \equiv \bar{c}$. We now explicitly build the strict Lyapunov-like function from Theorem 1. Since $j = 2$ and $a_2 = 1$, $U_2(\tilde{s}, \tilde{\gamma}) = \Upsilon_1 V_1(\tilde{s}, \tilde{\gamma}) + V_2(\tilde{s}, \tilde{\gamma})$ where $\Upsilon_1 = 1 + \bar{c}$. Since $\Omega_1(s) = [\gamma_M^2/v_\star + K(\gamma_M - \gamma_m)^2/2]s$ and $\Omega_2(s) \equiv s$

$$S(\tilde{s}, \tilde{\gamma}) = U_2(\tilde{s}, \tilde{\gamma}) + 2\left[\frac{\gamma_M^2}{v_\star} + \frac{K(\gamma_M - \gamma_m)^2}{2}\right]V_1(\tilde{s}, \tilde{\gamma})$$
$$= V_2(\tilde{s}, \tilde{\gamma}) + \left[\Upsilon_1 + \frac{2\gamma_M^2}{v_*} + K(\gamma_M - \gamma_m)^2\right]V_1(\tilde{s}, \tilde{\gamma}) \quad (17)$$

is a strict Lyapunov-like function for (14) in closed loop with $\nu(\tilde{s}) = -K\tilde{s}$. In fact

$$\dot{S} \leq -W(\tilde{s}, \tilde{\gamma}), \text{ where}$$
$$W(\tilde{s}, \tilde{\gamma}) = \mathcal{N}_2(\tilde{\gamma}) + \Upsilon_1\mathcal{N}_1(\tilde{s}) = \frac{v_*}{2}\tilde{\gamma}^2 + \Upsilon_1\tilde{s}^2 \quad (18)$$

along the closed loop trajectories.

Remark 3: Since V_1 is not globally proper on $\mathbb{R}^2$, we cannot construct the required explicit strong Lyapunov function for (14) using the results of [15]. Notice that (17) is a simple linear combination of V_1 and V_2. By contrast, the strong Lyapunov functions provided by [15, Theorem 3] for the $j = 2$ time invariant case have the form $S(x) = Q_1(V_1(x))V_1(x) + Q_2(V_1(x))V_2(x)$ where Q_1 is nonnegative, and where the positive definite function Q_2 needs to globally satisfy $Q_2(V_1) \leq \phi^{-1}(\omega(x)/\{2\rho(|x|)\})$ where $\nabla V_2(x)f(x) \leq -\mathcal{N}_2(x) + \phi(\mathcal{N}_1(x))\rho(|x|)$ for some $\phi \in \mathcal{K}_\infty$ and some positive nondecreasing function ρ and the positive definite function ω needs to satisfy $\mathcal{N}_1(x) + \mathcal{N}_2(x) \geq \omega(x)$ everywhere. In
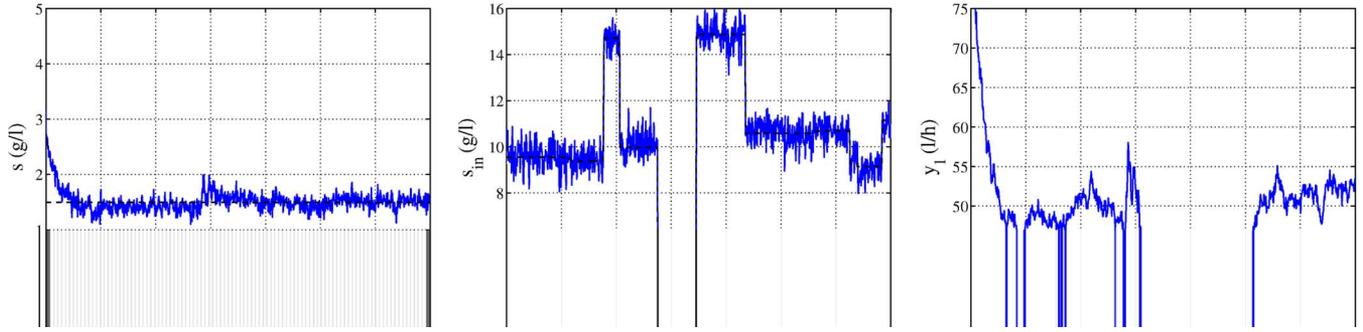
Fig. 1. Controlled substrate s (left), influent substrate concentration s_{in} (middle) and methane flow rate y_1 (right).

particular, we cannot take Q_2 to be constant to get a linear combination of the V_i's, so the construction of [15] is more complicated than the one we provide here. Similar remarks apply to the other constructions in [15].

