Strict Lyapunov Function Constructions Under LaSalle Conditions with an Application to Lotka-Volterra Systems

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***Abstract*—We provide new techniques for building explicit global strict Lyapunov functions for broad classes of periodic time varying nonlinear systems satisfying LaSalle conditions. We illustrate our work using the Lotka-Volterra model, which plays a fundamental role in bioengineering. We use our strict Lyapunov constructions to prove robustness of the Lotka-Volterra tracking dynamics to uncertainty in the death rates.**

***Index Terms*—Lyapunov function, LaSalle conditions, biological systems**

I. Introduction

LYAPUNOV functions provide vital tools for the analysis of, and controller design for, nonlinear systems [8], [9], [18]. The two main types of Lyapunov functions are *strict* Lyapunov functions (also called *strong* Lyapunov functions, having negative definite time derivatives along trajectories) and *nonstrict* Lyapunov functions (also called *weak* Lyapunov functions, whose time derivatives along the trajectories are negative *semi*definite); see Section II for precise definitions.

Strict Lyapunov functions are typically far more useful than nonstrict ones. In general, nonstrict Lyapunov functions can only be used to prove asymptotic stability, using, e.g., the LaSalle invariance principle. On the other hand, *strict* Lyapunov functions can often be used to show robustness properties, such as input-to-state stability (ISS) (but see [10] for an alternative stability proof based on weak Lyapunov functions for time-varying systems). Robustness is essential in engineering, due to uncertainty and controller noise. For this reason, it is important to construct strict Lyapunov functions, even for systems that are already known to be UGAS.

Moreover, many controller methods (e.g., forwarding [16], [18] and universal stabilizing controllers [19]) use strict Lyapunov functions. For example, if V is a strict Lyapunov function for a system $\dot{x} = f(t,x)$ for which $\alpha(x) := \inf_t\{-[V_t(t,x)+V_x(t,x)f(t,x)]\}$ is radially unbounded, with f and g locally Lipschitz, and with V, f, and g all periodic in t with the same period T, then $\dot{x} = f(t,x)+g(t,x)[K(t,x)+d]$ is input-to-state stable if $K(t,x) = -[V_x(t,x)g(t,x)]^\top$. In fact, along the trajectories of the closed loop system, the triangle inequality gives the ISS Lyapunov decay condition $\dot{V} \le -\alpha(x) - |V_x(t,x)g(t,x)|^2 + V_x(t,x)g(t,x)d \le -\alpha(x) - 0.5|V_x(t,x)g(t,x)|^2 + 0.5|d|^2 \le -\alpha(x) + 0.5|d|^2$. (The radial unboundedness of α is needed for the ISS Lyapunov function decay condition, since if α were only positive definite, then we could only guarantee integral ISS. On the other hand, one can first replace V with $\kappa(V)$ for a suitable function κ to guarantee that α is proper [20].) Consequently, when a global strict Lyapunov function is known, many important stabilization problems can be solved almost immediately.

In general, it is much easier to construct *nonstrict* Lyapunov functions, owing to the more restrictive decay condition for strict Lyapunov functions. For instance, when a passive nonlinear system is stabilized by linear output feedback, the energy (i.e., storage) function can typically be used as the weak Lyapunov function. This is useful for electro-mechanical systems. When a system is stabilized via the Jurdjevic-Quinn theorem, nonstrict Lyapunov functions are typically available, e.g., using the Hamiltonian for Euler Lagrange systems [5], [7], [13], [17]. If a system is known to be UGAS, then converse Lyapunov function theory typically guarantees the existence of a strict Lyapunov function. However, the Lyapunov functions provided by converse theory are often abstract and nonexplicit, and therefore may not always lend themselves to applications. This has also motivated a significant literature on constructing strict Lyapunov functions, e.g., [1], [5].

In this work, we present two new strict Lyapunov function constructions, based on transforming nonstrict Lyapunov functions into strict ones, under Lie derivative conditions. The assumptions for our first construction are more general than those of [15] and different from those of [12, Corollary 2]. This is because we allow *periodic time varying* systems, including cases where all of the higher order Lie derivatives are allowed to vanish at some points outside the equilibrium, on some time intervals. Our construction is simpler than the one in [15], even in the special case of time invariant systems.

Our second result uses the Matrosov approach. In general, Matrosov's method can be difficult to apply, because one needs to find the necessary auxiliary functions. Here we give simple sufficient conditions leading to a systematic design of auxiliary functions. The auxiliary functions we construct differ from the ones that are implicitly given in [12], and they lead to strict Lyapunov functions using the Matrosov construction from [14]. In fact, our strict Lyapunov functions have the property that their Lie derivatives are frequently

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bounded above by negative definite quadratic functions. Another important feature of our work is that it applies to cases where the state space of the system is a general subset of Euclidean space, instead of the whole Euclidean space. This is desirable for biological systems, whose state spaces are often restricted by the requirement that physical quantities need to be nonnegative. We illustrate our approach using an error dynamics associated with the Lotka-Volterra system, which plays a fundamental role in bioengineering.

II. Definitions and Assumptions

Throughout this work, $\mathcal{X}$ is any open subset of $\mathbb{R}^n$ containing the origin. Consider a nonlinear time varying dynamics

$$\dot{x} = f(t,x), \quad x \in \mathcal{X} \tag{1}$$

where $f : [0,\infty) \times \mathcal{X} \to \mathbb{R}^n$ is C^∞, $\mathcal{X}$ is positively invariant for (1), and $f(t,0) = 0$ for all $t \geq 0$. We always assume that (1) is *periodic* of period T in t, meaning there is a constant $T > 0$ so that $f(t+T,x) = f(t,x)$ for all $(t,x) \in [0,\infty)\times\mathcal{X}$. We further assume that (1) is *forward complete*, meaning for each initial condition $x(t_0) = x_0$ with $t_0 \geq 0$ and $x_0 \in \mathcal{X}$, the solution $x(t,t_0,x_0)$ for the corresponding initial value problem for (1) is uniquely defined on $[t_0,\infty)$. Set $\mathbb{N} = \{1,2,3,\ldots\}$. Given a C^∞ function $V : [0,\infty)\times\mathcal{X} \to \mathbb{R}$, we set

$$\begin{aligned} &a_0(t,x) = V(t,x)\,, \quad \text{and} \\ &a_r(t,x) = -\tfrac{\partial a_{r-1}}{\partial x}(t,x)f(t,x) - \tfrac{\partial a_{r-1}}{\partial t}(t,x) \ \ \forall r \in \mathbb{N}\,. \end{aligned} \tag{2}$$

If V and f are time invariant, then $a_r = (-1)^r L_f^r V$ for all $r \geq 1$, where L_f^r is the usual iterated Lie derivative defined by

$$\begin{aligned} &L_f^0 V = V, \ \ L_f V(x) = L_f^1 V(x) = \tfrac{\partial V}{\partial x}(x)f(x), \\ &\text{and } \ L_f^k V = L_f(L_f^{k-1}V) \ \ \forall k \geq 2\,. \end{aligned}$$

Also, if we use $\dot{G} = (\partial G/\partial t)(t,x) + (\partial G/\partial x)(t,x)f(t,x)$ for any C^1 function G, then $\dot{a}_r = -a_{r+1}$ for all $r \geq 1$. A continuous function $k : [0,\infty) \to [0,\infty)$ is of *class* $\mathcal{K}_\infty$ (written $k \in \mathcal{K}_\infty$) provided it is zero at zero, strictly increasing and unbounded. A function $G : [0,\infty)\times\mathcal{X} \to \mathbb{R}$ is *positive definite* (resp., *positive semi-definite*) on $\mathcal{X}$ provided $G(t,0) = 0$ for all t and $\inf\{G(t,x) : t \geq 0\} > 0$ (resp., ≥ 0) for all $x \in \mathcal{X}\setminus\{0\}$. A function G is *negative (semi-)definite* provided $-G$ is positive (semi-)definite. We use $|\cdot|$ to denote the usual Euclidean norm.

A function $V : [0,\infty)\times\mathcal{X} \to \mathbb{R}$ is called a *storage function* provided there are continuous positive definite functions $\alpha_1, \alpha_2 : \mathcal{X} \to [0,\infty)$ such that (I) for each i, $\alpha_i(q) \to +\infty$ whenever $|q| \to \infty$ with q remaining in $\mathcal{X}$ (i.e., $\alpha_i(q_j) \to \infty$ for each i and each unbounded sequence $\{q_j\}$ that remains in $\mathcal{X}$, which is true vacuously if $\mathcal{X}$ is bounded) and (II) $\alpha_1(x) \leq V(t,x) \leq \alpha_2(x)$ for all $(t,x) \in [0,\infty)\times\mathcal{X}$. A C^1 storage function V is called a *nonstrict* (resp., *strict*) *Lyapunov-like function* for (1) provided $-a_1(t,x)$ is negative semi-definite (resp., negative definite). If, in addition, for each i and each boundary point $\bar{q} \in \partial\mathcal{X}$, $\alpha_i(q) \to +\infty$ when $q \to \bar{q}$, then a nonstrict (resp., strict) Lyapunov-like function is called a *nonstrict* (resp., *strict*) *Lyapunov function*. The existence of strict Lyapunov functions is key to proving uniform global asymptotic stability (UGAS) [14].

III. First Construction: Iterated Lie Derivatives

To motivate our assumptions, suppose that a given C^∞ time invariant system $\dot{x} = f(x)$, satisfying $f(0) = 0$ and evolving on $\mathbb{R}^n$, admits a time invariant C^∞ *nonstrict* Lyapunov function V such that for each $q \in \mathbb{R}^n \setminus \{0\}$, there is an $i \in \mathbb{N}$ such that $L_f^i V(q) \neq 0$, e.g., conditions (i)-(ii) in Theorem 1 below hold, and (3) holds independently of t. If $L_f V(x(t,x_0)) \equiv 0$ along some trajectory $t \mapsto x(t,x_0)$ of the system, then we can differentiate repeatedly in time to get $L_f^k V(x(t,x_0)) \equiv 0$ for all $t \geq 0$ and all $k \in \mathbb{N}$, so $L_f^k V(x_0) = 0$ for all $k \in \mathbb{N}$. By assumption, this implies that $x_0 = 0$. Hence, UGAS follows from LaSalle invariance. However, it is not clear how to construct a global *strict* Lyapunov function. This motivates our (more general) hypotheses in the following theorem:

Theorem 1: Consider the periodic time varying system (1) with state space $\mathcal{X} = \mathbb{R}^n$ and some period $T > 0$ in t, where $f \in C^\infty$. Assume that there exists a C^∞ storage function $V : [0,\infty)\times\mathbb{R}^n \to [0,\infty)$ having period T in t such that the following hold: (i) V is a nonstrict Lyapunov function for (1) and (ii) there exist a constant $\tau \in (0,T]$, a constant $\ell \in \mathbb{N}$, and a positive definite continuous function ρ such that for all $x \in \mathbb{R}^n$ and all $t \in [0,\tau]$,

$$a_1(t,x) + \sum_{m=2}^{\ell} a_m^2(t,x) \geq \rho\big(V(t,x)\big)\,. \tag{3}$$

Then we can explicitly determine functions $\mathcal{F}_j$ and $\mathcal{G}$, with $\mathcal{G}$ periodic of period T in t, such that

$$\begin{aligned} &V^\sharp(t,x) = \sum_{j=1}^{\ell-1} \mathcal{F}_j\big(V(t,x)\big)A_j(t,x) + \mathcal{G}\big(t,V(t,x)\big), \\ &\text{where } \ A_j(t,x) = \sum_{m=1}^{j} a_{m+1}(t,x)a_m(t,x) \quad \forall j \end{aligned} \tag{4}$$

is a strict Lyapunov function for (1), giving UGAS of (1).

Remark 1: Theorem 1 remains true if V is merely $C^{\ell+1}$ (instead of C^∞). The assumptions of Theorem 1 are related to, but more general than, those of the strict Lyapunov function construction from [15] and different from those of [12, Corollary 2]. The assumptions of [15] are the special case of (i)-(ii) in which f and V are time invariant; in that case, (3) says there is a continuous positive definite function ρ so that $-L_f V(x) + \sum_{m=2}^{\ell}(L_f^m V(x))^2 \geq \rho(V(x))$ for all $x \in \mathbb{R}^n$. Our result is new, even in this special case, because the strict Lyapunov function construction in our proof of Theorem 1 is simpler than the one in [15]; see Remark 2 at the end of Section V for details. It is important to have strict Lyapunov functions that are as simple as possible for feedback design and robustness analysis. We prove Theorem 1 in Section V.

