Robustness of Nonlinear Systems with Respect to Delay and Sampling of the Controls *

Frédéric Mazenc [a], Michael Malisoff [b], and Thach N. Dinh [a]

[a] *EPI INRIA DISCO, Laboratoire des Signaux et Systèmes, L2S - CNRS - Supélec, 3 rue Joliot-Curie, 91192 Gif sur Yvette cedex, France*

[b] *Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918 USA*

Abstract

We consider continuous time nonlinear time varying systems that are globally asymptotically stabilizable by state feedbacks. We study the stability of these systems in closed loop with controls that are corrupted by both delay and sampling. We establish robustness results through a Lyapunov approach of a new type.

Key words: Nonlinear systems, sampling, delay, stability

1 Introduction

Sampling in controllers is a well known problem that has been studied in many contributions (Monaco & Normand-Cyrot, 2007; Monaco, Normand-Cyrot, Tiefense, 2011; Nesic & Teel, 2001; Stramigioli, Secchi, & Van der Schaft, 2002). Similarly, the last two decades have witnessed much research devoted to nonlinear systems with delay (Bekiaris-Liberis & Krstic, 2012; Karafyllis & Jiang, 2011; Mazenc & Malisoff, 2010; Mazenc, Niculescu, & Bekaik, 2011; Pepe, Karafyllis, & Jiang, 2008; Sharma, Gregory, & Dixon, 2011; Yeganefar, Pepe, & Dambrine, 2008). Although sampling and delay occur simultaneously in practice, not many papers consider systems with both delay and sampling in the controls. Notable exceptions are (Fridman, 2010; Jiang & Seret, 2010; Mazenc & Normand-Cyrot, 2012; Mirkin, 2007). Even more rare are results on *nonlinear* systems with delay and sampling; (Karafyllis & Krstic, 2012a) seems to be the only general result for this problem, and it relies on a prediction strategy that requires knowledge of the delay and the sampling interval.

Given a nonlinear time varying system with a uniformly globally asymptotically stabilizing time varying undelayed continuous time state controller, it is natural to search for conditions under which the closed loop system remains uniformly globally asymptotically stable (UGAS) when delays and sampling are introduced into the controller. To the best of our knowledge, the problem has never been addressed. However, implementing controls with measurement delays frequently leads to sampling of the control laws with delay. One real world practical motivation is in networked control systems in communication applications, and this has led to many significant papers; see (Heemels, Teel, van de Wouw, & Nesic, 2010) and many references therein.

Therefore, we consider a nonlinear system

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))u \tag{1}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ for any dimensions n and p where f and g are locally Lipschitz with respect to x and piecewise continuous in t. We assume that (1) is rendered UGAS by some C^1 controller $u_s(t, x(t))$. We give conditions under which the UGAS property is preserved when the input has sampling and delays, meaning the control value u entering (1) is $u_s(t_i - \tau, x(t_i - \tau))$ for all $t \in [t_i, t_{i+1})$ and $i = 0, 1, 2, \ldots$, where $\{t_i\}$ is a given sequence of sample times and $\tau > 0$ is the given positive delay. Our conditions give upper bounds for the delay and for the lengths of the sampling intervals. This differs from the literature on the emulation approach to sampled data and networked systems. In particular, (Laila, Nesic, & Teel, 2002) deals with a much more general class of systems with sampling but no delays, and in Remark 6 we discuss the value added by our work relative to the hybrid systems approach in (Heemels *et al.*, 2010). The paper (Teel, Nesic, & Kokotovic, 1998) seems to be the one closest to ours, but it differs from our work in several ways. In (Teel *et al.*, 1998), (i) there is an offset constant and a restriction on the initial states, which make its conclusions weaker than UGAS in the zero disturbance case unless special conditions are satisfied such

---

* Corresponding author: F. Mazenc. A summary excluding proofs is in preparation for possible inclusion in the 2013 American Control Conference. Supported by AFOSR Grant FA9550-09-1-0400 and NSF Grant ECCS-1102348.

Email addresses: `Frederic.MAZENC@lss.supelec.fr` (Frédéric Mazenc), `malisoff@lsu.edu` (Michael Malisoff), `Thach.DINH@lss.supelec.fr` (Thach N. Dinh).

as global exponential stability of the unsampled undelayed system, (ii) only time-invariant systems are considered, (iii) the main result is established via the Razumikhin theorem, (iv) the size of the largest admissible sampling interval is given by a condition on the gains while ours is expressed directly in terms of the dynamics, a controller, and a Lyapunov function, and (v) (Teel *et al.*, 1998) establishes ISS.

An important feature of our work is in our allowing perturbations of the sampling schedule. See, e.g., (Karafyllis & Krstic, 2012b) (which allows perturbed sampling schedules for an important class of feedforward systems based on discontinuous feedback) and (Karafyllis & Krstic, 2012c) (which uses prediction under perturbed sampling for strict-feedback systems and other systems where the solution map is available explicitly). The result of the present paper is novel and cannot be proven by adapting the proofs of (Herrmann, Spurgeon, & Edwards, 2001) or (Mazenc, Malisoff, & Lin, 2008). In fact, we show through examples that we establish our main result under conditions that do not imply the assumptions in (Mazenc *et al.*, 2008), including cases where the undelayed unsampled system is not locally exponentially stabilizable. To overcome this obstacle, we use a functional of Lyapunov type, which is reminiscent of the one used in (Fridman, Seuret, & Richard, 2004) to study time invariant linear systems and (Mazenc & Ito, 2012) to study neutral time delay systems. For simplicity, we only consider control affine systems, but extensions to systems that are not control affine can be obtained. We also conjecture that ISS results in the spirit of those of (Teel *et al.*, 1998) can be established and this may be the subject of further studies. We illustrate our work through several examples with input delays and sampled inputs, including a tracking problem for a model from (Jiang, Lefeber, & Nijmeijer, 2001).

2 Notation

Let $\mathcal{K}_\infty$ denote the set of all continuous functions $\rho : [0,+\infty) \to [0,+\infty)$ for which (i) $\rho(0) = 0$ and (ii) ρ is strictly increasing and unbounded. For any function $\phi : \mathcal{I} \to \mathbb{R}^p$ defined on any interval $\mathcal{I}$, let $|\phi|_{\mathcal{I}}$ denote its (essential) supremum over $\mathcal{I}$. Let $|\cdot|$ denote the Euclidean norm (or the induced matrix norm, depending on the context). For any continuous function $\varphi : \mathbb{R} \to \mathbb{R}^n$ and all $t \geq 0$, the function φ_t is defined by $\varphi_t(\theta) = \varphi(t+\theta)$ for all $\theta \in [-r, 0]$, where the constant $r > 0$ will depend on the context. We say that a function $\varphi(t, x)$ is uniformly bounded with respect to t provided there exists a function ρ of class $\mathcal{K}_\infty$ such that $|\varphi(t, x)| \leq \rho(1+|x|)$ for all (t, x) in the domain of φ. Throughout this paper, we assume that all of the time varying functions are uniformly bounded with respect to t. We set $\mathcal{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$. The notation will be simplified, e.g., by omitting arguments of functions, whenever no confusion can arise from the context.

