Reduction Model Approach for Linear Systems With Sampled Delayed Inputs

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***Abstract*—We propose a new construction of exponentially stabilizing sampled feedbacks for continuous-time linear time-invariant systems with an arbitrarily large constant pointwise delay in the inputs. Stability is guaranteed under an assumption on the size of the largest sampling interval. The proposed design is based on an adaptation of the reduction model approach. The stability of the closed loop systems is proved through a Lyapunov functional of a new type, from which is derived a robustness result.**

***Index Terms*—Sampling, delay, stabilization, reduction model.**

I. Introduction

The problem of asymptotically stabilizing linear dynamical systems with a pointwise delay in the inputs is one of the most important of the control theory of systems with delay. Several approaches are available in the literature. One of them is the so-called reduction model technique, also known as finite spectrum assignment. This technique originates in [16] and has been developed in particular in the contributions [23], [15], [13] and [1]. It applies to any system of the form:

$$\dot{x}(t) = Ax(t) + Bv(t-\tau), \tag{1}$$

where $\tau > 0$ is a pointwise delay, $x \in \mathbb{R}^n$ is the state, $v \in \mathbb{R}^p$ is the input and A and B are constant matrices. Thus, one of its remarkable features is that it provides with exponentially stabilizing feedbacks for the system (1), even in the difficult case where at least one of the eigenvalues of A has a positive real part and the size of delay τ precludes the stabilization of this system through a static feedback of the form $v(x(t-\tau))$. In several papers, notably in [25], and in [12, Section 2.6], it is explained how the reduction technique, when applied to the system (1), provides with stabilizing control laws of the type:

$$v(t) = K\left[e^{\tau A}x(t) + \int_{t-\tau}^{t} e^{A(t-\ell)}Bv(\ell)d\ell\right].$$

Two fundamental features of these feedbacks are that they are continuous functions of t and present a term that depends on the past values of v. Consequently, they cannot be used when only piecewise constant control laws can be used. This is a drawback in the sense that, in many cases, only sampled control laws can be utilized, most notably in control over networks (see for instance [26], [24]). Overcoming this obstacle does not seem to be an easy task: generally speaking, it is a well-known fact that the problem of controlling a system with delay in the inputs with discontinuous feedbacks is a difficult problem. It is worth noting that only a few contributions are devoted to it [3].

These remarks motivate the direct study of the system defined by

$$\dot{x}(t) \;=\; Ax(t) + Bu(t_i - \tau)\ , \quad \forall t \in [t_i, t_{i+1}), \tag{2}$$

where $\tau > 0$ is a pointwise delay, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input and t_i is an increasing sequence (which goes to the infinity) such that there exist two constants $\delta > 0$, $\epsilon > 0$ such that

$$t_0 = 0\ ,\ \epsilon \le t_{i+1} - t_i \le \delta\ ,\ \forall i \ge 0. \tag{3}$$

Since uncertainties occur in many dynamical systems and are frequently a source of instability, we consider in the present work the system (2) with an additive term $Cu(t_i - \tau)$, where C is an unknown matrix, which is small in a sense to be made precise later:

$$\dot{x}(t) \;=\; Ax(t) + (B + C)u(t_i - \tau)\ ,\ \forall t \in [t_i, t_{i+1}). \tag{4}$$

Systems with other types uncertainties can be considered too, but, for the sake of simplicity, we restrict ourselves to uncertainties of the type $Cu(t_i - \tau)$. For the system (4), we propose a new technique of design of piecewise constant stabilizing control laws which is based on an adaptation of the classical reduction model approach recalled above. The control laws we propose rely on a dynamic extension and are implemented through zero order holding devices. We show that they exponentially stabilize the origin of the system, provided the largest sampling interval has length δ smaller than a bound explicitly given by an explicit formula. To establish stability, we introduce functionals of a new type, which are similar to Lyapunov-Krasovskii functionals but do not belong to this family (defined for instance in [7], [14]).