IV. Second Construction: Matrosov Conditions

To simplify the notation in our next theorem, we only consider time invariant systems

$$\dot{x} = f(x), \quad x \in \mathcal{X} \tag{5}$$

for which $\mathcal{X} \subseteq \mathbb{R}^n$ is positively invariant, where $f(0) = 0$; see Remark 5 in Section VI for the generalization to (1). We use the Matrosov approach from [14] to construct global strict Lyapunov functions for (5). In addition to a nonstrict Lyapunov function, the Matrosov results in [14] require appropriate auxiliary functions, which can be difficult to find in practice. The paper [14] does not provide any general methods for constructing auxiliary functions.

Our next theorem provides a new mechanism for choosing auxiliary functions. However, its most important features are that (A) it applies to systems whose state space is only a subset of $\mathbb{R}^n$ and (B) it may yield Lyapunov functions that are simpler than other constructions, and that also have desirable local properties, such as local boundedness from below by positive definite quadratic functions; see Section VII-B. For the rest of this work, we assume that all of our functions are sufficiently smooth. Our Matrosov-type assumption is:

Assumption 1: There exist a storage function $V_1 : \mathcal{X} \to [0, \infty)$; functions $h_1, \ldots, h_m$ such that $h_j(0) = 0$ for all j; everywhere positive functions $r_1, \ldots, r_m$ and ρ; and an integer $N > 0$ for which

$$\begin{aligned} \nabla V_1(x) f(x) \ &\le\ -r_1(x) h_1^2(x) - \ldots - r_m(x) h_m^2(x) \\ \text{and}\ \sum_{l=0}^{N-1}\sum_{j=1}^{m} \left[L_f^l h_j(x)\right]^2 \ &\ge\ \rho(V_1(x))V_1(x) \end{aligned} \tag{6}$$

hold for all $x \in \mathcal{X}$. Moreover, f is defined on $\mathbb{R}^n$, and there is a function $\overline{\Gamma} \in \mathcal{K}_\infty$ such that

$$|f(x)| \ \le\ \overline{\Gamma}(|x|) \ \ \forall x \in \mathbb{R}^n. \tag{7}$$

Also, V_1 has a positive definite quadratic lower bound in some neighborhood of the origin.

In Section VI, we prove:

Theorem 2: If (5) satisfies Assumption 1, then one can determine explicit functions $k_i, \Omega_i \in \mathcal{K}_\infty \cap C^1$ and an everywhere positive function $\varrho \in C^1$ such that

$$S(x) = \sum_{i=1}^{N} \Omega_i\Big(k_i\big(V_1(x)\big) + V_i(x)\Big) \tag{8}$$

with the choices

$$V_i(x) = -\sum_{l=1}^{m} L_f^{i-2} h_l(x) L_f^{i-1} h_l(x)\ ,\ \ i = 2, \ldots, N \tag{9}$$

satisfies

$$S(x) \ge V_1(x)$$

and

$$\nabla S(x) f(x) \le -\varrho(x) V_1(x)$$

for all $x \in \mathcal{X}$. If, in addition, $\mathcal{X} = \mathbb{R}^n$, then the system (5) is GAS.

Throughout what follows, all inequalities should be understood to hold globally unless otherwise indicated, and we leave out the arguments of our functions when they are clear.

V. Proof of Theorem 1

1) Formula for Strict Lyapunov Function: We can assume that $\ell \ge 3$, by enlarging ℓ as needed without relabeling. We show that (1) admits the global strict Lyapunov function

$$\begin{aligned} &V^\sharp(t,x) = V(t,x) S_3(t,x) + \kappa\big(V(t,x)\big) V(t,x), \\ &\text{where}\ \ S_3(t,x) = S_1(t,x) + S_2(t,x)\ , \\ &S_1(t,x) = \sum_{p=1}^{\ell-1} k_p\big(V(t,x)\big) M_p(t,x) \\ &\qquad + k_0\big(V(t,x)\big) V(t,x), \\ &S_2(t,x) = G\big(V(t,x)\big) + \frac{1}{T}\left(\int_{t-T}^{t}\int_{s}^{t} q(r)\,\mathrm{d}r\,\mathrm{d}s\right) \\ &\qquad \times k_{\ell-1}\big(V(t,x)\big)\frac{\omega(V(t,x))}{K(V(t,x))},\ \ \text{and} \\ &M_p(t,x) = \sum_{m=1}^{p} a_{m+1}(t,x) a_m(t,x) \\ &\qquad + \int_0^{V(t,x)} \Gamma(r)\mathrm{d}r \end{aligned} \tag{10}$$

for $p = 1, 2, \ldots, \ell - 1$. Here $\kappa \in C^1$ is any strictly increasing function such that

$$\kappa(V(t,x)) \ge |S_3(t,x)| + 1$$

for all $(t,x) \in [0,\infty) \times \mathbb{R}^n$, $\Gamma \in C^1$ is any everywhere positive increasing function such that

$$\Gamma(V(t,x)) \ge (\ell+2)|a_m(t,x)| + 1$$

for all $m \in \{1, ..., \ell+1\}$ and all $(t,x) \in [0,\infty) \times \mathbb{R}^n$, $\omega \in \mathcal{K}_\infty \cap C^1$ and the strictly increasing everywhere positive function $K \in C^1$ are such that

$$\rho(r) \ge \frac{\omega(r)}{K(r)} \ \ \forall r \ge 0, \tag{11}$$

the C^1 positive definite functions $k_1, k_2, \ldots, k_{\ell-1}$ are

$$\begin{aligned} &k_{\ell-1}(v) = \omega^{2^{\ell-1}}(v) \ \ \text{and} \\ &k_p(v) = k_{\ell-1}(v)\Omega^{1-2^{\ell-p-1}}(v) \\ &\text{for } p = 1, 2, \ldots, \ell-2, \\ &\text{where}\ \ \Omega(v) = \frac{2\tau\omega(v)}{3T(\ell-2)\Gamma^2(v)K(v)}, \end{aligned} \tag{12}$$

k_0 is any C^1 increasing function such that

$$\begin{aligned} &k_0\big(V(t,x)\big) + k_0'\big(V(t,x)\big)V(t,x) \ \ge \\ &\sum_{p=1}^{\ell-1}\big|k_p'\big(V(t,x)\big)\big||M_p(t,x)| + 1\ , \end{aligned} \tag{13}$$

$q : \mathbb{R} \to [0,1]$ is any continuous function with period T that satisfies $q(t) = 0$ for all $t \in [\tau, T]$ and $q(t) = 1$ for all $t \in [\frac{\tau}{3}, \frac{2\tau}{3}]$, and G is any C^1 function such that

$$\begin{aligned} &G'(v) \ \ge \\ &T\left|k_{\ell-1}(v)\frac{\omega'(v)K(v) - \omega(v)K'(v)}{K^2(v)} + k_{\ell-1}'(v)\frac{\omega(v)}{K(v)}\right| \end{aligned} \tag{14}$$

for all $v \ge 0$. The functions ω and K can be obtained using Lemma A.1 below, Γ can be obtained by majorizing $s \mapsto 1 + \max\{(\ell+2)|a_m(t,x)| : t \ge 0, x \in \mathcal{X}, m \in \{1, 2, \ldots, \ell+1\}, V(t,x) \le s\}$ by a C^1 function, and k_0 can be obtained because each M_p is periodic in t and because V is a storage function that is also periodic in t. The theorem will then follow by collecting the functions involving V to produce (4).

2) Stability Analysis: To show that (10) is a strict Lyapunov function for (1), we use the everywhere nonnegative functions

$$N_j(t,x) = \sum_{m=2}^{j+1} a_m^2(t,x) + a_1(t,x)$$

for $1 \le j \le \ell - 1$, which have period T in t. By (3), (11), and the nonnegativity of $N_{\ell-1}$,

$$N_{\ell-1}(t,x) \;\ge\; q(t)\frac{\omega(V(t,x))}{K(V(t,x))} \tag{15}$$

for all $(t,x) \in [0,\infty) \times \mathbb{R}^n$. Since $a_1 = -\dot{V} \ge 0$, our choice of Γ and (10) give

$$\begin{aligned} \dot{M}_1 &= \dot{a}_2 a_1 - a_2^2 - \Gamma(V)a_1 \;\le\; -N_1\ , \text{ and} \\ \dot{M}_j &\le -\textstyle\sum_{m=1}^{j} a_{m+1}^2 + \sum_{m=2}^{j} |a_{m+2}||a_m| \\ &\quad + |a_3|a_1 - \Gamma(V)a_1 \\ &\le -\textstyle\sum_{m=1}^{j} a_{m+1}^2 + \sum_{m=2}^{j} |a_{m+2}||a_m| \\ &\quad + |a_3|a_1 - \big[(\ell+2)|a_3| + 1\big]a_1 \end{aligned} \tag{16}$$

for each $j \in \{2, ..., \ell-1\}$, since $\dot{a}_i = -a_{i+1}$ for all i. Recalling our choice of Γ gives

$$\begin{aligned} \dot{M}_j &\le -\textstyle\sum_{m=1}^{j} a_{m+1}^2 + \frac{\Gamma(V)}{\ell+2}\sum_{m=2}^{j} |a_m| \\ &\quad -\big[(\ell+1)|a_3| + 1\big]a_1 \quad \forall j \in \{2, \ldots, \ell-1\}. \end{aligned}$$

It follows from the Cauchy Inequality that for all $j \in \{2, ..., \ell-1\}$,

$$\begin{aligned} \dot{M}_j &\le -\textstyle\sum_{m=1}^{j} a_{m+1}^2 + \Gamma(V)\sqrt{\sum_{m=2}^{j} a_m^2} \\ &\quad -\big[(\ell+1)|a_3| + 1\big]a_1 \\ &\le -N_j + \Gamma(V)\sqrt{N_{j-1}}, \end{aligned} \tag{17}$$

since $a_1 = -\dot{V} \ge 0$ everywhere. Also,

$$\begin{aligned} \dot{S}_1 &= \textstyle\sum_{p=1}^{\ell-1} k_p(V)\dot{M}_p + \big[k_0(V) + k_0'(V)V\big]\dot{V} \\ &\quad + \Big[\textstyle\sum_{p=1}^{\ell-1} k_p'(V)M_p\Big]\dot{V} \\ &\le \textstyle\sum_{p=1}^{\ell-1} k_p(V)\dot{M}_p + \big[k_0(V) + k_0'(V)V\big]\dot{V} \\ &\quad \Big[\textstyle\sum_{p=1}^{\ell-1} |k_p'(V)||M_p|\Big]\big[-\dot{V}\big] \\ &\le \textstyle\sum_{p=1}^{\ell-1} k_p(V)\dot{M}_p\,, \end{aligned} \tag{18}$$

using (13) and the fact that $\dot{V}$ is nonpositive. Using (16)-(18), we deduce that

$$\begin{aligned} \dot{S}_1 &\le -k_1(V)N_1 \\ &\quad + \textstyle\sum_{p=2}^{\ell-1} k_p(V)\left[-N_p + \Gamma(V)\sqrt{N_{p-1}}\,\right] \\ &\le -\textstyle\sum_{p=1}^{\ell-1} k_p(V)N_p + \sum_{p=2}^{\ell-1} k_p(V)\Gamma(V)\sqrt{N_{p-1}}. \end{aligned}$$

It follows from (15) that

$$\begin{aligned} \dot{S}_1 &\le -k_{\ell-1}(V)q(t)\tfrac{\omega(V)}{K(V)} - \textstyle\sum_{p=1}^{\ell-2} k_p(V)N_p \\ &\quad + \textstyle\sum_{p=1}^{\ell-2} k_{p+1}(V)\Gamma(V)\sqrt{N_p}. \end{aligned} \tag{19}$$

Since $\int_{t-T}^{t}\int_{s}^{t} q(r)\,\mathrm{d}r\,\mathrm{d}s \le T^2$ and $\frac{d}{dt}\int_{t-T}^{t}\int_{s}^{t} q(r)\,\mathrm{d}r\,\mathrm{d}s = Tq(t) - \int_{t-T}^{t} q(r)\mathrm{d}r$, (14) gives

$$\begin{aligned} \dot{S}_2 &\le k_{\ell-1}(V)q(t)\tfrac{\omega(V)}{K(V)} \\ &\quad -\tfrac{1}{T}\left(\textstyle\int_{t-T}^{t} q(r)\mathrm{d}r\right)k_{\ell-1}(V)\tfrac{\omega(V)}{K(V)}. \end{aligned} \tag{20}$$