B. Robustness Result

It is important to assess the robustness of a control design to bounded uncertainties before implementing the controller. Indeed, biological systems are known to have highly uncertain dynamics. This is especially the case for waste water treatment processes made up of a complex mixture of bacteria. In [9], good performance of the controller was observed but could not be explained by a theoretical approach. Here we prove that an appropriate adaptive controller gives ISS of the relevant error dynamics to disturbances; see Section II for the definitions and motivations for ISS.

We focus on the system (9) for cases where s_{in} is replaced by $H_{in}(t) = s_{in} + \delta_1(t)$ and, for an arbitrary positive constant $K > 0$, the adaptive control is given by

$$ u = (\gamma + \delta_2(t)) y_1, \quad \dot{\gamma} = -K y_1 (\gamma - \gamma_m)(\gamma_M - \gamma)(\tilde{s} + \delta_3(t)) \quad (19) $$

where the disturbances $\delta_1(t)$ and $\delta_3(t)$ are bounded in absolute value by a constant $\bar{\delta}_1$ and the disturbance $\delta_2(t)$ is bounded by a constant $\bar{\delta}_2$; we specify the $\bar{\delta}_i$'s below.

We maintain the assumptions and notation from the preceding subsections. We also assume

$$ k/\lambda < (\gamma_m - \bar{\delta}_2)(s_{in} - \bar{\delta}_1), \quad \bar{\delta}_1 < s_{in}, \quad \text{and} \quad \bar{\delta}_2 < \gamma_m. \quad (20) $$

In particular, we keep the definitions of x_* and γ_* from (11) and the first sentence after (11) unchanged; we do not replace s_{in} by $H_{in}(t)$ in the expressions for x_* and γ_*. Our analysis will use the function S from (17) extensively. To specify our bounds $\bar{\delta}_i$, we use the constants

$$ \Xi = \left[\Upsilon_1 + 2\gamma_M^2/v_* + K(\gamma_M - \gamma_m)^2 \right] / \gamma_m \quad \text{and} $$
$$ \Upsilon_2 = \min\{\gamma_* - \gamma_m, \gamma_M - \gamma_*\} \quad (21) $$

where Υ_1 is from Section V-A. See Section V-C for an example with specific bounds $\bar{\delta}_i$.

Replacing s_{in} with H_{in} in (9), and using u from (19) and the expression for y_1, we get

$$ \begin{cases} \dot{s} = y_1 \left[(\gamma + \delta_2(t))(s_{in} + \delta_1(t) - s) - \frac{k}{\lambda} \right], \\ \dot{x} = y_1 \left[\frac{1}{\lambda} - \alpha(\gamma + \delta_2(t)) x \right], \\ \dot{\gamma} = -K y_1 (\gamma - \gamma_m)(\gamma_M - \gamma)(\tilde{s} + \delta_3(t)). \end{cases} \quad (22) $$

For simplicity, we restrict to the attractive and invariant domain where $0 < s < 2s_{in}$ which, in practice, is the domain of interest (but a result

on the entire set where $s > 0$ can be proved). Using (20) and arguing as we did to obtain Lemma 2, the time re-scaling $\tau = \int_{t_o}^{t} y_1(l)dl$ yields

$$ \begin{cases} \dot{\tilde{s}} = -\gamma\tilde{s} + \tilde{\gamma}v_* + \delta_2(t)(s_{in} - s) + (\gamma + \delta_2(t))\delta_1(t), \\ \dot{\tilde{x}} = -\alpha(\gamma + \delta_2(t))\tilde{x} - \alpha(\delta_2(t) + \tilde{\gamma})x_*, \\ \dot{\tilde{\gamma}} = -K(\gamma - \gamma_m)(\gamma_M - \gamma)\tilde{s} - K(\gamma - \gamma_m)(\gamma_M - \gamma)\delta_3(t) \end{cases} \quad (23) $$

evolving on $\hat{D} = (-s_*, 2s_{in} - s_*) \times (-x_*, +\infty) \times (\gamma_m - \gamma_*, \gamma_M - \gamma_*)$. The following ISS result implies that the trajectories of (22) satisfy $(x(t), s(t)) \to (x_*, s_*)$ globally uniformly, with an additional term that is small when the disturbances δ_i are small; see Section II for the precise ISS definition. For the proof of this result, see Appendix B below.