Despite the novelty of the contribution of the present paper, some important results entailing to the system (2) are available in the literature. In the case where $\delta = \epsilon$ this system is stabilized through a feedback of the form $Kx(t_i - \tau)$ in [8] and in the general case ($\delta \neq \epsilon$) in [26] and [24]. The results of these contributions are delay dependent, i.e. they rely on an assumption on the size of τ. The problem of asymptotically stabilizing the system (2) in the case where $\delta = \epsilon$ and τ is arbitrarily large admits at least two solutions. In [2, Chapt. 3],

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it is explained how it can be solved by constructing the zero-order-hold sampling of the system and adding extra variables. That way, a discrete-time system without delay is obtained, which can be stabilized using classical techniques. Notice that the size of the linear system obtained that way is proportional to $\frac{\tau}{\delta}$ and therefore it is very large when τ is large and δ is small. The second solution, which is based on an adaptation of the reduction approach of [1], [13] to discrete time systems, is proposed in the contributions [10], [6] where stabilization is achieved through a feedback of the form

$$u_i = Mx(i\delta - \tau) + \sum_{p=1}^{l+1} N_j u_{i-p}.$$

The main advantage of our main result relative to [8], [24] and [26] is that it applies for any delay $\tau \geq 0$. However, in contrast to [8], [24], the control laws we obtain depend on a dynamic extension. Its main advantage relative to the technique of [2, Chapt. 3] is that it does not require the introduction of many extra state variables and applies to the case of asynchronous sampling (i.e. $\delta \neq \epsilon$). Its main advantages relative to the technique of [10] and [6] are the following: (i) our approach applies to the case of asynchronous sampling, (ii) no sum of the type $\sum_{p=1}^{l+1} N_j u_{i-p}$ is present in the expression of the control laws we propose. The recent work [11] is closer to the present paper than [10]: in [11] asynchronous sampling is considered and a dynamic extension dependent on the input is used.

The contributions [4], [19] and [20] are devoted to systems with sampled inputs and not to systems with delay but their results can be adapted to the case where a delay is present in the input because they rely on a representation of the system where the control input has a piecewise continuous delay. This control strategy, which leads to controls without distributed terms, applies only when the delay is sufficiently small with respect to A and B and therefore the main advantages of our main result with respect to [4], [19] and [20] is that it applies for any delay $\tau \geq 0$. Observe also that the technique of proof we use does not lead to LMIs that depend on the delay or the largest sampling period, as for instance, lead the demonstrations in [4].

It is worth mentioning that our results are established under a restriction on the size of δ relative to the matrices A, B and the delay τ. A constraint of this type is needed in the sense that the problem may admit no solution if δ is larger than some threshold. However, the upper bound that we determine is probably smaller than the maximal admissible sampling period. This is a consequence of the Lyapunov approach we adopt: it makes it possible to establish exponential stability despite the presence of poorly known terms, like for instance $Cu(t_i-\tau)$ in (4), but it does not enable us to obtain an accurate estimate of the maximal admissible sampling period. Since the purpose that we pursue is to propose a stabilization technique that applies for any delay $\tau \geq 0$ and makes it possible to establish some robustness properties, and not the estimation of the largest possible value for δ, we did not try to improve the upper bound for δ that is deduced from our Lyapunov approach. However, improving this upper bound using, for instance, ideas borrowed from [19] and [20] may be the subject of further studies.

The paper is organized as follows. In Section II we adapt to sampling the classical reduction model approach. Conclusions are drawn in Section III.

Notation, definitions and basic result.

We denote by I the identity matrix in $\mathbb{R}^{n\times n}$. We denote $|\cdot|$ the Euclidean norm of matrices and vectors of any dimension. Let r be any positive integer. We denote $C_{\text{in}} = C([-\tau, 0], \mathbb{R}^r)$ the set of all continuous $\mathbb{R}^r$-valued functions defined on a given interval $[-\tau, 0]$. We denote $C^1 = C^1([-\tau, 0], \mathbb{R}^r)$ the set of all continuously differentiable $\mathbb{R}^r$-valued functions defined on a given interval $[-\tau, 0]$. For a continuous function $\varphi : [-\tau, +\infty) \to \mathbb{R}^k$, for all $t \geq 0$, the function φ_t is defined by $\varphi_t(\theta) = \varphi(t + \theta)$ for all $\theta \in [-\tau, 0]$ and $t \in [0, +\infty)$. The notation will be simplified whenever no confusion can arise from the context.