Since $\int_{t-T}^{t} q(r)\mathrm{d}r \ge \frac{\tau}{3}$ for all $t \in \mathbb{R}$, (19) and (20) give

$$\begin{aligned} \dot{S}_3 &\le -\tfrac{1}{T}\left(\textstyle\int_{t-T}^{t} q(r)\mathrm{d}r\right)k_{\ell-1}(V)\tfrac{\omega(V)}{K(V)} \\ &\quad -\textstyle\sum_{p=1}^{\ell-2} k_p(V)N_p \\ &\quad +\textstyle\sum_{p=1}^{\ell-2} k_{p+1}(V)\Gamma(V)\sqrt{N_p} \\ &\le -\tfrac{\tau}{3T}k_{\ell-1}(V)\tfrac{\omega(V)}{K(V)} - \textstyle\sum_{p=1}^{\ell-2} k_p(V)N_p \\ &\quad +\textstyle\sum_{p=1}^{\ell-2} k_{p+1}(V)\Gamma(V)\sqrt{N_p}. \end{aligned} \tag{21}$$

From the triangular inequality $c_1c_2 \le c_1^2 + \frac{1}{4}c_2^2$ for nonnegative values c_1 and c_2, we deduce that

$$\begin{aligned} &k_{p+1}(V)\Gamma(V)\sqrt{N_p} \\ &= \left\{\sqrt{k_p(V)N_p}\right\}\left\{\tfrac{\Gamma(V)k_{p+1}(V)}{\sqrt{k_p(V)}}\right\} \\ &\le k_p(V)N_p + \tfrac{\Gamma^2(V)k_{p+1}^2(V)}{4k_p(V)} \end{aligned} \tag{22}$$

for $p = 1, 2, \ldots, \ell-2$ when $V \neq 0$. Summing in (22) over p and substituting into (21) gives

$$\dot{S}_3 \;\le\; -\tfrac{\tau}{3T}k_{\ell-1}(V)\tfrac{\omega(V)}{K(V)} + \textstyle\sum_{p=1}^{\ell-2}\tfrac{\Gamma^2(V)k_{p+1}^2(V)}{4k_p(V)} \tag{23}$$

when $V \neq 0$. By our choices (12) of the k_p's and Ω, (23) gives

$$\begin{aligned} \dot{S}_3 &\le -\tfrac{\tau}{3T}k_{\ell-1}(V)\tfrac{\omega(V)}{K(V)} \\ &\quad +\textstyle\sum_{p=1}^{\ell-2}\tfrac{\Gamma^2(V)k_{\ell-1}^2(V)\Omega^{2(1-2^{\ell-p-2})}(V)}{4k_{\ell-1}(V)\Omega^{1-2^{\ell-p-1}}(V)} \\ &\le -\tfrac{\tau}{3T}k_{\ell-1}(V)\tfrac{\omega(V)}{K(V)} + (\ell-2)\tfrac{\Gamma^2(V)k_{\ell-1}(V)\Omega(V)}{4} \\ &\le -\tfrac{\tau}{6T}k_{\ell-1}(V)\tfrac{\omega(V)}{K(V)},\quad V \neq 0. \end{aligned}$$

Hence, $\dot{S}_3$ is negative definite. However, S_3 is not necessarily positive definite and radially unbounded, and so may not be a strict Lyapunov function.

To check that $V^\sharp$ from (10) is a strict Lyapunov function, first note that it is positive definite and radially unbounded because $V^\sharp(t,x) \ge V(t,x)$, by our choice of κ. Also, since $\dot{V} \le 0$ everywhere, $\dot{V}^\sharp = V\dot{S}_3 + \dot{V}S_3 + \big[\kappa'(V)V + \kappa(V)\big]\dot{V} \le -\tau k_{\ell-1}(V)\omega(V)V/\{6TK(V)\}$, which is the desired strict Lyapunov function decay condition. This proves Theorem 1.

Remark 2: For the special case of time-invariant systems, the assumptions from Theorem 1 agree with the assumptions from the strict Lyapunov function construction in [15, Theorem 3.1]. However, our proof of Theorem 1 is simpler than the arguments from [15]. The construction in [15, Theorem 3.1] requires a non-increasing function $\lambda : [0,\infty) \to (0,\infty)$ such that the function

$$U(x) = V(x)\left[1 + V(x) - \sum_{i=1}^{\ell-1} L_{f_\lambda}^{i}V(x)\cdot\left(L_{f_\lambda}^{i+1}V(x)\right)^{3^i}\right]$$

is a strict Lyapunov function for the system, where $f_\lambda(x) := \lambda(V(x))f(x)$. There is no analog of λ in our simpler proof of Theorem 1.

VI. Proof of Theorem 2

In the following proof, we omit the dependencies of the functions on x when they are clear from the context. To simplify our notation, we introduce the functions

$$\begin{aligned} \mathcal{N}_1(x) &= R(x)\sum_{l=1}^{m} h_l^2(x) \quad \text{and} \\ \mathcal{N}_i(x) &= \sum_{l=1}^{m}\left[L_f^{i-1}h_l(x)\right]^2 \quad \forall i \geq 2, \\ \text{where } R(x) &= \frac{\prod_{i=1}^{m} r_i(x)}{\prod_{i=1}^{m}[r_i(x)+1]}\ . \end{aligned} \tag{24}$$

Since R is everywhere positive and satisfies $R(x) \leq r_i(x)$ for all $x \in \mathbb{R}^n$ and all $i \in \{1, ..., m\}$, (6) and our choices (9) of the V_i's give

$$\begin{aligned} \nabla V_1(x)f(x) &\leq -\mathcal{N}_1 \quad \forall x \in \mathcal{X}, \text{ and} \\ \nabla V_i(x)f(x) &= -\sum_{l=1}^{m}\left[L_f^{i-1}h_l\right]^2 - \sum_{l=1}^{m} L_f^{i-2}h_l L_f^i h_l \\ &\leq -\mathcal{N}_i + \sum_{l=1}^{m}|L_f^{i-2}h_l||L_f^i h_l| \end{aligned}$$

for $i = 2, \ldots, N$ and all $x \in \mathcal{X}$. In particular, we have

$$\begin{aligned} \nabla V_2(x)f(x) &\leq -\mathcal{N}_2(x) + \sqrt{\mathcal{N}_1(x)}\textstyle\sum_{l=1}^{m}\frac{|L_f^2 h_l(x)|}{\sqrt{R(x)}}, \\ \nabla V_i(x)f(x) &\leq -\mathcal{N}_i(x) + \sqrt{\mathcal{N}_{i-1}(x)}\textstyle\sum_{l=1}^{m}|L_f^i h_l(x)| \end{aligned}$$

for $i = 3, 4, \ldots, N$. Our assumptions on f, V_1, and the h_i's allow us to determine a function $\underline{\alpha} \in \mathcal{K}_\infty$ such that $V_1(x) \geq \underline{\alpha}(|x|)$ for all $x \in \mathcal{X}$, and a continuous everywhere positive function ϕ_1 such that

$$\begin{aligned} \textstyle\sum_{l=1}^{m}\frac{|L_f^2 h_l(x)|}{\sqrt{R(x)}} &\leq \phi_1(V_1(x))\sqrt{V_1(x)} \quad \text{and} \\ \textstyle\sum_{l=1}^{m}|L_f^i h_l(x)| &\leq \phi_1(V_1(x))\sqrt{V_1(x)} \end{aligned} \tag{25}$$

for all $i \in \{3, \ldots, N\}$ and all $x \in \mathcal{X}$; see Appendix B. It follows that for all $i \geq 2$ and $x \in \mathcal{X}$,

$$\begin{aligned} &\nabla V_i(x)f(x) \\ &\leq -\mathcal{N}_i(x) + \phi_1\big(V_1(x)\big)\sqrt{\mathcal{N}_{i-1}(x)}\sqrt{V_1(x)}. \end{aligned} \tag{26}$$

Arguing as we did to get (25) gives an increasing everywhere nonnegative function $p_1 \in C^1$ such that $|V_i(x)| \leq p_1(V_1(x))V_1(x)$ for $i = 1, \ldots, N$ for all $x \in \mathcal{X}$ (by first finding an increasing positive function $\tilde{\alpha}$ so that $|V_i(x)| \leq \tilde{\alpha}(|x|)|x|^2$ for all $i \geq 2$ and all x near 0, using the fact that $h_r(0) = 0$ for all r to handle the $i = 2$ case). Finally, we can find a decreasing everywhere positive function $\underline{\rho}$ so that $R(x) \geq \underline{\rho}(\underline{\alpha}(|x|)) \geq \underline{\rho}(V_1(x))$ on $\mathcal{X}$, and then a continuous everywhere positive function $\tilde{\rho}$ so that

$$\sum_{i=1}^{N}\mathcal{N}_i(x) \ \geq\ \tilde{\rho}(V_1(x))V_1(x)$$

on $\mathcal{X}$, by (6). Hence, the assumptions of [14, Theorem 1] hold with $a_i \equiv \frac{1}{2}$, so [14, Theorem 1] constructs the necessary strict Lyapunov-like function.

Remark 3: As we noted above, the results of [14] do not provide general methods for building the auxiliary functions from the Matrosov conditions. Therefore, the novelty of the preceding proof lies in its general procedure for producing the functions V_i that satisfy the Matrosov conditions. On the other hand, given the functions V_i, ϕ_1, p_1, and $\tilde{\rho}$ we constructed above, the results of [14] readily produce the desired strict Lyapunov function S. The strict Lyapunov function takes the form

$$S(x) = \Omega_1\big(2V_1(x)\big) + \sum_{i=2}^{N}\Omega_i\big(U_i(x)\big) \tag{27}$$

where

$$U_i(x) = V_i(x) + V_1(x)[1 + p_1(V_1(x))] \tag{28}$$

for all $i \geq 2$, and where the functions $\Omega_i \in \mathcal{K}_\infty \cap C^1$ are recursively chosen as follows. We take $\Omega_N(r) = r$, and then choose Ω_i for $i = 1, 2, \ldots, N-1$ to satisfy

$$\Omega_i'(U_i) \geq (N-1)^2\frac{8\phi_1^2(V_1)}{\tilde{\rho}(V_1)}\sum_{r=1+i}^{N}\Omega_r'(U_r)^2, \tag{29}$$

with $\Omega_i' : [0, \infty) \to [1, \infty)$ continuous and increasing for each i. The formula (27) is the special case of the construction from [14, Theorem 1] when all of the exponents a_i in [14] are $\frac{1}{2}$.

Remark 4: A standard "smoothing of the corners" argument that is analogous to the one used to prove [4, Lemma 2.5] allows us to majorize (resp., minorize) any continuous increasing (resp., decreasing) function $\phi : [0, \infty) \to (0, \infty)$ by a C^∞ everywhere positive increasing (resp., decreasing) function. Therefore, when f and V_1 are both C^∞, we can use the argument from [14, Theorem 1] in conjunction with smoothing of the corners to guarantee that the functions Ω_i are C^∞, hence a C^∞ strict Lyapunov function that is consistent with the known converse Lyapunov function theory.

Remark 5: We can prove an analog of Theorem 2 for (1), under a periodic time varying version of Assumption 1. The periodic time varying analog of Assumption 1 is obtained by (A) replacing the arguments of f and the V_i's by (t, x) and (B) replacing $\nabla V_i(x)f(x)$ with $\frac{\partial V_i}{\partial t}(t, x) + \frac{\partial V_i}{\partial x}(t, x)f(t, x)$. The proof is then as before, using the periodic time varying extension in [14, Section IV].