3 Assumptions and Main Result

Consider the nonlinear system (1) and let $\{t_i\}$ be a sequence in $[0, +\infty)$ such that $t_0 = 0$ and such that there are two constants $\nu > 0$ and $\delta > \nu$ such that

$$t_{i+1} - t_i \in [\nu, \delta] \quad \forall i \in \mathcal{Z}_{\geq 0} \,. \tag{2}$$

Our closed loop system will then have the form $\dot{x}(t) = f(t, x(t)) + g(t, x(t))u_s(t_i - \tau, x(t_i - \tau))$ when $t \in [t_i, t_{i+1})$. For simplicity, we only consider initial conditions that are piecewise of class C^1 over $[-\tau, 0]$, but the case where the initial conditions are continuous over $[-\tau, 0]$ can be deduced easily from our result. Our first assumption is:

Assumption 1 *There exist a C^1 feedback $u_s(t, x)$, a C^1 positive definite and radially unbounded function V, and a continuous positive definite function W such that*

$$\begin{aligned} &W_b(t, x) = \\ &-\left[\tfrac{\partial V}{\partial t}(t, x) + \tfrac{\partial V}{\partial x}(t, x)\big(f(t, x) + g(t, x)u_s(t, x)\big)\right] \end{aligned} \tag{3}$$

satisfies

$$W_b(t, x) \geq W(x) \tag{4}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$. Also, $u_s(t, 0) = 0$ for all $t \in \mathbb{R}$.

Hence, V is a strict Lyapunov function for $\dot{x} = f(t, x) + g(t, x)u_s(t, x)$, and we can fix class $\mathcal{K}_\infty$ functions α_1 and α_2 such that $\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$. Define the function h by

$$\begin{aligned} h(t, x) \;=\; &\tfrac{\partial u_s}{\partial t}(t, x) + \tfrac{\partial u_s}{\partial x}(t, x)f(t, x) \\ &+ \tfrac{\partial u_s}{\partial x}(t, x)g(t, x)u_s(t, x). \end{aligned} \tag{5}$$

Our second assumption is:

Assumption 2 *There are constants $c_i > 0$ for $i = 1, 2, 3, 4$ such that*

$$\left|\tfrac{\partial u_s}{\partial x}(t, x)g(t, x)\right|^2 \;\leq\; c_1, \tag{6}$$

$$\left|\tfrac{\partial V}{\partial x}(t, x)g(t, x)\right|^2 \;\leq\; c_2 W(x), \tag{7}$$

$$|h(t, x)|^2 \;\leq\; c_3 W(x), \;\; \text{and} \tag{8}$$

$$\left|\tfrac{\partial V}{\partial x}(t, x)g(t, x)u_s(t, x)\right| \;\leq\; c_4[V(t, x) + 1] \tag{9}$$

hold for all $t \geq 0$ and $x \in \mathbb{R}^n$.

The preceding assumptions are not too restrictive. We verify them below in several examples, including a practical example from tracking where we can use a scaling argument to make the bound on the delay and sampling interval as large as desired. Note that if u_s is constant outside a given compact set, then (6) holds automatically, and (8) holds if and only if it is satisfied on a bounded compact set containing the origin. Our main result is:

Theorem 1 *Let the system (1) satisfy Assumptions 1 and 2. If δ and τ_* are any two positive constants such that*

$$\delta + \tau_* \leq \frac{1}{\sqrt{4c_1 + 8c_2c_3}} \tag{10}$$

and if $\tau \in (0, \tau_*]$, then the system (1) in closed loop with

$$u(t) = u_s(t_i - \tau, x(t_i - \tau)) \text{ when } t \in [t_i, t_{i+1}) \tag{11}$$

with the sequence $\{t_i\}$ from (2) is UGAS.

Remark 1 *Assumption 1 implies that the origin of (1)* *in closed loop with $u_s(t,x)$ without delay and sampling is UGAS. Assumptions 1 and 2 allow cases where W may not be radially unbounded; see the examples below.*

Remark 2 *The system (1) in closed loop with (11) admits solutions whose first derivatives are not continuous. However, their derivatives are piecewise continuous, and continuous over each interval of the form $[t_i, t_{i+1})$.*

Remark 3 *Theorem 1 can be extended to the case where the delay is time varying. Moreover, we could establish our result by representing the presence of delay and sampling as a time varying discontinuous feedback as was done in (Fridman, 2010). However, we did not make this choice because it does not help to simplify the forthcoming proof.*

Remark 4 *The requirement (9) is often satisfied in applications. We will see in the proof of Theorem 1 that it prevents the finite escape time phenomenon from occurring.*

Remark 5 *Theorem 1 applies to systems that are not necessarily locally exponentially stabilizable by continuous feedback. For example, take $n = 1$ and*

$$\dot{x} = \frac{x^2}{1+x^2}u \,, \tag{12}$$

with $u_s(x) = -x$ and $V(x) = \frac{1}{2}x^2$. Using the notation from above with the time dependency omitted, we have

$$f(x) = 0 \,, \; g(x) = \tfrac{x^2}{1+x^2} \,, \; \tfrac{\partial u_s}{\partial x}(x)g(x) = -\tfrac{x^2}{1+x^2} \,,$$
$$h(x) = \tfrac{x^3}{1+x^2} \,, \; L_gV(x) = \tfrac{x^3}{1+x^2} \,, \text{ and } W(x) = \tfrac{x^4}{1+x^2} \,.$$

Then Assumptions 1-2 hold with $c_1 = c_2 = c_3 = 1$, so Theorem 1 ensures that the corresponding input delayed sampled system is UGAS if $\delta + \tau < 1/(2\sqrt{3})$. The assumptions also hold with $n = 2$ and $\dot{x}_1 = -x_1 - x_1^9 + x_2$, $\dot{x}_2 = u + x_1$ with $u_s(t,x) = -x_1 - x_2$ and $V(t,x) = 0.1x_1^{10} + 0.5x_1^2 + 2x_2^2$.