We recall a classical result [5, Theorem 1.2]:

Lemma 1: The system

$$\dot{X}(t) = \mathcal{H}X(t), \tag{5}$$

where $X \in \mathbb{R}^n$ and $\mathcal{H} \in \mathbb{R}^{n\times n}$ is a constant matrix is asymptotically stable if and only if for any positive definite symmetric matrix $\mathcal{S} \in \mathbb{R}^{n\times n}$ there exists a unique positive definite symmetric matrix $\mathcal{P} \in \mathbb{R}^{n\times n}$ such that the matrix equality

$$\mathcal{H}^\top \mathcal{P} + \mathcal{P}\mathcal{H} = -\mathcal{S} \tag{6}$$

is satisfied.

II. New reduction model approach

The main result of the paper relies on the following classical assumption.

Assumption 1: The pair (A, B) in the system (4) is stabilizable and $B \neq 0$.

We are ready to state the following result:

Theorem 1: Assume that the system (4) with the sequence t_i satisfying (3), fulfills Assumption 1 and let $K \in \mathbb{R}^{p\times n}$, $K \neq 0$ and a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ be such that the matrix inequality

$$H^\top Q + QH \leq -I \tag{7}$$

with

$$H = A + BK \tag{8}$$

is satisfied. Assume that the largest sampling period $\delta > 0$ in (3) is such that the inequality

$$\delta \leq \Delta_M \tag{9}$$

with

$$\Delta_M = \frac{1}{4\sqrt{6}|B||K|} \min\left\{\frac{1}{|H|\,|Qe^{A\tau}|}, \frac{1}{|e^{A\tau}|}\right\} \tag{10}$$

is satisfied and the matrix $C \in \mathbb{R}^{n\times p}$ in (4) is such that

$$|C| \leq \min\left\{|B|, \varpi_1, \varpi_2\right\}, \tag{11}$$

with $\varpi_1 = \frac{1}{16\sqrt{2}|Q||e^{A\tau}||K|}$, $\varpi_2 = \frac{1}{16\sqrt{12}\Delta_M|Qe^{A\tau}||e^{A\tau}||B||K|^2}$. Then the origin of the system (4) is globally exponentially stabilized by the control law

$$u(t_i - \tau) = Kz(t_i - \tau), \tag{12}$$

with, for all $t \geq 0$,

$$z(t) = e^{A\tau}x(t) + \int_{t-\tau}^{t} e^{A(t-\ell)}BKz(\ell)d\ell \tag{13}$$

and $z(t) = z_0(t)$ for all $t \in [-\tau, 0]$, where $z_0 \in C_{\text{in}}$ is any function such that

$$z_0(0) = e^{A\tau}x(0) + \int_{-\tau}^{0} e^{-A\ell}BKz_0(\ell)d\ell. \tag{14}$$

Remark. Lemma 1 guarantees that if the system (4) satisfies Assumption 1, then there exist a matrix $K \in \mathbb{R}^{p\times n}$ and a symmetric positive definite matrix $Q \in \mathbb{R}^{n\times n}$ such that the matrix inequality (7) is satisfied.

Remark. We assume that $B \neq 0$ and $K \neq 0$ because these assumptions ensure that the constants in (10) and (11) are well-defined and the cases $B = 0$ and $K = 0$ have no interest.

Proof. Let us consider a solution $x(t)$ of (4) with the feedback (12)-(13)-(14). One can prove easily that this solution is defined over $[-\tau, +\infty)$.

Elementary calculations give, for all integer $i \geq 0$ and $t \in (t_i, t_{i+1})$,

$$\begin{aligned} \dot{z}(t) &= e^{A\tau}Ax(t) + e^{A\tau}(B + C)u(t_i - \tau) \\ &\quad + A\int_{t-\tau}^{t} e^{A(t-\ell)}BKz(\ell)d\ell + BKz(t) \\ &\quad - e^{A\tau}BKz(t-\tau) \\ &= Az(t) + e^{A\tau}(B+C)u(t_i - \tau) \\ &\quad + BKz(t) - e^{A\tau}BKz(t-\tau). \end{aligned} \tag{15}$$

From the definition of H in (8), it follows that for all $t \in (t_i, t_{i+1})$,

$$\begin{aligned} \dot{z}(t) &= Hz(t) + e^{A\tau}(B+C)u(t_i - \tau) \\ &\quad - e^{A\tau}BKz(t-\tau). \end{aligned} \tag{16}$$