VII. Illustrations

A. Periodic Time Varying System

To illustrate Theorem 1 in a reasonably simple way, we take the periodic time varying system

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& \cos(t)x_2 \\ \dot{x}_2 &=& -\cos(t)x_1 - x_2\ . \end{array}\right. \tag{30}$$

Along the trajectories of (30), the nonstrict Lyapunov function $V(x) = \frac{1}{2}|x|^2$ gives $\dot{V} = -x_2^2$. Using the notation from Theorem 1, we get $a_1(t, x) = x_2^2$, so $\dot{a}_1 = -2\cos(t)x_1x_2 - 2x_2^2$. This gives

$$\begin{aligned} a_2(t, x) &= 2\cos(t)x_1x_2 + 2x_2^2, \text{ hence} \\ \dot{a}_2 &= -2\cos^2(t)x_1^2 - 2\left[(3\cos(t) + \sin(t))x_1 \right. \\ &\quad \left. + (1 + \sin^2(t))x_2\right]x_2. \end{aligned}$$

Therefore,

$$\begin{aligned} &a_3(t, x) = 2\cos^2(t)x_1^2 \\ &+ 2\left[(3\cos(t) + \sin(t))x_1 + (1 + \sin^2(t))x_2\right]x_2. \end{aligned}$$

Using the relations $2p^2 + 2q^2 \geq (p+q)^2 \geq \frac{p^2}{10} - \frac{q^2}{9}$ for any $p \geq 0$ and $q \geq 0$ gives

$$\begin{array}{rcl} a_3^2(t,x) & \geq & \frac{2}{5}\cos^4(t)x_1^4 - \frac{4}{9}\left[\{3\cos(t)+\sin(t)\}x_1 \right. \\ & & \left. + \left\{1+\sin^2(t)\right\}x_2\right]^2 x_2^2 \\ & \geq & \frac{2}{5}\cos^4(t)x_1^4 - \frac{16}{9}\left[2|x_1|+|x_2|\right]^2 x_2^2 \\ & \geq & \frac{2}{5}\cos^4(t)x_1^4 - 29V(x)a_1(t,x)\ , \end{array} \tag{31}$$

hence

$$\begin{array}{l} \frac{1}{40(V(x)+1)}a_3^2(t,x) \\ \geq \frac{1}{100(V(x)+1)}\cos^4(t)x_1^4 - \frac{3}{4}a_1(t,x)\,. \end{array} \tag{32}$$

Noticing that $\frac{1}{40(V(x)+1)} < 1$ and $V^2(x) \leq \frac{1}{2}(x_1^4 + x_2^4)$, we deduce from (31)-(32) that

$$\begin{array}{l} a_1(t,x) + a_2^2(t,x) + a_3^2(t,x) \\ \geq \frac{\cos^4(t)}{100(V(x)+1)}x_1^4 + \frac{1}{4}a_1(t,x) \\ \geq \frac{\cos^4(t)}{100(V(x)+1)}x_1^4 + \frac{1}{8(V(x)+1)}x_2^4 \\ \geq \frac{\cos^4(t)}{100(V(x)+1)}[x_1^4 + x_2^4] \\ \geq \frac{4\cos^4(t)}{200(V(x)+1)}V^2(x)\ . \end{array}$$

Since $4\cos^4(t) \geq 1$ on $[0, \pi/4]$, we can satisfy the assumptions of Theorem 1 with $V(x) = \frac{1}{2}|x|^2$, $\ell = 3$, $T = 2\pi$, $\tau = \frac{\pi}{4}$, and $\rho(r) = r^2/\{200(r+1)\}$. See Appendix C for the construction of the strict Lyapunov function $V^\sharp$ that follows the proof of Theorem 1.

B. Lotka-Volterra Example

We illustrate Theorem 2 using the celebrated Lotka-Volterra Predator-Prey system

$$\left\{ \begin{array}{rcl} \dot{\chi} & = & \gamma\chi\left(1 - \frac{\chi}{L}\right) - a\chi\zeta \\ \dot{\zeta} & = & \beta\chi\zeta - \Delta\zeta \end{array} \right. \tag{33}$$

with positive constants a, β γ, Δ, and L. System (33) is a simple model of one predator feeding on one prey. The population of the predator is ζ, χ is the population of the prey, and the constants are related to the birth and death rates; see [6], [11] for an extensive analysis of this model and generalizations to several predators. We assume that the population levels are positive. While there are many Lyapunov constructions for Lotka-Volterra models available (based on computing the LaSalle invariant set), to the best of our knowledge, the result to follow is original and significant because we provide a *global strict Lyapunov function*.

1) Global Strict Lyapunov Function Construction: The time scaling, change of coordinates, and constants

$$\begin{array}{l} x(t) = \frac{1}{L}\chi\left(\frac{t}{\gamma}\right), \quad y(t) = \frac{a}{\beta L}\zeta\left(\frac{t}{\gamma}\right), \\ \alpha = \frac{\beta L}{\gamma} \ \text{ and } \ d = \frac{\Delta}{\gamma} \end{array} \tag{34}$$

give the simpler system

$$\left\{ \begin{array}{rcl} \dot{x} & = & x\,(1-x) - \alpha xy \\ \dot{y} & = & \alpha xy - dy. \end{array} \right. \tag{35}$$

We assume that we have imposed assumptions on the parameters such that $\alpha > d$. Let

$$x_* = \frac{d}{\alpha} \ \text{ and } \ y_* = \frac{1}{\alpha} - \frac{d}{\alpha^2}. \tag{36}$$

Then $x_* \in (0,1)$ and $y_* > 0$. Also, the new variables $\tilde{x} = x - x_*$ and $\tilde{y} = y - y_*$ have the dynamics

$$\left\{ \begin{array}{rcl} \dot{\tilde{x}} & = & -[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*) \\ \dot{\tilde{y}} & = & \alpha\tilde{x}(\tilde{y} + y_*)\ , \end{array} \right. \tag{37}$$

with state space

$$\mathcal{X} = (-x_*, +\infty) \times (-y_*, +\infty).$$

We do our Lyapunov function construction for (37), so we set

$$f(\tilde{x},\tilde{y}) = \left(\begin{array}{c} -[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*) \\ \alpha\tilde{x}(\tilde{y} + y_*) \end{array} \right). \tag{38}$$

We verify Assumption 1 with

$$\begin{array}{l} m = 1, \ \ N = 2, \ \ r_1 \equiv 1, \ \ h_1(\tilde{x},\tilde{y}) = \tilde{x}, \ \text{ and} \\ V_1(\tilde{x},\tilde{y}) = \tilde{x} - x_* \ln\left(1 + \frac{\tilde{x}}{x_*}\right) + \tilde{y} - y_* \ln\left(1 + \frac{\tilde{y}}{y_*}\right). \end{array} \tag{39}$$

One easily checks that V_1 is a (time invariant) storage function. Along the trajectories of (37),

$$\begin{array}{rcl} \dot{V}_1 & = & -\frac{\tilde{x}}{x_* + \tilde{x}}[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*) + \frac{\alpha\tilde{y}}{y_* + \tilde{y}}\tilde{x}(\tilde{y} + y_*) \\ & = & -\tilde{x}[\tilde{x} + \alpha\tilde{y}] + \alpha\tilde{y}\tilde{x} \ = \ -\tilde{x}^2. \end{array} \tag{40}$$

Also, $L_f h_1(\tilde{x},\tilde{y}) = -[\tilde{x} + \alpha\tilde{y}](\tilde{x} + x_*)$. One can check that V_1 has a positive definite quadratic lower bound near the origin; see Appendix D. A simple argument provides a positive constant $\underline{d}$ such that

$$\frac{1}{2}h_1^2(\tilde{x},\tilde{y}) + [L_f h_1(\tilde{x},\tilde{y})]^2 \ \geq \ \underline{d}\frac{V_1(\tilde{x},\tilde{y})}{1 + V_1^2(\tilde{x},\tilde{y})} \tag{41}$$

on $\mathcal{X}$; see Appendix E. Hence, Assumption 1 holds with

$$\rho(r) = \frac{\underline{d}}{1 + r^2}, \tag{42}$$

so Theorem 2 constructs the necessary global strict Lyapunov function for (37).

We construct the strict Lyapunov function from Theorem 2. Set

$$\mathcal{N}_1(\tilde{x},\tilde{y}) = \frac{1}{2}h_1^2(\tilde{x},\tilde{y}) \ \text{ and } \ \mathcal{N}_2(\tilde{x},\tilde{y}) = [L_f h_1(\tilde{x},\tilde{y})]^2\,.$$

Notice that

$$L_f^2 h_1(\tilde{x},\tilde{y}) = -\left(x_* + 2\tilde{x} + \alpha\tilde{y}\right)\dot{\tilde{x}} - \left(x_* + \tilde{x}\right)\alpha\dot{\tilde{y}}\,. \tag{43}$$

Therefore,

$$\begin{array}{l} |L_f^2 h_1(\tilde{x},\tilde{y})| \\ \leq \ (2 + \alpha + x_*)^3(1 + |\tilde{x}| + |\tilde{y}|)^2(|\tilde{x}| + |\tilde{y}|) \\ +\alpha^2(1 + x_* + y_*)^2(1 + |\tilde{x}|)(1 + |\tilde{y}|)|\tilde{x}|. \end{array} \tag{44}$$

On the other hand, Appendix D applied with $A = \tilde{x}/x_*$ gives

$$\begin{array}{rcl} \left|\frac{\tilde{x}}{x_*}\right| & \leq & 2\left\{\frac{V_1}{x_*} + \left[\frac{V_1}{x_*}\right]^2\right\}^{1/2} \\ & \leq & 2\left[\max\left\{\frac{1}{x_*}, \frac{1}{x_*^2}\right\}\{V_1 + V_1^2\}\right]^{1/2} \end{array} \tag{45}$$

and similarly for y, so

$$\begin{array}{l} \max\{|\tilde{x}|, |\tilde{y}|\} \leq J(V_1)\sqrt{V_1}, \\ \text{where } \ J(V_1) = 2(1 + x_* + y_*)\sqrt{V_1 + 1}. \end{array} \tag{46}$$

Combining (44) and (46) gives

$$\begin{aligned}&|L_f^2 h_1(\tilde{x},\tilde{y})| \leq \\ &\Big\{2J(V_1)(2+\alpha+x_*)^3\left[1+2J(V_1)\sqrt{V_1}\right]^2 \\ &+\alpha^2(1+x_*+y_*)^2 J(V_1)\left(1+J(V_1)\sqrt{V_1}\right)^2\Big\}\sqrt{V_1}.\end{aligned} \quad (47)$$

Therefore, we can satisfy (25)-(26) with $\phi_1(r) = 4(1+\alpha^2)(3+\alpha+y_*)^3\left(1+2J(r)\sqrt{r}\right)^2 J(r)$.

Since $V_2(\tilde{x},\tilde{y}) = \tilde{x}[\tilde{x}+\alpha\tilde{y}](\tilde{x}+x_*)$, we easily get

$$|V_2(\tilde{x},\tilde{y})| \leq 2(x_*+1)(1+\alpha)\left[\tilde{y}^4+|\tilde{x}|^3+\tilde{x}^2+\tilde{y}^2\right]. \quad (48)$$

Combining (46) and (48) and setting $\bar{d} = 1+x_*+y_*$, simple algebra gives

$$\begin{aligned}|V_2(\tilde{x},\tilde{y})| &\leq 4(x_*+1)(1+\alpha)\sum_{i=2}^4\left\{2\bar{d}\sqrt{V_1+V_1^2}\right\}^i \\ &\leq p_1\big(V_1(\tilde{x},\tilde{y})\big)V_1(\tilde{x},\tilde{y}),\end{aligned}$$

where $p_1(r) = 640(x_*+1)(\alpha+1)\bar{d}^4(1+r)^3$. Since we also have

$$\begin{aligned}&\textstyle\sum_{i=1}^2 \mathcal{N}_i(\tilde{x},\tilde{y}) \geq \tilde{\rho}(V_1(\tilde{x},\tilde{y}))V_1(\tilde{x},\tilde{y}), \quad \text{where} \\ &\tilde{\rho}(r) = \frac{d}{1+r^2},\end{aligned}$$

and since

$$\begin{aligned}U_2(\tilde{x},\tilde{y}) &= V_2(\tilde{x},\tilde{y}) + \left[p_1\big(V_1(\tilde{x},\tilde{y})\big)+1\right]V_1(\tilde{x},\tilde{y}) \\ &\geq V_1(\tilde{x},\tilde{y})\end{aligned}$$

everywhere, it follows from (27)-(29) that the desired strict Lyapunov-like function we get is

$$\begin{aligned}S(\tilde{x},\tilde{y}) &= V_2(\tilde{x},\tilde{y}) + \left[p_1\big(V_1(\tilde{x},\tilde{y})\big)+1\right]V_1(\tilde{x},\tilde{y}) \\ &\quad + \frac{8}{\underline{d}}\int_0^{2V_1(\tilde{x},\tilde{y})}(1+r^2)\phi_1^2(r)\,\mathrm{d}r.\end{aligned} \quad (49)$$

Moreover, S is a strict Lyapunov function because $V_1(\tilde{x},\tilde{y})$ goes to infinity when $\tilde{x}$ goes to $-x_*$ or $+\infty$, or when $\tilde{y}$ goes to $-y_*$ or $+\infty$.