Theorem 2: Assume that the system (23) satisfies (11) and (20). Define the constants

$$ \bar{\delta}_1 = \frac{99}{100}\Upsilon_2 \min\left\{ \frac{v_*}{4\left(\Xi v_* + \frac{5}{4}\gamma_M\right)}, \right. $$
$$ \left. \frac{4\sqrt{v_*}\sqrt{\Upsilon_1}}{K(\gamma_M - \gamma_m)^2 + 5\Xi\gamma_M} \right\} \quad \text{and} $$
$$ \bar{\delta}_2 = \frac{\gamma_M \bar{\delta}_1}{4(2s_{in} + \bar{\delta}_1)}. \quad (24) $$

Assume that δ_1 and δ_3 are bounded in absolute value by $\bar{\delta}_1$ and that the disturbance δ_2 is bounded in absolute value by $\bar{\delta}_2$. Then the closed loop error dynamics (23) is ISS on $\hat{D}$.[1]

C. Numerical Example

We simulated an anaerobic digestion process used to process waste water and produce biogas. We calibrated the model using real experimental data [2] to get realistic parameter values. We took the influent concentration s_{in} to be piecewise constant with respect to time, following the real profile experimented in [3] (see Fig. 1). Finally, we simulated white noises for the δ_i, introducing a 1% noise in the controller u and noises of standard deviation 0.5 g/l and 0.1 g/l to perturb s_{in} and s; see (19). Our set point s^* was 1.5 g/l. Following [9], we applied our controller successively on the intervals on which s_{in} is constant.

Our controlled simulation illustrates the robustness of the controller. The controller was successfully implemented on a real process in [9], where it was asserted that the feedback lent itself to realistic settings in which there are noisy signals. However, [9] did not provide any theoretical argument to prove this assertion. Our theory and simulation confirm and validate the robustness of the controller under realistic values of noise.

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[1]The constant 99/100 can be replaced by any constant in $(0,1)$ that is close enough to 1; we use the constant to get (A.7).

VI. Conclusion

We provided new strict Lyapunov function constructions for nonlinear systems that satisfy Matrosov's conditions. The advantages of our constructions lie in their simplicity and their applicability to the various examples whose stability can be established by the generalized Matrosov theorem. We demonstrated the efficacy of our methods through a class of biotechnological models with disturbances, which are of compelling engineering interest.

Appendix A
Two Lemmas

Lemma A.1: Let $k_1, \ldots, k_j \in \mathcal{K}_\infty \cap C^1$; Φ be a continuous nonnegative function defined on $[0, \infty)$; and $a_1, a_2, \ldots, a_j \in (0, 1]$ be constants. Then one can construct C^1 functions $\mu_1, \ldots, \mu_j :$ $[0, \infty) \to [1, \infty)$ such that for all $s \geq 0$, $\mu_j(s) = 1$ and $\mu_i(s) \geq 2\Phi(s) \sum_{l=1+i}^{j} \mu_l^{1/a_l}(2k_l(s))$ and $\mu_i'(s) \geq 0$ for $i = 1$ to $j - 1$.

Proof: Let $\overline{\Phi}$ be a positive increasing function such that $\overline{\Phi} \geq \Phi$ everywhere, and set $k = 2(k_1 + k_2 + \ldots k_j)$. To prove the lemma, it suffices to find μ_i's such that $\mu_i(s) \geq 2\overline{\Phi}(s) \sum_{l=1+i}^{j} \mu_l^{1/a_l}(k(s))$ and $\mu_i'(s) \geq 0$ for $i = 1$ to $j - 1$ and all $s \geq 0$, where $\mu_j(s) = 1$. Let us construct these functions by induction. Assume that m functions $\mu_j, \mu_{j-1}, \ldots, \mu_{j-m+1}$ are available. Then

$$\mu_{j-m}(s) := 1 + s + 2\overline{\Phi}(s) \sum_{l=1+j-m}^{j} \mu_l^{\frac{1}{a_l}}(k(s))$$

$$\geq 2\overline{\Phi}(s) \sum_{l=1+j-m}^{j} \mu_l^{\frac{1}{a_l}}(k(s)) \tag{A1}$$

and μ_{j-m} is nondecreasing, because $\overline{\Phi}$, k, and μ_l for $l = j$ to $j-m+1$ are nondecreasing. ∎