Remark 6 *As a corollary of Theorem 1, we get UGAS when $\tau = 0$ and $\delta \leq 1/\sqrt{4c_1 + 8c_2c_3}$, which we believe is a new result. As in (Heemels* et al.*, 2010), we can also prove UGAS properties when τ depends on the index i, which gives the control $u_s(t_i - \tau_i, x(t_i - \tau_i))$ when $t \in [t_i, t_{i+1})$ for all i. The proof is similar to the one in the next section. However, in terms of our notation, (Heemels* et al.*, 2010) requires the small delay condition $\tau_i \leq \min\{\tau_{\rm mad}, t_{i+1} - t_i\}$ for all i where $\tau_{\rm mad}$ is an upper bound on the delays, which we do not require here. The proofs in (Heemels* et al.*, 2010) use a hybrid systems method that does not apply unless the small delay condition holds; see (Heemels* et al.*, 2010, Remark II.4), which notes that dropping the small delay condition is a hard open problem. The main assumptions in (Heemels* et al.*, 2010) involve a set of gains in the decay conditions for Lyapunov-like functions for the discrete and continuous subsystems. Then (Heemels* et al.*, 2010) shows that the decay conditions hold for several important classes of networked systems. Therefore, two other notable features of our work are that (a) we do not require the small delay condition and (b) our conditions are expressed directly in terms of V, the dynamics, and the controller u_s.*

4 Proof of Theorem 1

Throughout the proof, all time derivatives are over all trajectories of $\dot{x}(t) = f(t,x(t)) + g(t,x(t))u_s(t_i - \tau, x(t_i - \tau))$ for all $t \in [t_i, t_{i+1})$ and all $i \in \mathcal{Z}_{\geq 0}$ with initial conditions over $[-\tau, 0]$ that are piecewise of class C^1. Moreover, to simplify the notation, we use x to denote solutions with initial conditions ϕ_x defined over $(-\infty, 0]$ and such that $\phi_x(s) = \phi_x(-\tau)$ for all $s \in (-\infty, -\tau]$.

Let $x(t)$ be any solution of the closed loop system and let t_c be a positive real number or $+\infty$ such that the maximal interval of definition of $x(t)$ includes $[-\tau, t_c)$. Then

$$\begin{aligned}\dot{V} \;=&\; -W_b(t,x(t)) - \tfrac{\partial V}{\partial x}(t,x)g(t,x)u_s(t,x(t)) \\ &+ \tfrac{\partial V}{\partial x}(t,x)g(t,x)u_s(t_i - \tau, x(t_i - \tau))\end{aligned} \tag{13}$$

for all $t \in [t_i, t_{i+1})$, $i \in \mathcal{Z}_{\geq 0}$, and $t \in [0, t_c)$. From (4), (7) and (9), we deduce that

$$\begin{aligned}\dot{V} \;\leq&\; -W(x(t)) + c_4[V(t,x(t)) + 1] \\ &+\sqrt{c_2 W(x(t))}|u_s(t_i - \tau, x(t_i - \tau))| \\ \leq&\; c_4 V(t,x(t)) + c_2|u_s(t_i - \tau, x(t_i - \tau))|^2 + c_4\end{aligned} \tag{14}$$

for all $t \in [t_i, t_{i+1})$, $i \in \mathcal{Z}_{\geq 0}$, and $t \in [0, t_c)$, where the last inequality used the triangle inequality. Therefore,

$$\begin{aligned}V(t,x(t)) \;\leq&\; e^{c_4(t-t_i)}V(t_i, x(t_i)) + \tfrac{e^{c_4(t-t_i)}-1}{c_4}\,[c_4 + \\ &\quad c_2|u_s(t_i - \tau, x(t_i - \tau))|^2] \\ \leq&\; e^{c_4\delta}V(t_i, x(t_i)) + \tfrac{e^{c_4\delta}-1}{c_4}\,[c_4 + \\ &\quad c_2|u_s(t_i - \tau, x(t_i - \tau))|^2]\end{aligned} \tag{15}$$

for all $t \in [t_i, t_{i+1})$, $i \in \mathcal{Z}_{\geq 0}$, and $t \in [0, t_c)$. This readily implies that the solutions are defined over $[-\tau, +\infty)$.

We now prove our UGAS result. Set $\Delta u_s(t) = u_s(t_i - \tau, x(t_i - \tau)) - u_s(t, x(t))$. Let E be the set of all piecewise C^1 functions $\phi : \mathbb{R} \to \mathbb{R}^n$. Define $\Omega : E \times \mathbb{R} \to E$ by $\Omega(\phi(\cdot), s) = \phi(s + \cdot)$, and $\psi : [0, \infty) \times E \to \mathbb{R}^p$ and $\Gamma : [0, \infty) \times E \to \mathbb{R}^p$ by

$$\begin{aligned}&\psi(t,\phi) = \tfrac{\partial u_s}{\partial t}(t, \phi(0)) + \tfrac{\partial u_s}{\partial x}(t, \phi(0))\dot{\phi}(0) \;\text{ and} \\ &\Gamma(t,\phi) = \tfrac{\epsilon}{\delta+\tau_*}\int_{-\delta-\tau_*}^{0}\int_{s}^{0}|\psi(t+r, \Omega(\phi, r))|^2\,{\rm d}r{\rm d}s,\end{aligned} \tag{16}$$

where $\epsilon > 0$ is a constant to be selected later. Then,

$$\begin{aligned}\Gamma(t, x_t) &= \tfrac{\epsilon}{\delta+\tau_*}\int_{-\delta-\tau_*}^{0}\int_{s}^{0}|\psi(t+r, \Omega(x_t, r))|^2\,{\rm d}r{\rm d}s \\ &= \tfrac{\epsilon}{\delta+\tau_*}\int_{-\delta-\tau_*}^{0}\int_{t+s}^{t}|\psi(m, x_m))|^2\,{\rm d}m{\rm d}s\end{aligned}$$

along all trajectories of (1). It follows that

$$\Gamma(t, x_t) = \frac{\epsilon}{\delta+\tau_*}\int_{t-\delta-\tau_*}^{t}\int_{\ell}^{t}|\psi(m, x_m)|^2\,{\rm d}m{\rm d}\ell. \tag{17}$$

Since $\dot{x}$ is a piecewise continuous function of t, the function $\Gamma(t, x_t)$ is piecewise differentiable in t and satisfies

$$\begin{aligned}&\tfrac{d}{dt}\big(\Gamma(t, x_t)\big) \\ &= \epsilon\,|\psi(t, x_t)|^2 - \int_{t-\delta-\tau_*}^{t}\tfrac{\epsilon|\psi(m,x_m)|^2}{\delta+\tau_*}{\rm d}m.\end{aligned} \tag{18}$$