Remark 6: Due to its restricted state space, it is not possible to apply Theorem 1 to the Lotka-Volterra example directly. However, a change of variables that transforms the state space to all of Euclidean space makes it possible to apply to Theorem 1. See Appendix F for details.

2) ISS and iISS: We can use our strict Lyapunov function constructions to quantify the effects of uncertainty in the Lotka-Volterra dynamics. We illustrate this by showing that the dynamics are (i)ISS with respect to additive uncertainty in the death rate Δ for the predator. Using the coordinate change and constants (34), this means that we replace the constant $d > 0$ with $d + \mathbf{u}$ in the dynamics (35), where $\mathbf{u} : [0,\infty) \to \mathbb{R}$ is a measurable essentially bounded uncertainty, and where d now represents the nominal (or estimated) value of the parameter. Later, we impose a bound on the allowable values for $|\mathbf{u}|_\infty$. We continue to use d in the formulas (36) for x_* and y_*; we do not introduce uncertainty in the equilibrium values. Set $\delta\mathcal{B}_r = \{x \in \mathbb{R}^r : |x| \leq \delta\}$ for any constants $r \in \mathbb{N}$ and $\delta > 0$.

We first define an appropriately restricted state space. Along the trajectories of (35), with d replaced by $d + \mathbf{u}$, we have $\dot{x} + \dot{y} = x(1-x) - (d+\mathbf{u})y$. Hence, if $|\mathbf{u}|_\infty \leq d/2$ with $(x,y) \in (0,\infty)^2$, then $\dot{x}+\dot{y} < 0$ when $x + y > 1 + \frac{2}{d}$ (by separately considering the cases $x > 1$ and $x \leq 1$). Therefore, we restrict to disturbances satisfying $|\mathbf{u}|_\infty \leq d/2$ and the forward invariant set

$$\begin{aligned}&\mathcal{S} = \left\{(x,y) \in (0,\infty)^2 : x + y \leq \mathcal{B}\right\}, \quad \text{where} \\ &\mathcal{B} = 1 + \tfrac{2}{d} + y_*\ ,\end{aligned} \quad (50)$$

where we added y_* to ensure that $(x_*,y_*) \in S$. The corresponding perturbed error dynamics

$$\left\{\begin{aligned}\dot{\tilde{x}} &= -[\tilde{x}+\alpha\tilde{y}](\tilde{x}+x_*) \\ \dot{\tilde{y}} &= \alpha\tilde{x}(\tilde{y}+y_*) - \mathbf{u}y\ ,\end{aligned}\right. \quad (51)$$

has the state space $\mathcal{X}^\flat = \{(\tilde{x},\tilde{y}) : (x,y) \in \mathcal{S}\}$ and a control set U we will specify. Our strategy is to build an appropriate strict Lyapunov function for (51) for the special case where $\mathbf{u} \equiv 0$ (i.e., (37)), which we then use to prove ISS of (51) with respect to the uncertainty $\mathbf{u}$.

To account for the restricted state space of the system, we use the following definitions. Given an open subset $\mathcal{D}$ of a Euclidean space that contains the origin, we say that a positive definite function $\bar{\alpha} : \mathcal{D} \to [0,\infty)$ is a modulus with respect to $\mathcal{D}$ provided $\bar{\alpha}(p) \to \infty$ as $|p| \to \infty$ or as $\mathrm{dist}(p,\partial\mathcal{D}) \to 0$ (with p remaining in $\mathcal{D}$). We say that (51) is ISS with respect to $\mathbf{u}$ provided there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, and a modulus with respect to $(-x_*,\infty) \times (-y_*,\infty)$, such that for each disturbance $\mathbf{u} : [0,\infty) \to U$ and each trajectory $(\tilde{x},\tilde{y}) : [0,\infty) \to \mathcal{X}^\flat$ of (51) corresponding to $\mathbf{u}$, we have

$$\left|(\tilde{x},\tilde{y})(t)\right| \leq \beta\Big(\bar{\alpha}\big((\tilde{x},\tilde{y})(0)\big), t\Big) + \gamma(|\mathbf{u}|_\infty) \quad \forall t \geq 0. \quad (52)$$

To simplify the statements of our results, we use the constants

$$\begin{aligned}&K_0 = 2\left[\frac{(3+\alpha)^2}{2}+\alpha^2\right]\mathcal{B}^2\ , \\ &K = \mathcal{B}^2\max\left\{(3+\alpha)^2+2\alpha^2, 2\max\{9,3\alpha^2\}\right\}\ , \\ &\hat{K} = \frac{\min\left\{32x_*, x_*^2\alpha^2 y_*\right\}}{16[K+\mathcal{B}^2\max\{9,3\alpha^2\}]}\ ,\ \bar{U} = \frac{\min\{\hat{K},\theta\}}{4(\alpha\mathcal{B}^3+K\mathcal{B})} \\ &\text{and}\ \ \theta = \min\left\{\frac{K_0 x_*^2}{8}, \frac{K_0 x_*^2 y_*^2\alpha^2}{8(x_*+2\sqrt{K_0})^2}\right\}\ ,\end{aligned} \quad (53)$$

where $\mathcal{B}$ is from (50). We continue to use the functions V_1 and V_2 from the preceding subsection. We prove the following (but see Remark 7 for integral ISS results under a less stringent disturbance bound, and Section VII-B3 for a specific numerical example):

Theorem 3: The system (51) is ISS with respect to disturbances $\mathbf{u}$ valued in $U = \bar{U}\mathcal{B}_1$.

Proof: The time derivatives of the functions V_1 and V_2 defined in Section VII-B, along the trajectories of (51) in $\mathcal{X}^\flat$, satisfy $\dot{V}_1 = -\tilde{x}^2 - \mathbf{u}\tilde{y} \leq -\tilde{x}^2 + \mathcal{B}|\mathbf{u}|$ and

$$\begin{aligned}\dot{V}_2 &= -(\tilde{x}+\alpha\tilde{y})^2x^2 \\ &\quad +\{-\tilde{x}(2\tilde{x}+x_*) - \alpha\tilde{y}\tilde{x}\}(\tilde{x}+\alpha\tilde{y})x \\ &\quad +\tilde{x}[\alpha^2\tilde{x} - \alpha\mathbf{u}]xy \\ &\leq -\tfrac{1}{2}(\tilde{x}+\alpha\tilde{y})^2x^2 + \tfrac{1}{2}\{\tilde{x}(2\tilde{x}+x_*)+\alpha\tilde{y}\tilde{x}\}^2 \\ &\quad +\alpha^2\tilde{x}^2xy - \alpha\tilde{x}\mathbf{u}xy\ ,\end{aligned} \quad (54)$$

by the triangle inequality. Since $(x,y)\in\mathcal{S}$, we deduce that

$$\begin{array}{rcl}\dot V_2 & \le & -\frac{1}{2}(\tilde x+\alpha\tilde y)^2x^2+\frac{(3+\alpha)^2\mathcal{B}^2}{2}\tilde x^2\\ & & +\left[\alpha^2\tilde x^2+\alpha\mathcal{B}|\mathbf{u}|\right]\mathcal{B}^2\\ & \le & -\frac{1}{2}(\tilde x+\alpha\tilde y)^2x^2+\left[\frac{(3+\alpha)^2}{2}+\alpha^2\right]\mathcal{B}^2\tilde x^2\\ & & +\alpha\mathcal{B}^3|\mathbf{u}|\ .\end{array}\tag{55}$$

On the other hand, according to (A.6) with the choices $A=\tilde x/x_*$ and then $A=\tilde y/y_*$, we have

$$\begin{array}{rcl}\tilde x-x_*\ln\left(1+\frac{\tilde x}{x_*}\right) & \ge & \frac{\tilde x^2}{2(2x_*+x)}\quad\text{and}\\ \tilde y-y_*\ln\left(1+\frac{\tilde y}{y_*}\right) & \ge & \frac{\tilde y^2}{2(2y_*+y)}\ ,\end{array}\tag{56}$$

and therefore

$$V_1(\tilde x,\tilde y)\ \ge\ \frac{\tilde x^2}{2(2x_*+x)}+\frac{\tilde y^2}{2(2y_*+y)}\ \ge\ \frac{\tilde x^2+\tilde y^2}{6\mathcal{B}}\tag{57}$$

for all $(x,y)\in\mathcal{S}$. Hence, for all $(x,y)\in\mathcal{S}$, we have

$$\begin{array}{rcl}|V_2(\tilde x,\tilde y)| & = & |\tilde x[\tilde x+\alpha\tilde y]x|\ \le\ (\tilde x^2+\alpha|\tilde x\tilde y|)\mathcal{B}\\ & \le & \mathcal{B}\left(\frac{3}{2}\tilde x^2+\frac{\alpha^2}{2}\tilde y^2\right)\\ & \le & \mathcal{B}\max\left\{\frac{3,\alpha^2}{2}\right\}\left(\tilde x^2+\tilde y^2\right)\\ & \le & \mathcal{B}^2\max\{9,3\alpha^2\}V_1(\tilde x,\tilde y)\ .\end{array}\tag{58}$$

Consider the function $\mathcal{U}_K(\tilde x,\tilde y)=V_2(\tilde x,\tilde y)+KV_1(\tilde x,\tilde y)$. It satisfies $\mathcal{U}_K(\tilde x,\tilde y)\ge\left[-\mathcal{B}^2\max\{9,3\alpha^2\}+K\right]V_1(\tilde x,\tilde y)\ge\mathcal{B}^2\max\{9,3\alpha^2\}V_1(\tilde x,\tilde y)$, by our choice of $K\ge K_0$ in (53), and $\dot{\mathcal{U}}_K\le-\mathcal{Q}(\tilde x,\tilde y)+\overline{\mathcal{B}}|\mathbf{u}|$, where $\overline{\mathcal{B}}=\alpha\mathcal{B}^3+K\mathcal{B}$ and $\mathcal{Q}(\tilde x,\tilde y)=\frac{1}{2}(\tilde x+\alpha\tilde y)^2x^2+\frac{K_0}{2}\tilde x^2$. We consider two cases:

First case: $\mathcal{Q}(\tilde x,\tilde y)\ge\theta$. Then

$$\dot{\mathcal{U}}_K\ \le\ -\theta\frac{\mathcal{U}_K(\tilde x,\tilde y)}{1+\mathcal{U}_K(\tilde x,\tilde y)}+\overline{\mathcal{B}}|\mathbf{u}|\ .\tag{59}$$

Second case: $\mathcal{Q}(\tilde x,\tilde y)\le\theta$. Then

$$|\tilde x|\le\sqrt{\frac{2}{K_0}\theta}\tag{60}$$

and therefore the inequality $\theta\le\frac{K_0x_*^2}{8}$ implies that

$$|\tilde x|\le\frac{x_*}{2},\ \text{ hence }\ \frac{x_*}{2}\le x\ .\tag{61}$$

Moreover, $|\tilde x+\alpha\tilde y|x\le\sqrt{2\theta}$. It follows from (61) that

$$|\tilde x+\alpha\tilde y|\le\frac{2}{x_*}\sqrt{2\theta},\ \text{ hence }\ |\tilde y|\le\frac{2}{x_*\alpha}\sqrt{2\theta}+\frac{1}{\alpha}|\tilde x|\ .\tag{62}$$

We deduce from (60) that

$$\begin{array}{rcl}|\tilde y| & \le & \frac{2}{x_*\alpha}\sqrt{2\theta}+\frac{1}{\alpha}\sqrt{\frac{2}{K_0}\theta}\\ & \le & \frac{\sqrt{2}\left(x_*+2\sqrt{K_0}\right)}{\sqrt{K_0}x_*\alpha}\sqrt{\theta}\ \le\ \frac{y_*}{2}\ ,\end{array}\tag{63}$$

by our choice of θ.