Now, replacing, for all $i \in \mathbb{N}$, $u(t_i - \tau)$ by its expression given in (12), we obtain, for all integer $i \geq 0$ and $t \in (t_i, t_{i+1})$,

$$\begin{aligned} \dot{z}(t) &= Hz(t) \\ &\quad + e^{A\tau}[(B+C)Kz(t_i-\tau) - BKz(t-\tau)] \\ &= Hz(t) + M[z(t_i - \tau) - z(t-\tau)] \\ &\quad + Nz(t-\tau), \end{aligned} \tag{17}$$

with

$$M = e^{A\tau}(B+C)K\ , \quad N = e^{A\tau}CK\ . \tag{18}$$

The next step of the proof consists in proving the global exponential stability of origin of the system (17).

First, observe that, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$,

$$\dot{z}(t) = Hz(t) - M\int_{t_i-\tau}^{t-\tau} \dot{z}(\ell)d\ell + Nz(t-\tau) \tag{19}$$

because $\dot{z}$ is a piecewise continuous function of t over $[0, +\infty)$. Now, we establish the exponential stability of the system (17) via a Lyapunov approach. Let $V_1 : \mathbb{R}^n \to [0, +\infty)$,

$$V_1(z) = z^\top Qz, \tag{20}$$

where $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix such that the linear matrix inequality (7) is satisfied. According to (19) and (7), the derivative of V_1 along the trajectories of (17) satisfies, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$,

$$\begin{aligned} \dot{V}_1(t) \leq & -|z(t)|^2 - 2z(t)^\top QM\int_{t_i-\tau}^{t-\tau} \dot{z}(\ell)d\ell \\ & + 2z(t)^\top QNz(t-\tau). \end{aligned} \tag{21}$$

By using the triangle inequality, we obtain

$$\begin{aligned} \dot{V}_1(t) \leq & -\tfrac{7}{16}|z(t)|^2 + 2q\left|\int_{t_i-\tau}^{t-\tau} \dot{z}(\ell)d\ell\right|^2 \\ & + r|z(t-\tau)|^2 \end{aligned} \tag{22}$$

with $q = |QM|^2$, $r = 16|QN|^2$. Now, we introduce the functional $V_2 : C^1 \to [0, +\infty)$,

$$\begin{aligned} V_2(\phi) &= V_1(\phi(0)) + \tfrac{1}{16}\int_{-\tau}^{0} |\phi(m)|^2 dm \\ &\quad + \tfrac{1}{16\tau}\int_{-\tau}^{0}\int_{m}^{0} |\phi(\ell)|^2 d\ell dm. \end{aligned} \tag{23}$$

Then, along the trajectories of the system (17), the equality

$$\begin{aligned} V_2(z_t) &= V_1(z(t)) + \tfrac{1}{16}\int_{t-\tau}^{t} |z(m)|^2 dm \\ &\quad + \tfrac{1}{16\tau}\int_{t-\tau}^{t}\int_{m}^{t} |z(\ell)|^2 d\ell dm \end{aligned} \tag{24}$$

is satisfied for $t \geq \tau + \delta$, and, keeping in mind the inequality (22), we deduce that, the derivative of V_2 along the solutions of the system (17) satisfies, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$,

$$\begin{aligned} \dot{V}_2(t) \leq & -\tfrac{5}{16}|z(t)|^2 + 2q\left|\int_{t_i-\tau}^{t-\tau} \dot{z}(\ell)d\ell\right|^2 \\ & + \left(r - \tfrac{1}{16}\right)|z(t-\tau)|^2 \\ & - \tfrac{1}{16\tau}\int_{t-\tau}^{t} |z(\ell)|^2 d\ell. \end{aligned} \tag{25}$$

Since

$$r = 16|QN|^2 = 16\left|Qe^{A\tau}CK\right|^2 \leq 16|C|^2|Q|^2\left|e^{A\tau}\right|^2|K|^2,$$

we deduce from (11) that $r - \frac{1}{16} \leq -\frac{1}{32}$. It follows that, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$,

$$\begin{aligned} \dot{V}_2(t) \leq & -\tfrac{5}{16}|z(t)|^2 - \tfrac{1}{32}|z(t-\tau)|^2 \\ & + 2q\left|\int_{t_i-\tau}^{t-\tau} \dot{z}(\ell)d\ell\right|^2 \\ & - \tfrac{1}{16\tau}\int_{t-\tau}^{t} |z(\ell)|^2 d\ell. \end{aligned} \tag{26}$$