Lemma A.2: Let $X(t)$ be a solution of a system $\dot{X} = F(X)$ defined over $[0, +\infty)$. Let M be a positive definite function and c_a, c_b, and c_c be positive constants such that for each $t \geq 0$, the time derivative of M along the solution $X(t)$ satisfies either $\dot{M} \leq -c_a$ or $\dot{M} \leq -c_b M(X(t)) + c_c$. Then

$$M(X(t)) \leq e^{-\min\left\{c_a, \frac{c_b}{2}\right\}t} \left(e^{M(X(0))} - 1\right) + \frac{e^{\frac{2c_c}{c_b}} c_c}{\min\left\{c_a, \frac{c_b}{2}\right\}} \tag{A2}$$

is satisfied for all $t \geq 0$.

Proof: The time derivative of the function $\Omega(X) = e^{M(X)} - 1$ along the solution $X(t)$ is $\dot{\Omega} = e^{M(X)} \dot{M}$. Therefore, either $\dot{\Omega} \leq -e^{M(X)} c_a \leq -\Omega(X) c_a$ or $\dot{\Omega} \leq -e^{M(X)} c_b M(X) + e^{M(X)} c_c$. In the latter case, $M(X) \geq (2c_c/c_b) \Rightarrow \dot{\Omega} \leq -(1/2)e^{M(X)} c_b M(X)$ while $M(X) \leq (2c_c/c_b) \Rightarrow \dot{\Omega} \leq -e^{M(X)} c_b M(X) + e^{2c_c/c_b} c_c$. Hence, either $\dot{\Omega} \leq -e^{M(X)} c_a \leq -\Omega(X) c_a$ or

$$\dot{\Omega} \leq -\frac{c_b}{2} e^{M(X)} M(X) + e^{\frac{2c_c}{c_b}} c_c. \tag{A3}$$

Notice that $e^A - 1 = \int_0^A e^m dm \leq A e^A$ for all $A \geq 0$. We deduce that if (A3) holds, then $\dot{\Omega} \leq -(c_b/2)(e^{M(X)} - 1) + e^{2c_c/c_b} c_c = -(c_b/2)\Omega(X) + e^{2c_c/c_b} c_c$. Therefore, in the two cases, we have $\dot{\Omega} \leq -\min\{c_a, (c_b/2)\}\Omega(X) + e^{2c_c/c_b} c_c$. By integrating over any interval $[0, t]$ and using the definition of Ω, we deduce that for all $t \geq 0$

$$M(X(t)) \leq \ln\left(+ e^{-\min\left\{c_a, \frac{c_b}{2}\right\}t} \left(e^{M(X(0))} - 1\right) \right.$$
$$\left. + \frac{e^{\frac{2c_c}{c_b}} c_c}{\min\left\{c_a, \frac{c_b}{2}\right\}} \right). \tag{A4}$$

Noting that $\ln(1 + A) \leq A$ is valid for all $A \geq 0$, we deduce that (A2) is satisfied. ∎

Appendix B
Proof of Theorem 2

Since the $\tilde{x}$ sub-dynamics in (23) is ISS when $(\tilde{\gamma}, \delta_2)$ is viewed as its disturbance (because $(d/dt)\tilde{x}^2 \leq -\underline{b}\tilde{x}^2 + \overline{b}(|\delta_2| + |\tilde{\gamma}|)$ along its trajectories for appropriate constants $\underline{b}, \overline{b} > 0$ combined with standard ISS arguments [18]), it suffices to check that the $(\tilde{s}, \tilde{\gamma})$ sub-dynamics is ISS with respect to δ [18]. (In other words, the serial connection of ISS systems is ISS.) Hence, we focus on the $(\tilde{s}, \tilde{\gamma})$ sub-dynamics in the rest of the proof.