Next, we define

$$U(t,x_t) = V(t,x(t)) + \Gamma(t,x_t) \tag{19}$$

along all trajectories of (1), and $\dot U$ will mean $\frac{d}{dt}(U(t,x_t))$. Combining (13), (18), and the definition of h in (5) gives

$$\begin{aligned}\dot U =& -W_b(t,x(t)) - \tfrac{\epsilon}{\delta+\tau_*}\int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm \\ &+\epsilon\left|h(t,x(t)) + \tfrac{\partial u_s}{\partial x}(t,x(t))g(t,x(t))\Delta u_s(t)\right|^2 \\ &+\tfrac{\partial V}{\partial x}(t,x)g(t,x)\Delta u_s(t)\end{aligned} \tag{20}$$

along all trajectories of the delayed sampled dynamics. Using the inequality $(a+b)^2 \le 2a^2+2b^2$ which is valid for all $a\in\mathbb{R}$ and $b\in\mathbb{R}$, (4), and Assumptions 1-2, we get

$$\begin{aligned}\dot U \le& -W(x(t)) - \tfrac{\epsilon}{\delta+\tau_*}\int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm \\ &+2\epsilon|h(t,x(t))|^2 + \left|\tfrac{\partial V}{\partial x}(t,x)g(t,x)\right||\Delta u_s(t)| \\ &+2\epsilon\left|\tfrac{\partial u_s}{\partial x}(t,x(t))g(t,x(t))\right|^2|\Delta u_s(t)|^2 \\ \le& -W(x(t)) - \tfrac{\epsilon}{\delta+\tau_*}\int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm \\ &+2\epsilon c_3 W(x(t)) + 2\epsilon c_1|\Delta u_s(t)|^2 \\ &+\sqrt{c_2 W(x(t))}|\Delta u_s(t)|.\end{aligned} \tag{21}$$

From the triangle inequality, we deduce that

$$\sqrt{c_2W(x(t))}|\Delta u_s(t)| \le \tfrac14 W(x(t)) + c_2|\Delta u_s(t)|^2. \tag{22}$$

Combining (21) and (22), we get

$$\begin{aligned}\dot U \le& \left(-\tfrac34 + 2\epsilon c_3\right)W(x(t)) + (2\epsilon c_1 + c_2)|\Delta u_s(t)|^2 \\ &-\tfrac{\epsilon}{\delta+\tau_*}\int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm \\ =& \left(-\tfrac34 + 2\epsilon c_3\right)W(x(t)) \\ &-\tfrac{\epsilon}{\delta+\tau_*}\int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm \\ &+(2\epsilon c_1 + c_2)\left|\int_{t_i-\tau}^t \psi(m,x_m)dm\right|^2.\end{aligned} \tag{23}$$

From Jensen's inequality, it follows that

$$\left|\int_{t_i-\tau}^t \psi(m,x_m)dm\right|^2 \le (t-t_i+\tau)\int_{t_i-\tau}^t |\psi(m,x_m)|^2 dm,$$

and $t - t_i + \tau \le \delta + \tau_*$, so we have

$$\begin{aligned}\dot U \le& \left(-\tfrac34 + 2\epsilon c_3\right)W(x(t)) \\ &-\tfrac{\epsilon}{\delta+\tau_*}\int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm \\ &+(2\epsilon c_1 + c_2)(\delta+\tau_*)\int_{t_i-\tau}^t |\psi(m,x_m)|^2 dm.\end{aligned} \tag{24}$$

By grouping terms and using the fact that $t_i - \tau \ge t - \tau_* - \delta$ when $t\in[t_i,t_{i+1})$ to upper bound the second integral in (24) by the first integral, and then taking $\epsilon = \frac{1}{4c_3}$, we get

$$\begin{aligned}\dot U \le& -\tfrac14 W(x(t)) + \tfrac{1}{\delta+\tau_*}\left[-\tfrac{1}{4c_3} + \left(\tfrac{c_1}{2c_3} + c_2\right)(\delta+\tau_*)^2\right] \\ &\times\int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm.\end{aligned}$$

From the bound (10) on $\delta+\tau_*$, we deduce that

$$\begin{aligned}\dot U \le& -\tfrac14 W(x(t)) \\ &-\tfrac{1}{8c_3(\delta+\tau_*)}\int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm.\end{aligned} \tag{25}$$

Let κ be a C^1 function of class $\mathcal{K}_\infty$ such that κ' is nondecreasing and $\kappa'(0) = 8c_3(\delta+\tau_*)$. Then $U_\kappa = \kappa(U)$ satisfies

$$\begin{aligned}\dot U_\kappa \le& -\tfrac14\kappa'(U(t))W(x(t)) \\ &-\tfrac{1}{8c_3(\delta+\tau_*)}\kappa'(U(t))\int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm.\end{aligned}$$

One can choose κ such that there exists a function $\rho\in\mathcal{K}_\infty$ satisfying $\rho(V(t,x(t))) \le \frac14\kappa'(V(t,x(t)))W(x(t))$ for all t; see (Malisoff & Mazenc, 2009, Lemma A.7, p.354). We may assume that $\rho(s)\le s$ for all $s\ge 0$. (Otherwise, replace it by $\min\{s,\rho(s)\}$ which is also of class $\mathcal{K}_\infty$.) Hence,

$$\begin{aligned}\dot U_\kappa &\le -\rho(V(t,x(t))) - \int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm \\ &\le -\rho(V(t,x(t))) - \rho\left(\int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm\right) \\ &\le -\rho\left(\tfrac12\left[V(t,x(t)) + \int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm\right]\right) \\ &\le -\rho\left(\tfrac{1}{2(1+\varepsilon)}\left[V(t,x(t)) + \epsilon\int_{t-\delta-\tau_*}^t |\psi(m,x_m)|^2 dm\right]\right) \\ &\le -\rho\left(\tfrac{V(t,x(t))+\Gamma(t,x_t)}{2(1+\epsilon)}\right) = -\rho\left(\tfrac{\kappa^{-1}(U_\kappa(t))}{2(1+\epsilon)}\right).\end{aligned}$$

Since $s\to\rho(\kappa^{-1}(s)/[2(1+\epsilon)])$ is of class $\mathcal{K}_\infty$, and $U(t)\ge V(t,x(t)) \ge \alpha_1(|x(t)|)$ for all t, and there is a function $\bar U\in\mathcal{K}_\infty$ such that $U(t,x_0)\le \bar U(|x_0|_{[-\tau_*-\delta,0]})$ for all (t,x_0) (by (8) and our assumption that $u_s(t,0)=0$ for all $t\in\mathbb{R}$), we get the UGAS estimate (Malisoff & Mazenc, 2009).