Using the fact that the inequalities (3) guarantee that, for all $t \in [t_i, t_{i+1})$, $(t-\tau) - (t_i - \tau) \in [0, \delta]$, and the Cauchy-Schwarz's inequality, we deduce that

$$\begin{aligned} \dot{V}_2(t) \leq & -\tfrac{5}{16}|z(t)|^2 - \tfrac{1}{32}|z(t-\tau)|^2 \\ & + 2q\delta\int_{t_i-\tau}^{t-\tau} |\dot{z}(\ell)|^2 d\ell \\ & - \tfrac{1}{16\tau}\int_{t-\tau}^{t} |z(\ell)|^2 d\ell. \end{aligned} \tag{27}$$

The presence of a nonnegative term in the right hand side of (27) leads us to consider the functional $V_3 : C^1 \to [0, +\infty)$,

$$V_3(\phi) = V_2(\phi) + 3q\delta \int_{-\tau-\delta}^{-\tau} \int_m^0 \left|\dot{\phi}(\ell)\right|^2 d\ell dm, \qquad (28)$$

which is such that, along the trajectories of the system (17), for all $t \geq \tau + \delta$,

$$V_3(z_t) = V_2(z_t) + 3q\delta \int_{t-\tau-\delta}^{t-\tau} \int_m^t |\dot{z}(\ell)|^2 d\ell dm. \qquad (29)$$

It follows from (27) that the derivative of V_3 along the solutions of the system (17) satisfies, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$,

$$\begin{array}{rcl} \dot{V}_3(t) & \leq & -\frac{5}{16}|z(t)|^2 - \frac{1}{32}|z(t-\tau)|^2 \\ & & +2q\delta \int_{t_i-\tau}^{t-\tau} |\dot{z}(\ell)|^2 d\ell \\ & & -\frac{1}{16\tau} \int_{t-\tau}^{t} |z(\ell)|^2 d\ell \\ & & +3q\delta^2 |\dot{z}(t)|^2 + 3q\delta \int_{t-\tau}^{t} |\dot{z}(\ell)|^2 d\ell \\ & & -3q\delta \int_{t-\tau-\delta}^{t} |\dot{z}(\ell)|^2 d\ell \\ & \leq & -\frac{5}{16}|z(t)|^2 - \frac{1}{32}|z(t-\tau)|^2 \\ & & -q\delta \int_{t-\tau-\delta}^{t-\tau} |\dot{z}(\ell)|^2 d\ell \\ & & -\frac{1}{16\tau} \int_{t-\tau}^{t} |z(\ell)|^2 d\ell + 3q\delta^2 |\dot{z}(t)|^2 . \end{array} \qquad (30)$$

Using (19) again, we deduce that, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$,

$$\begin{array}{rcl} |\dot{z}(t)|^2 & \leq & 2|H|^2|z(t)|^2 \\ & & +4|M|^2 \left(\int_{t-\delta-\tau}^{t-\tau} |\dot{z}(\ell)| d\ell\right)^2 \\ & & +4|N|^2|z(t-\tau)|^2. \end{array} \qquad (31)$$

From the Cauchy-Schwarz's inequality, we deduce that, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$ the inequality

$$\begin{array}{rcl} |\dot{z}(t)|^2 & \leq & 2|H|^2|z(t)|^2 \\ & & +4|M|^2\delta \int_{t-\delta-\tau}^{t-\tau} |\dot{z}(\ell)|^2 d\ell \\ & & +4|N|^2|z(t-\tau)|^2 \end{array} \qquad (32)$$

holds. Combining (30) and (32), we obtain, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$

$$\begin{array}{rcl} \dot{V}_3(t) & \leq & -\frac{5}{16}|z(t)|^2 - \frac{1}{32}|z(t-\tau)|^2 \\ & & -q\delta \int_{t-\tau-\delta}^{t-\tau} |\dot{z}(\ell)|^2 d\ell \\ & & -\frac{1}{16\tau} \int_{t-\tau}^{t} |z(\ell)|^2 d\ell \\ & & +6q\delta^2|H|^2|z(t)|^2 \\ & & +12q\delta^2|M|^2\delta \int_{t-\delta-\tau}^{t-\tau} |\dot{z}(\ell)|^2 d\ell \\ & & +12q\delta^2|N|^2|z(t-\tau)|^2. \end{array} \qquad (33)$$