It follows from elementary calculations and (18) that along the trajectories of (23)

$$\dot{S} \leq -W(\tilde{s}, \tilde{\gamma}) + T_1(\tilde{s}, \tilde{\gamma}) + T_2(\tilde{s}, \tilde{\gamma}) \tag{A5}$$

with $T_1(\tilde{s}, \tilde{\gamma}) = |(\partial S/\partial \tilde{s})(\tilde{s}, \tilde{\gamma})|[\overline{\delta}_2|s_{in} - s| + (\gamma + \overline{\delta}_2)\overline{\delta}_1]$ and $T_2(\tilde{s}, \tilde{\gamma}) = |(\partial S/\partial \tilde{\gamma})(\tilde{s}, \tilde{\gamma})|K(\gamma - \gamma_m)(\gamma_M - \gamma)\overline{\delta}_1$. Recall the constants Ξ and Υ_2 defined in (21). One can use the formulas for $\partial S/\partial \tilde{s}$ and $\partial S/\partial \tilde{\gamma}$ to get

$$T_1(\tilde{s}, \tilde{\gamma}) \leq |\tilde{\gamma} - \Xi\tilde{s}| \left[\overline{\delta}_2|s_{in} - s| + \gamma\overline{\delta}_1 + \overline{\delta}_1\overline{\delta}_2\right]$$
$$\leq |\tilde{\gamma} - \Xi\tilde{s}| \left[\overline{\delta}_2(2s_{in} + \overline{\delta}_1) + \gamma_M\overline{\delta}_1\right],$$
$$T_2(\tilde{s}, \tilde{\gamma}) \leq \left|\tilde{s} - \Xi\frac{v_*}{K} \frac{\tilde{\gamma}}{(\tilde{\gamma} + \gamma_* - \gamma_m)(\gamma_M - \gamma_* - \tilde{\gamma})}\right|$$
$$\times K(\gamma - \gamma_m)(\gamma_M - \gamma)\overline{\delta}_1$$
$$\leq \frac{K}{4}(\gamma_M - \gamma_m)^2 \overline{\delta}_1|\tilde{s}| + \Xi v_* \overline{\delta}_1|\tilde{\gamma}|. \tag{A6}$$

Therefore, $T_1(\tilde{s}, \tilde{\gamma}) + T_2(\tilde{s}, \tilde{\gamma}) \leq E_1|\tilde{\gamma}| + E_2|\tilde{s}|$ with $E_1 = \Xi v_* \overline{\delta}_1 + (2s_{in} + \overline{\delta}_1)\overline{\delta}_2 + \gamma_M\overline{\delta}_1$ and $E_2 = (K/4)(\gamma_M - \gamma_m)^2\overline{\delta}_1 + \Xi[\overline{\delta}_2(2s_{in} + \overline{\delta}_1) + \gamma_M\overline{\delta}_1]$. From (24), we deduce that $E_1 \leq (99/100)\Upsilon_2(v_*/4)$ and $E_2 \leq ((K/4)(\gamma_M - \gamma_m)^2 + \Xi(5\gamma_M/4))\overline{\delta}_1 \leq (99/100)\Upsilon_2\sqrt{v_*}\sqrt{\Upsilon_1}$. Therefore, (A5) and (A6) give

$$\dot{S} \leq -\frac{v_*}{2}\tilde{\gamma}^2 - \Upsilon_1\tilde{s}^2 + \frac{99}{100}\Upsilon_2\frac{v_*}{4}|\tilde{\gamma}| + \frac{99}{100}\Upsilon_2\sqrt{v_*}\sqrt{\Upsilon_1}|\tilde{s}| \tag{A7}$$

by our choice (18) of W. We introduce the constants

$$\Upsilon_3 = \frac{397v_*}{160000}\Upsilon_2^2, \quad \Upsilon_4 = \frac{1}{2} + \Xi\frac{v_*}{K}\frac{2(10^4)}{(\gamma_* - \gamma_m)(\gamma_M - \gamma_*)},$$
$$\Upsilon_5 = \frac{\min\left\{\frac{v_*}{2}, \Upsilon_1\right\}}{2\max\left\{\frac{1+\Xi}{2}, \Upsilon_4\right\}},$$
$$\text{and } \Upsilon_6 = \frac{1}{v_*}\left(\Xi v_* + \frac{5\gamma_M}{4}\right)^2$$
$$+ \frac{1}{\Upsilon_1}\left[\frac{K}{4}(\gamma_M - \gamma_m)^2 + \Xi\frac{5\gamma_M}{4}\right]^2. \tag{A8}$$