Next, one can easily prove that for all $A\in\left[-\frac{1}{2},\frac{1}{2}\right]$, we have $A-\ln(1+A)\le A^2$, and therefore, when $|\tilde x|\le\frac{x_*}{2}$ and $|\tilde y|\le\frac{y_*}{2}$, we have

$$V_1(\tilde x,\tilde y)\le\frac{\tilde x^2}{x_*}+\frac{\tilde y^2}{y_*}\ .\tag{64}$$

Since the definition of $\mathcal{U}_K$ and (58) imply that $\mathcal{U}_K(\tilde x,\tilde y)\le\left(K+\mathcal{B}^2\max\left\{9,3\alpha^2\right\}\right)V_1(\tilde x,\tilde y)$, we get

$$\mathcal{U}_K(\tilde x,\tilde y)\le\overline{K}\left[\frac{\tilde x^2}{x_*}+\frac{\tilde y^2}{y_*}\right]\ ,\tag{65}$$

where $\bar K=K+\mathcal{B}^2\max\{9,3\alpha^2\}$. Also, the inequality $x\ge\frac{x_*}{2}$ from (61) implies that

$$\mathcal{Q}(\tilde x,\tilde y)\ \ge\ \frac{x_*^2}{8}(\tilde x+\alpha\tilde y)^2+\frac{K_0}{2}\tilde x^2\ .\tag{66}$$

By separately considering the possibilities $|\tilde x|\ge\frac{1}{4}\alpha|\tilde y|$ and $|\tilde x|\le\frac{1}{4}\alpha|\tilde y|$ and noting that $K_0\ge 9\mathcal{B}^2\ge 9$, it follows from (66) that

$$\begin{array}{rcl}\mathcal{Q}(\tilde x,\tilde y) & \ge & \frac{x_*^2}{16}\alpha^2\tilde y^2+2\tilde x^2\\ & \ge & \min\left\{\frac{x_*^2}{16}\alpha^2y_*,2x_*\right\}\left[\frac{\tilde x^2}{x_*}+\frac{\tilde y^2}{y_*}\right]\ .\end{array}\tag{67}$$

Combining (65) and (67) yields

$$\mathcal{U}_K(\tilde x,\tilde y)\ \le\ \overline{K}\frac{\mathcal{Q}(\tilde x,\tilde y)}{\min\left\{2x_*,\frac{x_*^2}{16}\alpha^2y_*\right\}}\ .\tag{68}$$

Recalling the estimate $\dot{\mathcal{U}}_K\le-\mathcal{Q}(\tilde x,\tilde y)+\overline{\mathcal{B}}|\mathbf{u}|$, we deduce that $\dot{\mathcal{U}}_K\le-\widehat K\mathcal{U}_K(\tilde x,\tilde y)+\overline{\mathcal{B}}|\mathbf{u}|$.

We deduce that, in both cases,

$$\dot{\mathcal{U}}_K\le-\mho\frac{\mathcal{U}_K(\tilde x,\tilde y)}{1+\mathcal{U}_K(\tilde x,\tilde y)}+\overline{\mathcal{B}}|\mathbf{u}|\ ,\tag{69}$$

where $\mho=\min\{\widehat K,\theta\}$. Let $\overline{\mathcal{U}_K}(\tilde x,\tilde y)=\mathcal{U}_K(\tilde x,\tilde y)e^{\mathcal{U}_K(\tilde x,\tilde y)}$. Then

$$\begin{array}{rcl}\dot{\overline{\mathcal{U}_K}} & = & e^{\mathcal{U}_K(\tilde x,\tilde y)}\left[1+\mathcal{U}_K(\tilde x,\tilde y)\right]\dot{\mathcal{U}}_K\\ & \le & e^{\mathcal{U}_K(\tilde x,\tilde y)}\left[\left\{-\mho+\overline{\mathcal{B}}|\mathbf{u}|\right\}\mathcal{U}_K(\tilde x,\tilde y)+\overline{\mathcal{B}}|\mathbf{u}|\right]\ .\end{array}$$

Therefore, when $|\mathbf{u}|_\infty\le\frac{\mho}{2\overline{\mathcal{B}}}$, we have

$$\begin{array}{rcl}\dot{\overline{\mathcal{U}_K}} & \le & e^{\mathcal{U}_K(\tilde x,\tilde y)}\left[-\frac{\mho}{2}\mathcal{U}_K(\tilde x,\tilde y)+\overline{\mathcal{B}}|\mathbf{u}|\right]\\ & \le & -\frac{\mho}{2}\overline{\mathcal{U}_K}(\tilde x,\tilde y)+\overline{\mathcal{B}}|\mathbf{u}|e^{\mathcal{U}_K(\tilde x,\tilde y)}\\ & \le & -\frac{\mho}{2}\overline{\mathcal{U}_K}(\tilde x,\tilde y)+\overline{\mathcal{B}}|\mathbf{u}|\left[e^{\mathcal{U}_K(\tilde x,\tilde y)}-1\right]\\ & & +\overline{\mathcal{B}}|\mathbf{u}|\\ & \le & -\frac{\mho}{2}\overline{\mathcal{U}_K}+\overline{\mathcal{B}}|\mathbf{u}|\overline{\mathcal{U}_K}+\overline{\mathcal{B}}|\mathbf{u}|\end{array}\tag{70}$$

where the last inequality used the condition $e^a-1\le ae^a$ for all $a\ge 0$. Therefore, when $\overline{\mathcal{B}}|\mathbf{u}|\le\frac{\mho}{4}$, we obtain

$$\dot{\overline{\mathcal{U}_K}}\ \le\ -\frac{\mho}{4}\overline{\mathcal{U}_K}(\tilde x,\tilde y)+\overline{\mathcal{B}}|\mathbf{u}|\ .\tag{71}$$

The desired ISS inequality (52) now follows from a slight variant of the standard arguments [20]. ∎

Remark 7: The Lyapunov function construction in the preceding proof can be used to explicitly construct the functions β, γ, and $\bar\alpha$ in the ISS estimate (52). The inequality (69) implies that $\mathcal{U}_K$ is an iISS Lyapunov function for the Lotka-Volterra errors dynamics (51) when the disturbance $\mathbf{u}$ satisfies the less stringent bound $|\mathbf{u}|_\infty\le\frac{d}{2}$; see [20] for the original treatment of iISS, and see [2], [3], [21] for extensive iISS discussions and results. In fact, a slight variant of the iISS arguments from [3] in conjunction with (69) and the growth properties of $\mathcal{U}_K$ can be used to explicitly construct functions

$\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, a constant $\bar{G} > 0$, and a modulus $\bar{\alpha}$ with respect to $(-x_*, \infty) \times (-y_*, \infty)$, such that for each disturbance $\mathbf{u} : [0, \infty) \to [-d/2, d/2]$ and each trajectory $(\tilde{x}, \tilde{y}) : [0, \infty) \to \mathcal{X}^\flat$ of (51) corresponding to $\mathbf{u}$, we have

$$\begin{array}{l}\gamma\Big(|(\tilde{x},\tilde{y})(t)|\Big) \leq \\ \beta\Big(\bar{\alpha}\big((\tilde{x},\tilde{y})(0)\big), t\Big) + \bar{G}\int_0^t |\mathbf{u}(r)|\mathrm{d}r \;\; \forall t \geq 0.\end{array} \quad (72)$$

We next illustrate these ideas in simulations.

3) Simulations: To illustrate our findings, we simulated the dynamics (51) using the parameter values $\alpha = 2$, $d = 1$, $x_* = 0.5$, and $y_* = 0.25$, corresponding to the parameter choices

$$a = \gamma = \beta = \Delta = 0.5 \;\text{ and }\; L = 2 \quad (73)$$

in the original model. Hence, (51) is iISS with respect to disturbances that are bounded by 0.5. We chose the disturbance $\mathbf{u}(t) = 0.49e^{-t}$. In Figures 1-2, we plotted the corresponding levels of ζ and χ, which are related to x and y in terms of the coordinate changes (34).

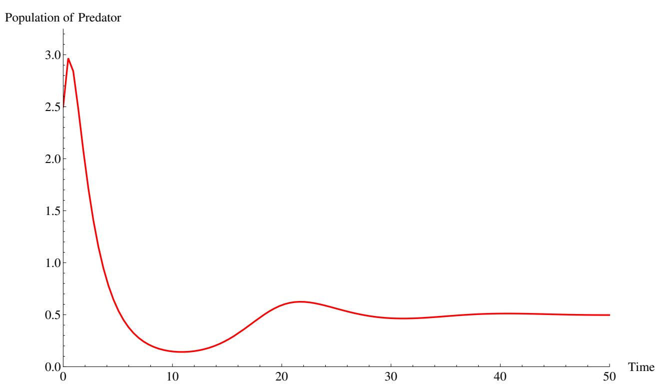

Fig. 1. Population of Predator ζ in Lotka-Volterra Dynamics (51) with Parameters (73) and $\mathbf{u}(t) = 0.49e^{-t}$

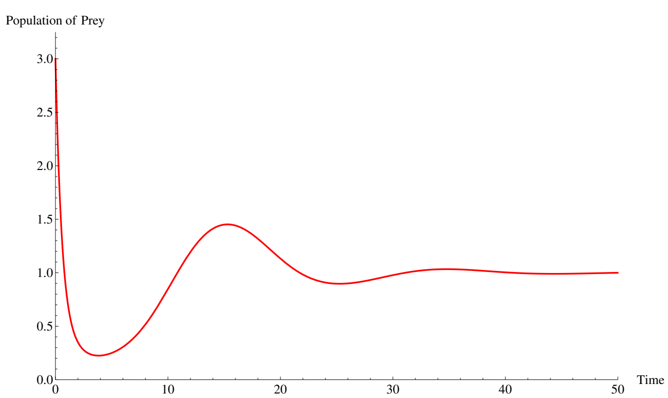

Fig. 2. Population of Prey χ in Lotka-Volterra Dynamics (51) with Parameters (73) and $\mathbf{u}(t) = 0.49e^{-t}$

If $x(t) \to x_* = 0.5$ and $y(t) \to y_* = 0.25$, then the coordinate changes (34) give

$$\zeta(t) \to 0.25\frac{\beta L}{a} = 0.5 \;\text{ and }\; \chi(t) \to 0.5L = 1, \quad (74)$$

which is what we see in the figures. This shows the robustness of the convergence in the face of the disturbance $\mathbf{u}$.

VIII. Conclusion

We gave new methods for building global strict Lyapunov functions under LaSalle conditions. The novelty of our first result is in the generality of its assumptions. The novelty of our second is in the local properties of our strict Lyapunov functions and its applicability on general state spaces. As a byproduct, we exhibited a general class of auxiliary functions for which the Matrosov theorem from [14] applies. We illustrated our work using a robustness analysis for the Lotka-Volterra model.

Appendix A
A Useful Lower Bound

We used the following simple lemma in Section V:

Lemma A.1: For each continuous positive definite function $\rho : [0, \infty) \to [0, \infty)$, we can find a function $\omega \in \mathcal{K}_\infty \cap C^1$ and a strictly increasing everywhere positive function $K \in C^1$ such that

$$\rho(r) \geq \frac{\omega(r)}{K(r)} \quad (A.1)$$

for all $r \geq 0$.

Proof: By replacing ρ with

$$\rho_{\text{new}}(r) = \begin{cases} r\min\{\rho(q) : r \leq q \leq 1\}, & 0 \leq r \leq 1 \\ \min\{\rho(q) : 1 \leq q \leq r\}, & r > 1 \end{cases}$$

without relabeling and noting that $\rho_{\text{new}}(r) \leq \rho(r)$ for all $r \geq 0$, we can assume that ρ is strictly increasing on $[0, 1]$ and nonincreasing on $[1, \infty)$. Notice that

$$\rho(r) \;=\; \frac{\omega_0(r)}{K_0(r)} \quad (A.2)$$

for all $r \geq 0$, where ω_0 and K_0 are the increasing continuous functions

$$\begin{array}{l}\omega_0(r) = \begin{cases} \frac{\rho(r)}{\rho(1)}, & 0 \leq r \leq 1 \\ r, & r \geq 1 \end{cases} \quad \text{and} \\ K_0(r) = \begin{cases} \frac{1}{\rho(1)}, & 0 \leq r \leq 1 \\ \frac{r}{\rho(r)}, & r \geq 1 \end{cases} \quad . \end{array} \quad (A.3)$$

We can then satisfy (A.1) by picking any function $\omega \in \mathcal{K}_\infty \cap C^1$ such that $\omega(r) \leq \omega_0(r)$ for all $r \geq 0$ and any strictly increasing C^1 function K such that $K(r) \geq K_0(r)$ for all $r \geq 0$. ■

Appendix B
Verifying Estimates (25)

We only show the second estimate in (25); the other estimate in (25) is handled similarly. We maintain the notation from Section VI. Since $f(0) = 0$, all of the functions $L_f^i h_l(x)$ are zero at the origin and sufficiently smooth for all $i \in \mathbb{N}$. Also, Assumption 1 provides a positive definite quadratic lower bound for V_1 near the origin. Moreover, the fact that V_1 is a storage function implies that there exists a function $\underline{\alpha} \in \mathcal{K}_\infty$ such that $V_1(x) \geq \underline{\alpha}(|x|)$ for all $x \in \mathcal{X}$. Hence,

$$\begin{array}{rcl}\sum_{l=1}^m |L_f^i h_l(x)| & \leq & |x|\mathcal{G}_1(|x|) \\ & \leq & \bar{\kappa}\sqrt{V_1(x)}\mathcal{G}_1\big(\underline{\alpha}^{-1}(V_1(x))\big)\end{array}$$

for all $i \in \{3, \ldots, N\}$ for some increasing everywhere positive function $\mathcal{G}_1$ and some constant $\bar{\kappa} > 0$ in some neighborhood $\mathcal{O}$ of the origin (by our choice of $\underline{\alpha}$ and the fact that V_1 is bounded from below by a positive definite quadratic function near 0). We can also find a $\mathcal{G}_2 \in \mathcal{K}_\infty$ so that

$$\frac{\sum_{l=1}^{m} |L_f^i h_l(x)|}{\sqrt{\underline{\alpha}(|x|)}} \leq \mathcal{G}_2(|x|) \ \ \forall i \in \{3, \ldots, N\}$$

on $\mathbb{R}^n \setminus \mathcal{O}$. Hence, we can take $\phi_1(r) = 1 + \bar{\kappa}\mathcal{G}_1(\underline{\alpha}^{-1}(r)) + \mathcal{G}_2(\underline{\alpha}^{-1}(r))$.