By grouping the terms, the inequality rewrites, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$,

$$\begin{array}{rcl} \dot{V}_3(t) & \leq & \left(-\frac{5}{16} + 6q\delta^2|H|^2\right)|z(t)|^2 \\ & & + \left(-\frac{1}{32} + 12|N|^2q\delta^2\right)|z(t-\tau)|^2 \\ & & -\frac{1}{16\tau} \int_{t-\tau}^{t} |z(\ell)|^2 d\ell \\ & & + \left(12\delta^2|M|^2 - 1\right) q\delta \int_{t-\tau-\delta}^{t-\tau} |\dot{z}(\ell)|^2 d\ell. \end{array} \qquad (34)$$

From the inequalities (9), (11), we deduce that,

$$\delta \leq \frac{1}{4\sqrt{6}|B||K|} \min\left\{\frac{1}{|H|\,|Qe^{A\tau}|}, \frac{1}{|e^{A\tau}|}\right\} \qquad (35)$$

and

$$\delta|C| \leq \frac{1}{16\sqrt{12}\,|Qe^{A\tau}|\,|e^{A\tau}||B||K|^2}. \qquad (36)$$

It follows that

$$\delta \quad \leq \quad \min\{\varsigma_1, \varsigma_2, \varsigma_3\} \qquad (37)$$

with $\varsigma_1 = \frac{1}{4\sqrt{6}|H||Qe^{A\tau}||B||K|}$, $\varsigma_2 = \frac{1}{16\sqrt{12}|Qe^{A\tau}||e^{A\tau}||B||C||K|^2}$ and $\varsigma_3 = \frac{1}{4\sqrt{6}|e^{A\tau}||B||K|}$. Since (11) ensures that $|C| \leq |B|$, the inequality

$$\delta \quad \leq \quad \min\{\varsigma_4, \varsigma_5, \varsigma_6\} \qquad (38)$$

with $\varsigma_4 = \frac{1}{2\sqrt{6}|H||Qe^{A\tau}||B+C||K|}$, $\varsigma_5 = \frac{1}{16\sqrt{3}|Qe^{A\tau}||(B+C)K||e^{A\tau}C||K|}$ and $\varsigma_6 = \frac{1}{2\sqrt{6}|e^{A\tau}(B+C)||K|}$ is satisfied. From the definitions of M and N in (18), we deduce that the inequalities

$$\begin{array}{c} |QM|\delta \leq \frac{1}{2\sqrt{6}|H|}\ , \quad |QM||N|\delta \leq \frac{1}{\sqrt{768}}\ , \\ |M|\delta \leq \frac{1}{2\sqrt{6}}\ , \end{array} \qquad (39)$$

are satisfied. Using again the definitions of M and N, we obtain the inequalities

$$6q\delta^2|H|^2 \leq \frac{1}{4},\ 12|N|^2q\delta^2 \leq \frac{1}{64},\ 12\delta^2|M|^2 \leq \frac{1}{2}. \qquad (40)$$

These inequalities in combination with (34) give, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$,

$$\begin{array}{rcl} \dot{V}_3(t) & \leq & -\frac{1}{16}|z(t)|^2 - \frac{1}{64}|z(t-\tau)|^2 \\ & & -\frac{1}{16\tau} \int_{t-\tau}^{t} |z(\ell)|^2 d\ell \\ & & -\frac{q\delta}{2} \int_{t-\tau-\delta}^{t-\tau} |\dot{z}(\ell)|^2 d\ell. \end{array} \qquad (41)$$