We consider two cases. Case 1: If $\tilde{\gamma} \in (\gamma_m - \gamma_*, (199/200)(\gamma_m - \gamma_*)]$ or $\tilde{\gamma} \in [(199/200)(\gamma_M - \gamma_*), \gamma_M - \gamma_*)$, then $|\tilde{\gamma}| \geq (199/200)\Upsilon_2$. From $pq \leq p^2 + q^2/4$ with $p = \sqrt{\Upsilon_1}|\tilde{s}|$ and $q = .99\Upsilon_2\sqrt{v_*}$ and (A7)

$$\dot{S} \leq -\frac{v_*}{4}\left(\frac{199}{200}\right)^2\Upsilon_2^2 - \Upsilon_1\tilde{s}^2 + \frac{99}{100}\Upsilon_2\sqrt{v_*}\sqrt{\Upsilon_1}|\tilde{s}|$$
$$\leq -\frac{v_*}{4}\left[\left(\frac{199}{200}\right)^2 - \left(\frac{99}{100}\right)^2\right]\Upsilon_2^2 \leq -\Upsilon_3. \tag{A9}$$

Case 2: If $\tilde{\gamma} \in [(199/200)(\gamma_m - \gamma_*), (199/200)(\gamma_M - \gamma_*)]$, then[2]

$$\int_0^{\tilde{\gamma}} \frac{l}{(l + \gamma_* - \gamma_m)(\gamma_M - \gamma_* - l)} dl \leq \frac{2(10^4)\tilde{\gamma}^2}{(\gamma_* - \gamma_m)(\gamma_M - \gamma_*)}.$$

By our formula (17) for S, we get $S(\tilde{s}, \tilde{\gamma}) \leq (1 + \Xi/2)\tilde{s}^2 + \Upsilon_4 \tilde{\gamma}^2 \leq \max\{(1 + \Xi)/2, \Upsilon_4\}[\tilde{s}^2 + \tilde{\gamma}^2]$. We also have $W(\tilde{s}, \tilde{\gamma}) \geq \min\{(v_*/2), \Upsilon_1\}(\tilde{s}^2 + \tilde{\gamma}^2)$. Therefore, $(1/2)W(\tilde{s}, \tilde{\gamma}) \geq \Upsilon_5 S(\tilde{s}, \tilde{\gamma})$. It follows from (A5) that

$$\begin{aligned}
\dot{S} &\leq -\Upsilon_5 S(\tilde{s}, \tilde{\gamma}) - \frac{v_*}{4}\tilde{\gamma}^2 - \frac{1}{2}\Upsilon_1 \tilde{s}^2 + \left\{\frac{E_1}{\sqrt{v_\star}}\right\}\{|\tilde{\gamma}|\sqrt{v_\star}\} \\
&\quad + \left\{\frac{E_2}{\sqrt{\Upsilon_1}}\right\}\{|\tilde{s}|\sqrt{\Upsilon_1}\} \\
&\leq -\Upsilon_5 S(\tilde{s}, \tilde{\gamma}) + \frac{E_1^2}{v_*} + \frac{E_2^2}{\Upsilon_1} \\
&\leq -\Upsilon_5 S(\tilde{s}, \tilde{\gamma}) + \Upsilon_6 \overline{\delta}_1^2. \qquad (A10)
\end{aligned}$$

(We used the formulas for E_1 and E_2 and (24) and applied $pq \leq p^2 + (1/4)q^2$ to the terms in braces.) Therefore, in the first case we get (A9), while (A10) holds in the second case. Hence, along the closed-loop trajectories, we have either $\dot{S} \leq -\Upsilon_3$ or $\dot{S} \leq -\Upsilon_5 S(\tilde{s}, \tilde{\gamma}) + \Upsilon_6 \overline{\delta}_1^2$. Using (16) and (17) and the fact that $V_2 \geq (1 - \Upsilon_1)V_1$ everywhere, one easily checks that S admits a constant $Q_o > 0$ such that $Q_o(\tilde{s}^2 + \tilde{\gamma}^2) \leq S(\tilde{s}, \tilde{\gamma})$ on D. We readily conclude by applying Lemma A.2, combined with the relation $\sqrt{p + q} \leq \sqrt{2p} + \sqrt{2q}$ (by taking square roots of both sides of (A2) with $M = S$).