APPENDIX C
STRICT LYAPUNOV FUNCTION FOR (30)

We construct the functions needed for the strict Lyapunov function construction for (30). We use the notation from Section VII-A and the proof of Theorem 1. Then

$$\begin{aligned} \dot{a}_3 = & -2\sin(2t)x_1^2 + 2(-3\sin(t) + \cos(t))x_1x_2 \\ & +4\cos^3(t)x_1x_2 \\ & +2(\sin(2t) + \cos(t)\,[3\cos(t) + \sin(t)])x_2^2 \\ & -2\left[(3\cos(t)+\sin(t))x_1 + 2(1+\sin^2(t))x_2\right] \\ & \times[\cos(t)x_1 + x_2] \ . \end{aligned} \tag{A.4}$$

Applying the relation $|x_1x_2| \leq \frac{1}{2}(x_1^2 + x_2^2)$ and collecting coefficients of x_1^2 and x_2^2 readily gives $\max\{|a_i(t,x)| : t \in \mathbb{R}, 1 \leq i \leq 4\} \leq 64V(x)$, so $\Gamma(r) = 320r + 1$ satisfies our requirements. Since $\ell = 3$, $T = 2\pi$, and $\tau = \frac{\pi}{4}$, taking $\omega(r) = r^2$ and $K(r) = 200(r+1)$ gives

$$\begin{aligned} \Omega(v) &= \tfrac{2\tau\omega(v)}{3T(\ell-2)\Gamma^2(v)K(v)} = \tfrac{v^2}{2400(v+1)(320v+1)^2}, \\ \text{so } k_2(v) &= \omega^{2^{\ell-1}}(v) = v^8 \ , \ \text{ and} \\ k_1(v) &= \tfrac{k_2(v)}{\Omega(v)} = 2400v^6(v+1)(320v+1)^2 \ . \end{aligned} \tag{A.5}$$

Since $\omega(v) = v^2$, our requirement (14) on G is satisfied if $G'(v) \geq \frac{2\pi}{K(v)}\big(k_2(v)[2vK(v) - 200v^2] + k_2'(v)v^2\big)$ for all $v \geq 0$, which holds if $G(v) = 5\pi v^9$.

Since $|M_1(t,x)| = |a_1(t,x)a_2(t,x) + 160V^2(x) + V(x)| \leq 172V^2(x) + V(x)$ and $|M_2(t,x)| = |a_1(t,x)a_2(t,x) + a_2(t,x)a_3(t,x) + 160V^2(x) + V(x)| \leq 268V^2(x) + V(x)$, we can satisfy (13) using $k_0(v) = (k_1'(v) + k_2'(v))(268v^2 + v) + 1$. Also, $|S_2(t,x)| \leq G(V(x)) + 2\pi V^{10}(x)$ and $|S_1(t,x)| \leq \max\{k_1(V(x)), k_2(V(x))\}[268V^2(x) + V(x)] + k_0(V(x))V(x)$. Therefore, we can take

$$\kappa(v) = [k_1(v) + k_2(v)](268v^2 + v) + k_0(v)v + G(v) + 2\pi v^{10} + 1.$$

The formula for $V^\sharp$ is now immediate from plugging the preceding functions into (10) from Theorem 1.

APPENDIX D
USEFUL INEQUALITIES

We used the following simple lemma in Section VII-B:

Lemma A.2: For all $A \in (-1, \infty)$, the following inequalities hold:

$$\begin{aligned} & A - \ln(1+A) \geq \tfrac{A^2}{2(1+|A|)} \quad \text{and} \\ & |A| \leq 2\sqrt{[A - \ln(1+A)] + [A - \ln(1+A)]^2} \ . \end{aligned} \tag{A.6}$$

Proof: First assume that $A \in (-1, 0)$. Then

$$A - \ln(1+A) = \int_0^A \frac{m}{1+m}\mathrm{d}m \geq \frac{A^2}{2} \ .$$

If, on the other hand, $A \geq 0$, then

$$\begin{aligned} A - \ln(1+A) &= \int_0^A \frac{m}{1+m}\mathrm{d}m \\ &\geq \int_0^A \frac{m}{1+A}\mathrm{d}m = \frac{A^2}{2(1+A)} \ , \end{aligned}$$

which gives the first condition in (A.6).

To prove the second condition in (A.6), notice that the first condition implies that for all $A > -1$,

$$2[A - \ln(1+A)] + 2|A|[A - \ln(1+A)] \ \geq \ A^2 \ . \tag{A.7}$$

Applying the triangle inequality $c_1c_2 \leq \frac{c_1^2}{2} + \frac{c_2^2}{2}$ gives

$$\{|A|\}\{2[A - \ln(1+A)]\} \leq \frac{1}{2}A^2 + 2[A - \ln(1+A)]^2,$$

which we can combine with (A.7) to deduce the second condition (A.6). ■

APPENDIX E
VERIFYING (41) FOR LOTKA-VOLTERRA EXAMPLE

We show that (41) is satisfied for an appropriate constant $\underline{d} > 0$. We continue to use the notation of Section VII-B. Consider the function

$$E(p,q) \ = \ p - q\ln\left(1 + \frac{p}{q}\right),$$

which is defined for all $p > -q$ when $q > 0$. For each $q > 0$,

$$E(p,q) \to \infty \ \text{ as } \ p \to -q^+. \tag{A.8}$$

Also, our choice (39) of V_1 says $V_1(\tilde{x}, \tilde{y}) = E(\tilde{x}, x_*) + E(\tilde{y}, y_*)$. Setting $\mathcal{N}_1(\tilde{x}, \tilde{y}) = 0.5h_1^2(\tilde{x}, \tilde{y})$ and $\mathcal{N}_2(\tilde{x}, \tilde{y}) = [L_f h_1(\tilde{x}, \tilde{y})]^2$ as before, we claim that we can find a constant

$$\delta \in \left(0, \tfrac{1}{2}\min\{x_*, y_*\}\right] \tag{A.9}$$

so that

$$\begin{aligned} \textstyle\sum_{i=1}^{2} \mathcal{N}_i(\tilde{x}, \tilde{y}) &= \tfrac{1}{2}\tilde{x}^2 + [(\tilde{x} + \alpha\tilde{y})(\tilde{x} + x_*)]^2 \\ &\geq \tfrac{\delta^2 V_1(\tilde{x}, \tilde{y})}{1 + V_1^2(\tilde{x}, \tilde{y})} \end{aligned} \tag{A.10}$$

for all $(\tilde{x}, \tilde{y})$ in the set

$$\mathcal{D} = \{(\tilde{x}, \tilde{y}) \in \mathcal{X} : \tilde{x} \leq -x_* + \delta \text{ or } \tilde{y} \leq -y_* + \delta\}.$$

To check this claim, first note that for any δ satisfying (A.9), $\sum_{i=1}^{2} \mathcal{N}_i(\tilde{x}, \tilde{y})$ is bounded from below on $\mathcal{D}$ by a positive constant depending on δ. (Indeed, if $\tilde{x} \leq -x_* + \delta$, then

$$\sum_{i=1}^{2} \mathcal{N}_i(\tilde{x}, \tilde{y}) \geq \frac{1}{8}x_*^2.$$

If on the other hand $\tilde{x} \geq -x_* + \delta$, then

$$\sum_{i=1}^{2} \mathcal{N}_i(\tilde{x}, \tilde{y}) \geq \frac{1}{2}\delta^2\tilde{x}^2 + \delta^2(\tilde{x} + \alpha\tilde{y})^2,$$

since $\delta \leq x_* < 1$. Since $\alpha > 0$, $\frac{1}{2}\tilde{x}^2 + (\tilde{x} + \alpha\tilde{y})^2$ is a positive definite quadratic form and so admits a constant $c_* \in (0, 1)$

so that

$$\frac{1}{2}\tilde{x}^2 + (\tilde{x}+\alpha\tilde{y})^2 \geq c_*(\tilde{x}^2+\tilde{y}^2)$$

on $\mathbb{R}^2$, which is bounded below by $c_* y_*^2/4$ when $\tilde{y} \leq -y_* + \delta$. Hence,

$$\sum_{i=1}^{2} \mathcal{N}_i(\tilde{x},\tilde{y}) \;\geq\; \delta^2 \frac{c_*}{8} \min\left\{x_*^2, y_*^2\right\} \;=:\; \underline{m}(\delta) \qquad \text{(A.11)}$$

on $\mathcal{D}$.) Reducing $\delta > 0$ and recalling (A.8) guarantees that

$$\frac{\delta^2 V_1(\tilde{x},\tilde{y})}{1+V_1^2(\tilde{x},\tilde{y})} \;\leq\; \underline{m}(\delta) \qquad \text{(A.12)}$$

on $\mathcal{D}$. The claim now follows by combining (A.11)-(A.12). Fix a constant $\delta > 0$ satisfying the preceding requirements.

We next consider points in $\mathcal{X}\setminus\mathcal{D}$. For each constant $q > 0$, we can find a constant $c(q) > 1$ such that

$$E(p,q) \leq c(q)p^2 \quad \forall p \geq -q+\delta,$$

by applying L'Hôpital's rule to $E(p,q)/p^2$ as $p \to 0$ or $p \to \infty$. Therefore,

$$\begin{aligned} V_1(\tilde{x},\tilde{y}) &= \tilde{x}^2\left(\frac{E(\tilde{x},x_*)}{\tilde{x}^2}\right) + \tilde{y}^2\left(\frac{E(\tilde{y},y_*)}{\tilde{y}^2}\right) \\ &\leq [c(x_*)+c(y_*)](\tilde{x}^2+\tilde{y}^2) \end{aligned}$$

on $\mathcal{X}\setminus\mathcal{D}$ when neither $\tilde{x}$ nor $\tilde{y}$ is zero. Similar reasoning gives

$$V_1(\tilde{x},\tilde{y}) \leq [c(x_*)+c(y_*)](\tilde{x}^2+\tilde{y}^2)$$

on all of $\mathcal{X}\setminus\mathcal{D}$ (by separately considering the possibilities $\tilde{x}=0$ and $\tilde{x}\neq 0$ and similarly for $\tilde{y}$). Moreover, we can find a constant $\underline{c} > 0$ so that

$$\mathcal{N}_1(\tilde{x},\tilde{y}) + \mathcal{N}_2(\tilde{x},\tilde{y}) \geq \underline{c}(\tilde{x}^2+\tilde{y}^2)$$

on $\mathcal{X}\setminus\mathcal{D}$, because $(\tilde{x}+x_*)^2 \geq \delta^2$ on $\mathcal{X}\setminus\mathcal{D}$ and

$$\frac{1}{2}\tilde{x}^2 + \delta^2[\tilde{x}+\alpha\tilde{y}]^2$$

is a positive definite quadratic function (again using the fact that $\alpha > 0$). Therefore,

$$\begin{aligned} &\sum_{i=1}^{2} \mathcal{N}_i(\tilde{x},\tilde{y}) \\ &\geq \left(\frac{\underline{c}}{c(x_*)+c(y_*)}\right)[c(x_*)+c(y_*)](\tilde{x}^2+\tilde{y}^2) \qquad \text{(A.13)} \\ &\geq \left(\frac{\underline{c}}{c(x_*)+c(y_*)}\right) V_1(\tilde{x},\tilde{y}) \end{aligned}$$

on $\mathcal{X}\setminus\mathcal{D}$. It follows from (A.10) that we can take

$$\underline{d} = \min\left\{\tfrac{\underline{c}}{c(x_*)+c(y_*)}, \delta^2\right\}. \qquad \text{(A.14)}$$

Appendix F
Strict Lyapunov Function for Lotka-Volterra Dynamics Using Theorem 1

To further illustrate Theorem 1, we show how it applies to the Lotka-Volterra dynamics, after a change of variables. We take the change of coordinates

$$\xi = \ln(x) \;\text{ and }\; \psi = \ln(y).$$

Taking $x_* = \frac{d}{\alpha} \in (0,1)$ and $y_* = \frac{1-x_*}{\alpha} > 0$ as before, we also set

$$\xi_* = \ln(x_*) \;\text{ and }\; \psi_* = \ln(y_*).$$

This and (35) give

$$\left\{\begin{aligned} \dot{\tilde{\xi}} &= x_*\left[1-e^{\tilde{\xi}}\right] + \theta_1\left[1-e^{\tilde{\psi}}\right] \\ \dot{\tilde{\psi}} &= \theta_2\left[e^{\tilde{\xi}}-1\right] \end{aligned}\right. \qquad \text{(A.15)}$$

for the error variables $\tilde{\xi} = \xi - \xi_*$ and $\tilde{\psi} = \psi - \psi_*$, where

$$\theta_1 = \alpha y_* \;\text{ and }\; \theta_2 = \alpha x_*.$$

The state space for (A.15) is $\mathbb{R}^2$.