From this inequality, one can easily deduce that $|z(t)|$ converges to zero when the time goes to the infinity. In order to establish exponential stability and to determine a lower bound for the decay rate of $|z(t)|$, we introduce the functional $V_4 : C^1 \to [0, +\infty)$,

$$V_4(\phi) \quad = \quad V_3(\phi) + \rho \int_{-\tau-\delta}^{0} \int_m^0 \left|\dot{\phi}(\ell)\right|^2 d\ell dm \qquad (42)$$

with $\rho = \frac{1}{64(\tau+\delta)} \min\left\{\frac{1}{|H|^2}, \frac{1}{2\tau\delta|M|^2}, \frac{1}{8|N|^2}\right\}$. Then along the trajectories of the system (17), for all $t \geq \tau + \delta$, the equality

$$V_4(z_t) = V_3(z_t) + \rho \int_{t-\tau-\delta}^{t} \int_m^t |\dot{z}(\ell)|^2 d\ell dm \qquad (43)$$

is satisfied and the derivative of V_4 along the trajectories of (17) satisfies, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$,

$$\begin{array}{rcl} \dot{V}_4(t) & \leq & -\frac{1}{16}|z(t)|^2 - \frac{1}{64}|z(t-\tau)|^2 \\ & & -\frac{1}{16\tau}\int_{t-\tau}^{t} |z(\ell)|^2 d\ell \\ & & -\frac{q\delta}{2}\int_{t-\tau-\delta}^{t-\tau} |\dot{z}(\ell)|^2 d\ell \\ & & +\rho(\tau+\delta)|\dot{z}(t)|^2 - \rho\int_{t-\tau-\delta}^{t} |\dot{z}(\ell)|^2 d\ell. \end{array} \tag{44}$$

Using the inequality (32), we obtain, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$,

$$\begin{array}{rcl} \dot{V}_4(t) & \leq & -\frac{1}{16}|z(t)|^2 - \frac{1}{64}|z(t-\tau)|^2 \\ & & -\frac{1}{16\tau}\int_{t-\tau}^{t} |z(\ell)|^2 d\ell \\ & & -\frac{q\delta}{2}\int_{t-\tau-\delta}^{t-\tau} |\dot{z}(\ell)|^2 d\ell \\ & & +2\rho(\tau+\delta)|H|^2|z(t)|^2 \\ & & +4\rho(\tau+\delta)|M|^2\delta\int_{t-\delta-\tau}^{t-\tau} |\dot{z}(\ell)|^2 d\ell \\ & & +4\rho(\tau+\delta)|N|^2|z(t-\tau)|^2 \\ & & -\rho\int_{t-\tau-\delta}^{t} |\dot{z}(\ell)|^2 d\ell. \end{array} \tag{45}$$

From the definition of the constant ρ, we deduce that, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$

$$\begin{array}{rcl} \dot{V}_4(t) & \leq & -\frac{1}{32}|z(t)|^2 - \frac{1}{128}|z(t-\tau)|^2 \\ & & -\frac{1}{32\tau}\int_{t-\tau}^{t} |z(\ell)|^2 d\ell \\ & & -\rho\int_{t-\tau-\delta}^{t} |\dot{z}(\ell)|^2 d\ell. \end{array} \tag{46}$$

On the other hand, the functional V_4 satisfies, for all $\phi \in C^1$,

$$\begin{array}{rcl} V_4(\phi) & = & \phi(0)^\top Q\phi(0) + \frac{1}{16}\int_{-\tau}^{0} |\phi(m)|^2 dm \\ & & +\frac{1}{16\tau}\int_{-\tau}^{0}\int_{m}^{0} |\phi(\ell)|^2 d\ell dm \\ & & +3q\delta\int_{-\tau-\delta}^{-\tau}\int_{m}^{0} \left|\dot{\phi}(\ell)\right|^2 d\ell dm \\ & & +\rho\int_{-\tau-\delta}^{0}\int_{m}^{0} \left|\dot{\phi}(\ell)\right|^2 d\ell dm \\ & \leq & |Q||\phi(0)|^2 + \frac{1}{8}\int_{-\tau}^{0} |\phi(m)|^2 dm \\ & & +[3q\delta^2 + \rho(\tau+\delta)]\int_{-\tau-\delta}^{0} \left|\dot{\phi}(\ell)\right|^2 d\ell. \end{array} \tag{47}$$