5 Examples

5.1 *Saturating Controller*

Our work (Mazenc *et al.*, 2008) used Lyapunov-Krasovskii functionals to prove robustness of closed loop control affine systems with respect to small enough input delays. We next give an example that satisfies our Assumptions 1-2 and so is covered by Theorem 1, but does not satisfy the assumptions imposed to establish the main result in (Mazenc *et al.*, 2008). It will be key to the higher dimensional tracking dynamics in the next subsection. Take $\dot x = u$, where the state x and input u are one dimensional. This is rendered UGAS and locally exponentially stable by

$$u_s(x) = -\frac{\xi x}{\sqrt{1+x^2}}, \tag{26}$$

where ξ is any positive constant. Then, with the notation of Section 3, we have $f(x)=0$ and $g(x)=1$. We choose the positive definite radially unbounded function $V(x) = \sqrt{1+x^2}-1$. Then Assumption 1 is satisfied with $W_b(x) = W(x) = \xi x^2/(1+x^2)$. Omitting the time dependency,

$$\begin{aligned}&\left|\tfrac{\partial u_s}{\partial x}(x)g(x)\right|^2 \le \xi^2,\ \ |LgV(x)|^2 = \tfrac{1}{\xi}W(x), \\ &|h(x)|^2 = \tfrac{\xi^4x^2}{(1+x^2)^4} \le \xi^3 W(x), \text{ and} \\ &|LgV(x)u_s(x)| \le \tfrac{\xi x^2}{1+x^2} \le \xi[V(x)+1]\end{aligned} \tag{27}$$

so Assumption 2 holds. Hence, Theorem 1 applies to the system $\dot x = u$ with the feedback (26). The upper bound for $\delta+\tau_*$ is then $1/(2\sqrt3\xi)$, which can be made arbitrarily large if ξ is sufficiently small. However, this example violates (Mazenc *et al.*, 2008, Assumption H). This follows from:

Lemma 1 *If a system $\dot{x} = f(x) + u$ with f bounded is rendered GAS on $\mathbb{R}^n$ by a bounded feedback $u_s(x)$, then for each Lyapunov function $V(t,x)$ of the closed loop system, the requirements (Mazenc* et al.*, 2008, Assumption H) on the delayed system $\dot{x}(t) = f(x(t)) + u_s(x(t-\tau))$ fail to hold.*

Proof. Suppose the contrary. Then Assumption H provides a function $\sigma \in \mathcal{K}_\infty$ such that $V_t(t,x) + V_x(t,x)[f(x) + u_s(x)] \leq -\sigma^2(\sqrt{n}|x|)$ along all trajectories of the undelayed system. We claim that for each $x \in \mathbb{R}^n$, we can find a value $t_x \geq 0$ such that $|V_t(t_x,x)| \leq 0.5\sigma^2(\sqrt{n}|x|)$. To prove this claim, we can assume that there is no $t_x \geq 0$ such that $V_t(t_x,x) = 0$ and therefore that $V_t(t,x) < 0$ for all $t \geq 0$ for our given x, or that $V_t(t,x) > 0$ for all $t \geq 0$ for our chosen x. In the former case, we have $0 < -\int_0^t V_t(s,x)\mathrm{d}s = V(0,x) - V(t,x) \leq V(0,x)$ for all $t > 0$, so letting $t \to +\infty$ gives $V_t(s,x) \to 0$ as $s \to +\infty$ by the divergence test. The case where $V_t(t,x) > 0$ for all t is handled similarly, since there is a function $\bar{\alpha} \in \mathcal{K}_\infty$ such that $V(t,x) \leq \bar{\alpha}(|x|)$ for all $t \geq 0$ and $x \in \mathbb{R}^n$. Therefore, $V_x(t_x,x)[f(x) + u_s(x)] \leq -0.5\sigma^2(\sqrt{n}|x|)$ for all $x \in \mathbb{R}^n$. Moreover, Assumption H gives a constant $K_1 > 0$ such that $|V_x(t_x,x)| \leq K_1\sigma(|x|)$ for all $x \in \mathbb{R}^n$. Since f and u_s are bounded, this provides a constant $\bar{K} > 0$ such that $\sigma^2(\sqrt{n}|x|) \leq \bar{K}\sigma(|x|)$ and so also $\sigma(\sqrt{n}|x|) \leq \bar{K}\sigma(|x|)/\sigma(\sqrt{n}|x|) \leq \bar{K}$ for all nonzero $x \in \mathbb{R}^n$, contradicting the unboundedness of σ. $\square$

5.2 Tracking Example

We consider the system

$$\begin{cases} \dot{x}_1 = \omega x_2 \\ \dot{x}_2 = -\omega x_1 + \lambda \\ \dot{x}_3 = \omega, \end{cases} \tag{28}$$

where λ and ω are the controls, which is obtained from the kinematics of a wheeled mobile robot after a change of coordinates; see (Jiang *et al.*, 2001). Let $\zeta > 0$ be any constant. We aim to track the periodic trajectory $(0, 0, -\cos(\zeta t))^\top$, using sampled delayed feedback. Hence, the change of coordinates $z = x_3 + \cos(\zeta t)$ gives the tracking dynamics

$$\begin{cases} \dot{x}_1 = \omega x_2 \\ \dot{x}_2 = -\omega x_1 + \lambda \\ \dot{z} = -\zeta\sin(\zeta t) + \omega. \end{cases} \tag{29}$$

Case 1: The change of feedback $\omega = \zeta\sin(\zeta t) + \mu$ leads to

$$\begin{cases} \dot{x}_1 = (\zeta\sin(\zeta t) + \mu)x_2 \\ \dot{x}_2 = -(\zeta\sin(\zeta t) + \mu)x_1 + \lambda \\ \dot{z} = \mu. \end{cases} \tag{30}$$

(See Case 2 below for the case where the full feedback ω has sampling, i.e., $\omega(t_i - \tau) = \zeta\sin(\zeta(t_i - \tau)) + \mu(t_i - \tau)$.) The z subsystem of (30) can be stabilized easily using

$$\mu(z(t_i - \tau)) = -\zeta\frac{\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}, \tag{31}$$

where Ψ is any positive constant, since Section 5.1 shows that

$$\dot{z}(t) = -\frac{\zeta\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}} \tag{32}$$

is UGAS if $\delta + \tau_* < 1/(2\sqrt{3}\zeta\Psi)$. Assume that $0 < \Psi \leq \frac{1}{30}$. Next, we choose the control $\lambda(x(t)) = -\zeta x_2(t)$ to get

$$\begin{cases} \dot{x}_1 = \zeta\left(\sin(\zeta t) - \frac{\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right)x_2 \\ \dot{x}_2 = -\zeta\left(\sin(\zeta t) - \frac{\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right)x_1 - \zeta x_2 \\ \dot{z} = -\frac{\zeta\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}. \end{cases} \tag{33}$$