REFERENCES

[1] G. Bastin and D. Dochain, *On-Line Estimation and Adaptive Control of Bioreactors*.   Amsterdam, The Netherlands: Elsevier, 1990.

[2] O. Bernard, B. Chachuat, A. Hélias, and J. Rodriguez, "Can we assess the model complexity for a bioprocess? Theory and example of the anaerobic digestion process," *Water Sci. Technol.*, vol. 53, no. 1, pp. 85–92, 2006.

[3] O. Bernard, Z. Hadj-Sadok, D. Dochain, A. Genovesi, and J.-P. Steyer, "Dynamical model development and parameter identification for an anaerobic wastewater treatment process," *Biotechnol. Bioeng.*, vol. 75, no. 4, pp. 424–438, 2001.

[4] L. Faubourg and J.-B. Pomet, "Control Lyapunov functions for homogeneous "Jurdjevic-Quinn" systems," *ESAIM: Control, Optim. Calculus Variations*, vol. 5, pp. 293–311, 2000.

[5] S. B. Hsu, "Limiting behavior for competing species," *SIAM J. Appl. Math.*, vol. 34, no. 4, pp. 760–763, 1978.

[6] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*.   New York: Wiley, 1995.

[7] J. P. LaSalle, "Stability theory for ordinary differential equations," *J. Diff. Eq.*, vol. 4, pp. 57–65, 1968.

[8] A. Loria, E. Panteley, D. Popović, and A. Teel, "A nested Matrosov theorem and persistency of excitation for uniform convergence in stable non-autonomous systems," *IEEE Trans. Autom. Control*, vol. 50, no. 2, pp. 183–198, Feb. 2005.

[9] L. Mailleret, O. Bernard, and J.-P. Steyer, "Nonlinear adaptive control for bioreactors with unknown kinetics," *Automatica*, vol. 40, no. 8, pp. 1379–1385, Aug. 2004.

[10] V. M. Matrosov, "On the theory of stability of motion," *J. Appl. Math. Mech.*, vol. 26, no. 6, pp. 1560–1572, 1962.

[11] F. Mazenc and M. Malisoff, "Further constructions of control-Lyapunov functions and stabilizing feedbacks for systems satisfying the Jurdjevic-Quinn conditions," *IEEE Trans. Automat. Control*, vol. 51, no. 2, pp. 360–365, Feb. 2006.

[12] F. Mazenc and M. Malisoff, "Lyapunov function constructions for slowly time-varying systems," *Math. Control Signals Syst.*, vol. 19, no. 1, pp. 1–21, 2007.

[13] F. Mazenc, M. Malisoff, and P. De Leenheer, "On the stability of periodic solutions in the perturbed chemostat," *Math. Biosci. Eng.*, vol. 4, no. 2, pp. 319–338, 2007.

[14] F. Mazenc and D. Nesic, "Strong Lyapunov functions for systems satisfying the conditions of LaSalle," *IEEE Trans. Automat. Control*, vol. 49, no. 6, pp. 1026–1030, Jun. 2004.

[15] F. Mazenc and D. Nesic, "Lyapunov functions for time-varying systems satisfying generalized conditions of Matrosov theorem," *Math. Control Signals Syst.*, vol. 19, no. 2, pp. 151–182, 2007.

[16] H. L. Smith and P. Waltman, *The Theory of the Chemostat*.   Cambridge, U.K.: Cambridge University Press, 1995.

[17] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Automat. Control*, vol. 34, no. 4, pp. 435–443, Apr. 1989.

[18] E. D. Sontag, "Input-to-state stability: Basic concepts and results," in *Nonlinear and Optimal Control Theory*, P. Nistri and G. Stefani, Eds. Berlin, Germany: Springer, 2008, pp. 163–220.

[19] J.-P. Steyer, J.-C. Bouvier, T. Conte, P. Gras, and P. Sousbie, "Evaluation of a four year experience with a fully instrumented anaerobic digestion process," *Water Sci. Technol.*, vol. 45, no. 4–5, pp. 495–502, 2002.

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[2]The constant $2(10^4)$ enters the analysis by finding the minimum values of the denominator in the integrand.