We show how (A.15) is covered by Theorem 1. Due to space constraints, we do not construct the strict Lyapunov function for (A.15) from Theorem 1, since we already constructed the strict Lyapunov function (49) for the Lotka-Volterra error dynamics using Theorem 2. Let

$$V(\tilde{\xi},\tilde{\psi}) \;=\; \theta_2\left[e^{\tilde{\xi}}-1-\tilde{\xi}\right] + \theta_1\left[e^{\tilde{\psi}}-1-\tilde{\psi}\right]. \qquad \text{(A.16)}$$

Then V is a storage function whose time derivative along the trajectories of (A.15) satisfies

$$\begin{aligned} \dot{V} &= \theta_2\left[e^{\tilde{\xi}}-1\right]\left[x_*\left(1-e^{\tilde{\xi}}\right)+\theta_1\left(1-e^{\tilde{\psi}}\right)\right] \\ &\quad +\theta_1\left[e^{\tilde{\psi}}-1\right]\theta_2\left[e^{\tilde{\xi}}-1\right] \\ &= -\theta_3\left[e^{\tilde{\xi}}-1\right]^2, \end{aligned}$$

where $\theta_3 = \theta_2 x_*$, so V is a nonstrict Lyapunov function for the error dynamics (A.15). Also,

$$a_1 = \theta_3[e^{\tilde{\xi}}-1]^2 \geq 0.$$

Taking $a_2 = -\dot{a}_1$ as before gives

$$\dot{a}_1 \;=\; 2\theta_3 e^{\tilde{\xi}}\left[-x_*\left(1-e^{\tilde{\xi}}\right)^2 - \theta_1\left(e^{\tilde{\xi}}-1\right)\left(e^{\tilde{\psi}}-1\right)\right],$$

so

$$\begin{aligned} \dot{a}_2 &= 2\theta_3 e^{\tilde{\xi}}\dot{\tilde{\xi}}\left[x_*\left(1-e^{\tilde{\xi}}\right)^2+\theta_1\left(e^{\tilde{\xi}}-1\right)\left(e^{\tilde{\psi}}-1\right)\right] \\ &\quad +2\theta_3 e^{\tilde{\xi}}\left[2x_*\left(e^{\tilde{\xi}}-1\right)e^{\tilde{\xi}}\dot{\tilde{\xi}}+\theta_1\left(e^{\tilde{\psi}}-1\right)e^{\tilde{\xi}}\dot{\tilde{\xi}}\right. \\ &\quad \left. +\theta_1\left(e^{\tilde{\xi}}-1\right)e^{\tilde{\psi}}\dot{\tilde{\psi}}\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \tfrac{\dot{a}_2}{2\theta_3 e^{\tilde{\xi}}} &= \dot{\tilde{\xi}}\left[x_*\left(1-e^{\tilde{\xi}}\right)^2+\theta_1\left(e^{\tilde{\xi}}-1\right)\left(e^{\tilde{\psi}}-1\right)\right] \\ &\quad +2x_*\left(e^{\tilde{\xi}}-1\right)e^{\tilde{\xi}}\dot{\tilde{\xi}} \\ &\quad +\theta_1\left(e^{\tilde{\psi}}-1\right)e^{\tilde{\xi}}\dot{\tilde{\xi}}+\theta_1\left(e^{\tilde{\xi}}-1\right)e^{\tilde{\psi}}\dot{\tilde{\psi}} \\ &= \dot{\tilde{\xi}}\left[\tfrac{a_2}{2\theta_3 e^{\tilde{\xi}}}+2x_*\left(e^{\tilde{\xi}}-1\right)e^{\tilde{\xi}}\right] \\ &\quad +\theta_1\theta_2\left(e^{\tilde{\xi}}-1\right)^2 e^{\tilde{\psi}} \\ &\quad +\theta_1\left(e^{\tilde{\psi}}-1\right)e^{\tilde{\xi}}\left[x_*\left(1-e^{\tilde{\xi}}\right)+\theta_1\left(1-e^{\tilde{\psi}}\right)\right] \end{aligned}$$

which gives

$$\begin{aligned}\frac{\dot{a}_2}{2\theta_3 e^{\tilde{\xi}}} &= \left[x_*\left(1-e^{\tilde{\xi}}\right)+\theta_1\left(1-e^{\tilde{\psi}}\right)\right] \\ &\quad \times\left[\frac{a_2}{2\theta_3 e^{\tilde{\xi}}}+2x_*\left(e^{\tilde{\xi}}-1\right)e^{\tilde{\xi}}\right] \\ &\quad +\theta_1\theta_2\left(e^{\tilde{\xi}}-1\right)^2 e^{\tilde{\psi}} \\ &\quad +x_*\theta_1 e^{\tilde{\xi}}\left(e^{\tilde{\psi}}-1\right)\left(1-e^{\tilde{\xi}}\right) \\ &\quad -\theta_1^2\left(e^{\tilde{\psi}}-1\right)^2 e^{\tilde{\xi}}\ .\end{aligned}$$

Then

$$\begin{aligned}\dot{a}_2 &= \left[x_*\left(1-e^{\tilde{\xi}}\right)+\theta_1\left(1-e^{\tilde{\psi}}\right)\right] \\ &\quad \times\left[a_2+4\theta_3 x_*\left(e^{\tilde{\xi}}-1\right)e^{2\tilde{\xi}}\right]+2\theta_1\theta_2 a_1 e^{\tilde{\xi}}e^{\tilde{\psi}} \\ &\quad +2\theta_3 x_*\theta_1 e^{2\tilde{\xi}}\left(e^{\tilde{\psi}}-1\right)\left(1-e^{\tilde{\xi}}\right) \\ &\quad -2\theta_1^2\theta_3\left(e^{\tilde{\psi}}-1\right)^2 e^{2\tilde{\xi}}\end{aligned}$$

and so also

$$\begin{aligned}\dot{a}_2 &= \left[x_*\left(1-e^{\tilde{\xi}}\right)+\theta_1\left(1-e^{\tilde{\psi}}\right)\right]a_2 \\ &\quad +\left[-4x_*^2 e^{2\tilde{\xi}}+2\theta_1\theta_2 e^{\tilde{\xi}}e^{\tilde{\psi}}\right]a_1 \\ &\quad +6\theta_3 x_*\theta_1 e^{2\tilde{\xi}}\left(e^{\tilde{\psi}}-1\right)\left(1-e^{\tilde{\xi}}\right) \\ &\quad -2\theta_1^2\theta_3\left(e^{\tilde{\psi}}-1\right)^2 e^{2\tilde{\xi}}\ .\end{aligned}$$

Taking $a_3 = -\dot{a}_2$ as before, it readily follows that

$$\mathcal{M}(\tilde{\xi},\tilde{\psi}) := a_1(\tilde{\xi},\tilde{\psi}) + a_2^2(\tilde{\xi},\tilde{\psi}) + a_3^2(\tilde{\xi},\tilde{\psi})$$

is positive definite, so (A.15) satisfies the assumptions of Theorem 1.

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REFERENCES

[1] D. Angeli. Input-to-state stability of PD-controlled robotic systems. *Automatica J. IFAC*, 35(7):1285–1290, 1999.

[2] D. Angeli, B. Ingalls, E.D. Sontag, and Y. Wang. Separation principles for input-output and integral-input to state stability. *SIAM J. Control Optim.*, 43(1):256–276, 2004.

[3] D. Angeli, E.D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *IEEE Trans. Automat. Contr.*, 45:1082–1097, June 2000.

[4] F. Clarke, Yu. Ledyaev, and R. Stern. Asymptotic stability and smooth Lyapunov functions. *J. Differential Equations*, 149(1):69-114, 1998.

[5] L. Faubourg and J-B. Pomet. Control Lyapunov functions for homogeneous "Jurdjevic-Quinn" systems. *ESAIM Control Optim. Calc. Var.*, 5:293–311, 2000.

[6] S. Hsu. Limiting behavior for competing species. *SIAM J. Appl. Math*, 34(4):760–763, 1978.

[7] V. Jurdjevic and J.P. Quinn. Controllability and stability. *J. Differential Equations*, 28(3):381–389, 1978.

[8] H. Khalil. *Nonlinear Systems, Third Edition*. Prentice Hall, Upper Saddle River, NJ, 2002.

[9] M. Krstić, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear and Adaptive Control Design*. Wiley, New York, 1995.

[10] T-C. Lee and Z-P. Jiang. A generalization of Krasovskii-LaSalle theorem for nonlinear time-varying systems: converse results and applications. *IEEE Trans. Automat. Contr.*, 50:1147-1163, August 2005.

[11] A. Leung. Limiting behavior for several interacting populations. *Math. Biosci.*, 29(1-2):85–98, 1976.

[12] A. Loría, E. Panteley, D. Popović, and A. Teel. A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems. *IEEE Trans. Automat. Contr.*, 50:183–198, February 2005.

[13] F. Mazenc and M. Malisoff. Further constructions of control-Lyapunov functions and stabilizing feedbacks for systems satisfying the Jurdjevic-Quinn conditions. *IEEE Trans. Automat. Contr.*, 51:360–365, February 2006.

[14] F. Mazenc, M. Malisoff, and O. Bernard. A simplified design for strict Lyapunov functions under Matrosov conditions. *IEEE Trans. Automat. Contr.*, 54:177-183, January 2009.

[15] F. Mazenc and D. Nešić. Strong Lyapunov functions for systems satisfying the conditions of La Salle. *IEEE Trans. Automat. Contr.*, 49:1026–1030, June 2004.

[16] F. Mazenc and L. Praly. Adding integrations, saturated controls, and stabilization for feedforward systems. *IEEE Trans. Automat. Contr.*, 41:1559–1578, November 1996.

[17] R. Outbib and J. Vivalda. On the stabilization of smooth nonlinear systems. *IEEE Trans. Automat. Contr.*, 44:200–202, January 1999.

[18] R. Sepulchre, M. Janković, and P. V. Kokotović. *Constructive Nonlinear Control*. Springer-Verlag, Berlin, 1997.

[19] E.D. Sontag. A "universal" construction of Artstein's theorem on nonlinear stabilization. *Systems Control Lett.*, 13(2):117–123, 1989.

[20] E.D. Sontag. Comments on integral variants of ISS. *Systems and Control Letters*, 34(1-2):93–100, 1998.

[21] E.D. Sontag. Input-to-state stability: Basic concepts and results. In P. Nistri and G. Stefani, editors, *Nonlinear and Optimal Control Theory. Lectures Given at the C.I.M.E. Summer School Held in Cetraro, Italy June 19–29, 2004*, volume 1932 of *Lecture Notes in Mathematics*, pages 163–220. Springer, Berlin, Germany, 2008.

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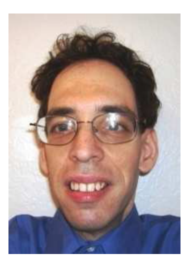

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