By Lemma A.1 in the appendix, the time derivative of the positive definite and proper quadratic function $Q_\zeta(t,x)$ in (A.1) along all trajectories of (33) satisfies

$$\dot{Q}_\zeta \leq -\frac{\zeta}{4}[x_1^2 + x_2^2]. \tag{34}$$

We replace the control λ by the delayed sampled controller $\lambda(x(t_i - \tau)) = -\zeta x_2(t_i - \tau)$. This gives

$$\begin{cases} \dot{x}_1 = \zeta\left(\sin(\zeta t) - \frac{\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right)x_2 \\ \dot{x}_2 = -\zeta\left(\sin(\zeta t) - \frac{\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right)x_1 \\ \qquad -\zeta x_2(t_i - \tau) \\ \dot{z} = -\frac{\zeta\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}. \end{cases} \tag{35}$$

We study (35) using the following strategy. We fix a particular solution of the z subsystem and focus on the system

$$\begin{cases} \dot{x}_1 = \zeta(\sin(\zeta t) + \gamma(t))x_2 \\ \dot{x}_2 = -\zeta(\sin(\zeta t) + \gamma(t))x_1 + \lambda, \end{cases} \tag{36}$$

where

$$\gamma(t) = -\frac{\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}. \tag{37}$$

Then, we apply Theorem 1 to (36), with $\lambda(x_2)$ playing the role of $u(x_2)$ in (1) and $\lambda(x_2(t_i - \tau)) = -\zeta x_2(t_i - \tau$ the role of u_s.

With the notation of (1) we choose $f(t,x) = (\zeta(\sin(\zeta t) + \gamma(t))x_2, -\zeta(\sin(\zeta t) + \gamma(t))x_1)^\top$ and $g(t,x) = (0,1)^\top$, $V(t,x) = Q_\zeta(t,x)$ and $u_s(t,x) = -\zeta x_2$, so $h(t,x) = \zeta^2[(\sin(\zeta t) + \gamma(t))x_1 + x_2]$. We check that Theorem 1 applies to (36). Since (34) holds along all trajectories of (36), Assumption 1 holds with $W(x) = \zeta|x|^2/4$. Condition (6) holds with $c_1 = \zeta^2$. We have $V_x(t,x)g(t,x) = (9/2)x_2 + 2\sin(\zeta t)x_1$, so (7) is satisfied with $c_2 = 117/\zeta$. Also, (8) holds with $c_3 = 8\zeta^3(\Psi+1)^2$, by the formula for h. Finally, it is clear that (9) is satisfied. Therefore Theorem 1 provides an upper bound on $\delta + \tau_*$ that is independent of the choice of the solution z. Combining this with our analysis of the z subsystem (32) gives the upper bound

$$\delta + \tau_* < \frac{1}{2\zeta}\min\left\{\frac{1}{\sqrt{3}\Psi}, \frac{1}{44(\Psi+1)}\right\}, \tag{38}$$

which ensures that (35) is UGAS.

Case 2: Next consider the case where $\omega = \zeta\sin(\zeta(t_i-\tau)) +$

$\mu(t_i-\tau)$, which we substitute into (28) to get

$$\begin{cases} \dot{x}_1 = [\zeta\sin(\zeta(t_i-\tau))+\mu]x_2 \\ \dot{x}_2 = -[\zeta\sin(\zeta(t_i-\tau))+\mu]x_1+\lambda \\ \dot{x}_3 = \zeta\sin(\zeta(t_i-\tau))+\mu. \end{cases} \tag{39}$$

We assume that $t_i=\delta i$ where $\delta=\pi/(\zeta L)$ for any positive integer L, and we set $\varphi(t)=t_i-\tau$ for all $t\in[t_i,t_{i+1})$ and $i\in\mathcal{Z}_{\geq 0}$. Setting $z=x_3-\int_0^t\zeta\sin(\zeta\varphi(m))\mathrm{d}m$, this gives

$$\begin{cases} \dot{x}_1 = [\zeta\sin(\zeta(t_i-\tau))+\mu]x_2 \\ \dot{x}_2 = -[\zeta\sin(\zeta(t_i-\tau))+\mu]x_1+\lambda \\ \dot{z} = \mu\ . \end{cases} \tag{40}$$

Also, Lemma A.2 in the appendix ensures that $z(t)-2\pi\leq x_3(t)\leq z(t)+2\pi$ for all t, so if $z(t)$ is bounded, then $x_3(t)$ is bounded as well. Next, we choose

$$\mu(z(t_i-\tau)) = -\frac{\zeta\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}, \tag{41}$$

where Ψ is such that $0<\Psi\leq\frac{1}{60}$. Then we have

$$\begin{cases} \dot{x}_1 = \zeta\left[\sin(\zeta(t_i-\tau))-\frac{\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right]x_2 \\ \dot{x}_2 = -\zeta\left[\sin(\zeta(t_i-\tau))-\frac{\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right]x_1+\lambda \\ \dot{z} = -\frac{\zeta\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}. \end{cases} \tag{42}$$

We rewrite the (x_1,x_2) subsystem of (42) as

$$\begin{cases} \dot{x}_1 = \zeta\left[\sin(\zeta t)+\omega(t)\right]x_2 \\ \dot{x}_2 = -\zeta\left[\sin(\zeta t)+\omega(t)\right]x_1+\lambda \end{cases} \tag{43}$$

with

$$\omega(t)=\sin(\zeta(t_i-\tau))-\sin(\zeta t)-\frac{\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}. \tag{44}$$

We have $|\omega(t)|\leq\zeta(\delta+\tau_*)+\Psi$. Therefore, if $\delta+\tau_*\leq 1/(60\zeta)$, then $|\omega(t)|\leq 1/30$. Hence, our analysis of (36) applies to (43) with the sampled feedback $\lambda=-\zeta x_2(t_i-\tau)$ to give a bound on the admissible values of $\tau_*+\delta$. Hence,

$$\begin{cases} \dot{x}_1 = \zeta\left[\sin(\zeta(t_i-\tau))-\frac{\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right]x_2 \\ \dot{x}_2 = -\zeta\left[\sin(\zeta(t_i-\tau))-\frac{\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}}\right]x_1 \\ \qquad -\zeta x_2(t_i-\tau) \\ \dot{z} = -\frac{\zeta\Psi z(t_i-\tau)}{\sqrt{1+z^2(t_i-\tau)}} \end{cases} \tag{45}$$

is also UGAS when (38) is satisfied.

6 Conclusions

Taking delays and sampling in the inputs into account is a challenging, central problem that has been studied by several authors using a variety of methods. We considered nonlinear control affine systems with feedbacks corrupted by delay and sampling. We gave conditions on the size of the delay and the maximal sampling interval that ensure uniform global asymptotic stability, using a new Lyapunov approach to obtain a new result. We applied our result to a tracking problem, where the bound on the sampling interval and delay can be arbitrarily large. Extensions to non-affine systems are possible. We conjecture that our main result can also be adapted to systems that can be locally but not globally asymptotically stabilized.

Appendix: Two Technical Lemmas

We used the following in our analysis of (30):

Lemma A.1 *Let $\zeta>0$ be any constant. Then: (a) The time derivative of*

$$Q_\zeta(t,x)=\frac{9}{4}|x|^2+2\sin(\zeta t)x_1x_2-\sin(\zeta t)\cos(\zeta t)x_1^2 \tag{A.1}$$

along all trajectories of the two dimensional system

$$\begin{cases} \dot{x}_1(t) = \zeta\sin(\zeta t)x_2(t) \\ \dot{x}_2(t) = -\zeta\sin(\zeta t)x_1(t)-\zeta x_2(t) \end{cases} \tag{A.2}$$

satisfies $\dot{Q}_\zeta(t,x)\leq-\zeta|x|^2/2$. (b) For any piecewise continuous function χ satisfying $|\chi(t)|\leq 1/30$ for all $t\geq 0$, the time derivative of (A.1) along all trajectories of

$$\begin{cases} \dot{x}_1(t) = \zeta\sin(\zeta t)x_2(t)+\zeta\chi(t)x_2(t) \\ \dot{x}_2(t) = -\zeta\sin(\zeta t)x_1(t)-\zeta x_2(t)-\zeta\chi(t)x_1(t) \end{cases} \tag{A.3}$$

satisfies $\dot{Q}_\zeta(t,x)\leq-\frac{\zeta}{4}|x|^2$. Also, $5|x|^2\geq Q_\zeta(t,x)\geq\frac{1}{4}|x|^2$ for all $t\in\mathbb{R}$ and $x\in\mathbb{R}^2$.

Proof. We only prove the case where $\zeta=1$. The general case will then follow from a scaling argument. Part (a) follows because along all trajectories of (A.2), we have

$$\begin{aligned} \dot{Q}_1 &= \left(-9/2+2\sin^2(t)\right)x_2^2-x_1^2 \\ &\quad +2\left(\cos(t)-\sin(t)-\sin^2(t)\cos(t)\right)x_1x_2 \\ &= \left(-9/2+2\sin^2(t)\right)x_2^2 \\ &\quad +2\left(\cos^3(t)-\sin(t)\right)x_1x_2-x_1^2 \\ &\leq \left[-9/2+2\sin^2(t)\right. \\ &\quad \left.+2\left(\cos^3(t)-\sin(t)\right)^2\right]x_2^2-\tfrac{1}{2}x_1^2 \\ &\leq -\tfrac{1}{2}[x_1^2+x_2^2], \end{aligned} \tag{A.4}$$

where the last inequality used $\max_t\left\{\sin^2(t)+(\cos^3(t)-\sin(t))^2\right\}=2$. Therefore, along all trajectories of (A.3),

$$\begin{aligned} &\dot{Q}_1\leq-\tfrac{1}{2}[x_1^2+x_2^2] \\ &+\left(9x_1/2-2\sin(t)\cos(t)x_1+2\sin(t)x_2\right)\chi(t)x_2 \\ &-\left(9x_2/2+2\sin(t)x_1\right)\chi(t)x_1. \end{aligned} \tag{A.5}$$

Since $|\chi(t)|\leq\frac{1}{30}$ for all $t\geq 0$, we deduce that

$$\begin{aligned} \dot{Q}_1 &\leq -\tfrac{1}{2}[x_1^2+x_2^2]+11\tfrac{1}{30}|x_1x_2|+2\tfrac{1}{30}x_2^2+2\tfrac{1}{30}x_1^2 \\ &\leq -\tfrac{1}{4}[x_1^2+x_2^2], \end{aligned} \tag{A.6}$$

which proves the decay estimate in part (b). □

We used the following in Case 2 in our tracking example:

Lemma A.2 *Let* $t_i = i\delta$, *where* $\delta = \pi/(\zeta L)$, L *is any positive integer, and* $\zeta > 0$ *is any constant. Let* $\varphi(t) = t_i - \tau$ *for all* $t \in [t_i, t_{i+1})$ *and all* $i \in \mathcal{Z}_{\geq 0}$. *Then*

$$\sin(\zeta\varphi(t + \pi/\zeta)) = -\sin(\zeta\varphi(t)) \text{ for all } t \in \mathbb{R}, \quad \text{(A.7)}$$

$$\int_0^{2\pi/\zeta} \sin(\zeta\varphi(m))\mathrm{d}m = 0 , \quad \text{(A.8)}$$

and $\sin(\zeta\varphi(t))$ *is periodic of period* $2\pi/\zeta$.

Proof. Let t and $i \in \mathcal{Z}_{\geq 0}$ be such that $t \in [t_i, t_{i+1})$. Then $t+\pi/\zeta \in [t_i + \pi/\zeta, t_{i+1} + \pi/\zeta)$, so our formulas for δ and t_i give $t+\pi/\zeta \in [t_{i+L}, t_{i+L+1})$. Hence, $\varphi(t+\pi/\zeta) = t_{i+L}-\tau = t_i - \tau + \pi/\zeta = \varphi(t) + \pi/\zeta$. Therefore, (A.7) holds, which implies that $\sin(\zeta\varphi(t))$ is periodic of period $\frac{2\pi}{\zeta}$. Next notice that

$$\int_0^{\frac{2\pi}{\zeta}} \sin(\zeta\varphi(m))\mathrm{d}m = \sum_{i=0}^{2L-1} \int_{t_i}^{t_{i+1}} \sin(\zeta\varphi(m))\mathrm{d}m. \quad \text{(A.9)}$$

Since φ is constant on $[t_i, t_{i+1})$, we get

$$\int_0^{\frac{2\pi}{\zeta}} \sin(\varphi(m))\mathrm{d}m = \delta \sum_{i=0}^{2L-1} \sin\left(\zeta\varphi\left(\frac{i\pi}{\zeta L}\right)\right). \quad \text{(A.10)}$$

Then we deduce successively that

$$\begin{aligned}
&\textstyle\int_0^{\frac{2\pi}{\zeta}} \sin(\varphi(m))\mathrm{d}m \\
&= \delta \sum_{i=0}^{L-1} \sin\left(\zeta\varphi\left(\frac{i\pi}{\zeta L}\right)\right) + \delta \sum_{i=L}^{2L-1} \sin\left(\zeta\varphi\left(\frac{i\pi}{\zeta L}\right)\right) \\
&= \delta \sum_{i=0}^{L-1} \sin\left(\zeta\varphi\left(\frac{i\pi}{\zeta L}\right)\right) \\
&\quad + \delta \sum_{j=0}^{L-1} \sin\left(\zeta\varphi\left(\frac{j\pi}{\zeta L} + \frac{\pi}{\zeta}\right)\right) \\
&= \delta \sum_{i=0}^{L-1} \sin\left(\zeta\varphi\left(\frac{i\pi}{\zeta L}\right)\right) - \delta \sum_{j=0}^{L-1} \sin\left(\zeta\varphi\left(\frac{j\pi}{\zeta L}\right)\right),
\end{aligned}$$

where the last equality used (A.7). Hence, all terms cancel and the lemma follows. □

References

Bekiaris-Liberis, N., & Krstic, M. (2012). Compensation of time varying input and state delays for nonlinear systems. *Journal of Dynamical Systems, Measurement, and Control*, 134(011009).

Fridman, E. (2010). A refined input delay approach to sampled-data control. *Automatica*, 46(2), 421-427.

Fridman, E., Seuret, A., & Richard, J.-P. (2004). Robust sampled-data stabilization of linear systems: An input delay approach. *Automatica*, 40(8), 1441-1446.

Heemels, M., Teel, A., van de Wouw, N., and Nesic, D. (2010). Networked control systems with communication constraints: tradeoffs between transmission intervals, delays and performance. *IEEE Transactions on Automatic Control*, 55(8), 1781-1796.

Herrmann, G., Spurgeon, S., & Edwards, C. (2001). A new approach to discretization applied to a continuous, nonlinear, sliding-mode-like control using nonsmooth analysis. *IMA Journal of Mathematical Control and Information*, 18(2), 161-187.

Jiang, Z.-P., Lefeber, E., & Nijmeijer, H. (2001). Saturated stabilization and tracking of a nonholonomic mobile robot. *Systems & Control Letters*, 42(5), 327-332.

Jiang, W., & Seuret, A. (2010). Improving stability analysis of networked control systems under asynchronous sampling and input delay. In *Proceedings of the 2nd IFAC Workshop on Distributed Estimation and Control in Networked Systems, Annecy, France* (pp. 79-84).

Karafyllis, I., & Jiang, Z.-P. (2011). *Stability and Stabilization of Nonlinear Systems.* London: Springer.

Karafyllis, I., & Krstic, M. (2012a). Nonlinear stabilization under sampled and delayed measurements, and with inputs subject to delay and zero-order hold. *IEEE Transactions on Automatic Control*, 57(5), 1141-1154.

Karafyllis, I., & Krstic, M. (2012b). Global stabilization of feedforward systems under perturbations in sampling schedule. *SIAM Journal on Control and Optimization*, 50(3), 1389-1412.

Karafyllis, I., & Krstic, M. (2012c). Predictor-based output feedback for nonlinear delay systems. *Preprint*, arXiv:1108.4499.

Laila, D.S., Nesic D., & Teel, A. (2002). Open and closed loop dissipation inequalities under sampling and controller emulation. *European Journal of Control*, 18(2), 109-125.

Malisoff, M., & Mazenc, F. (2009). *Constructions of Strict Lyapunov Functions.* London, UK: Springer-Verlag London Ltd.

Mazenc, F., & Ito, H. (2012). Lyapunov technique and backstepping for nonlinear neutral systems. *IEEE Transactions on Automatic Control*, to appear. [Preliminary Version: Mazenc, F., & Ito, H. (2012). New stability analysis technique and backstepping for neutral nonlinear systems. In *Proceedings of the 2012 American Control Conference, Montreal, Canada* (pp. 4673- 4678).]

Mazenc, F., & Malisoff, M. (2010). Stabilization of a chemostat model with Haldane growth functions and a delay in the measurement. *Automatica*, 46(9), 1428-1436.

Mazenc, F., Malisoff, M., & Lin, Z. (2008). Further results on input-to-state stability for nonlineal systems with delayed feedbacks. *Automatica*, 44(9), 2415-2421.

Mazenc, F., Niculescu, S.-I., & Bekaik, M. (2011). Backstepping for nonlinear systems with delay in the input revisited. *SIAM Journal on Control and Optimization*, 49(6), 2263-2278.

Mazenc, F., & Normand-Cyrot, D. (2012). Reduction model approach for linear systems with sampled delayed inputs. *IEEE Transactions on Automatic Control*, to appear. [Preliminary version: Mazenc, F., & Normand-Cyrot, D. (2012). Reduction model approach for linear systems with sampled delayed inputs. In *Proceedings of the 51st Conference on Decision and Control (Hawaii,* *USA)*, to appear.]

Mirkin, L. (2007). Some remarks on the use of time varying delay to model sample-and-hold circuits. *IEEE Transactions on Automatic Control*, 52(6), 1109-1112.

Monaco, S., & Normand-Cyrot, D. (2007). Advanced tools for nonlinear sampled-data systems' analysis and control. *European Journal of Control*, 13(2-3), 221-241.

Monaco, S., Normand-Cyrot, D., & Tiefense, F. (2011). Sampled-data stabilization: A PBC approach. *IEEE Transactions on Automatic Control*, 56(4), 907-912.

Nesic, D., & Teel, A. (2001). Sampled-data control of nonlinear systems: and overview of recent results. In *Perspectives in Robust Control*, R.S.O. Moheimani, Ed. New York: Springer-Verlag, 221-239.

Pepe, P., Karafyllis, I., & Jiang, Z.-P. (2008). On the Liapunov-Krasovskii methodology for the ISS of systems described by coupled delay differential and difference equations. *Automatica*, 44(9), 2266-2273.

Sharma, N., Gregory, C.M., & Dixon, W.E. (2011). Predictor-based compensation for electromechanical delay during neuromuscular electrical stimulation. *IEEE Transactions on Neural Systems and Rehabilitation Engineering*, 19(6), 601-611.

Stramigioli, S., Secchi, C., & Van der Schaft, A. (2002). Sampled data systems passivity and discrete port-Hamiltonian systems. *IEEE Transactions on Automatic Control*, 21(4), 574-587.

Teel, A., Nesic, D., & Kokotovic, P.V. (1998). A note on input-to-state stability of sampled-data nonlinear systems. In *Proceedings of the IEEE Conference on Decision and Control, Tampa, FL* (pp. 2473-2478).

Yeganefar, N., Pepe, P., & Dambrine, M. (2008). Input-to-state stability of time delay systems: A link with exponential stability. *IEEE Transactions on Automatic Control*, 53(6), 1526-1531.