We deduce that, for all $t \geq \tau + \delta$ and $t \in (t_i, t_{i+1})$,

$$\dot{V}_4(t) \leq -gV_4(z_t) \tag{48}$$

with $g = \min\left\{\frac{1}{32|Q|}, \frac{1}{4\tau}, \frac{\rho}{3q\delta^2+\rho(\tau+\delta)}\right\} > 0$. Since V_4 is a nonnegative functional, it follows that, for all $t \geq \tau + \delta$,

$$V_4(z_t) \leq e^{-g(t-\tau-\delta)}V_4(z_{\tau+\delta}). \tag{49}$$

Since, for all $\phi \in C^1$, the inequality $V_4(\phi) \geq V_1(\phi(0))$ holds, it follows that, for all $t \geq \tau + \delta$,

$$V_1(z(t)) \leq e^{-g(t-\tau-\delta)}V_4(z_{\tau+\delta}). \tag{50}$$

From the definition of V_1 and the fact that Q is a symmetric positive definite matrix, we deduce that there exists a constant $h > 0$ such that, for all $t \geq \tau + \delta$,

$$|z(t)| \leq he^{-\frac{g}{2}t}\sqrt{V_4(z_{\tau+\delta})}. \tag{51}$$

Now, observe that, for all $t \geq 0$,

$$x(t) = e^{-A\tau}z(t) - \int_{t-\tau}^{t} e^{A(t-\ell-\tau)}BKz(\ell)d\ell. \tag{52}$$

From (51), it follows that, for all $t \geq 2\tau + \delta$, the inequalities

$$\begin{array}{rcl} |x(t)| & \leq & |e^{-A\tau}|he^{-\frac{g}{2}t}\sqrt{V_4(z_{\tau+\delta})} \\ & & +e^{|A|\tau}|BK|\int_{t-\tau}^{t} he^{-\frac{g}{2}\ell}\sqrt{V_4(z_{\tau+\delta})}d\ell \\ & \leq & |e^{-A\tau}|he^{-\frac{g}{2}t}\sqrt{V_4(z_{\tau+\delta})} \\ & & +e^{|A|\tau}|BK|h\tau e^{-\frac{g}{2}t-\tau}\sqrt{V_4(z_{\tau+\delta})} \end{array} \tag{53}$$

are satisfied. Thus the solution $x(t)$ converges exponentially to the origin when the time goes to the infinity with a decay rate larger than $\frac{g}{2}$.

A. Some computational aspects

(i) Determining a continuous function $z_0 \in C_{\text{in}}$ such that the equality (14) is satisfied is an easy task. For instance, let us choose a negative constant c such that the matrix $J = I - \int_{-\tau}^{0} e^{c\ell}e^{-A\ell}BKd\ell$ is invertible and let

$$z_0(\ell) = e^{c\ell}J^{-1}e^{A\tau}x(0)\ ,\ \forall \ell \in [-\tau, 0].$$

Then the equality (14) is satisfied. Indeed, the definition of J implies that

$$\left[I - \int_{-\tau}^{0} e^{c\ell}e^{-A\ell}BKd\ell\right]J^{-1} = I.$$

It follows that the equality

$$J^{-1} = I + \int_{-\tau}^{0} e^{c\ell}e^{-A\ell}BKJ^{-1}d\ell, \tag{54}$$

holds, which implies that the equality

$$\begin{array}{rcl} J^{-1}e^{A\tau}x(0) & = & e^{A\tau}x(0) \\ & + & \int_{-\tau}^{0} e^{-A\ell}BKe^{c\ell}J^{-1}e^{A\tau}x(0)d\ell \end{array} \tag{55}$$

holds as well. It is equivalent to (14).

(ii) From a practical point of view, to obtain numerically the function z over $(0, +\infty)$, it may be worth solving the equation defined, for all $t \in [t_i, t_{i+1})$ by (17) with the initial condition $z(\ell) = z_0(\ell)$ for all $\ell \in [-\tau, 0]$.

(iii) Assuming that, for all $t \in [t_i, t_{i+1})$ only the measurement $x(t_i)$ is available, then the extension (13) cannot be used because the value of $x(t)$ with $t \in (t_i, t_{i+1})$ is unknown. However, in the case where $C = 0$ that we consider for the sake of simplicity, the following arguments can be used. For all $t \in [t_i, t_{i+1})$, the equality

$$x(t) = e^{A(t-t_i)}x(t_i) + B^{t-t_i}u(t_i - \tau)\ ,\ B^r := \int_0^r e^{A\ell}Bd\ell$$

is satisfied. Therefore, we can still stabilize the system by using (12) and replacing (13) by z defined, for all $t \in [t_i, t_{i+1})$, by

$$\begin{aligned} z(t) &= e^{A(t-t_i+\tau)}x(t_i) + e^{\tau A}B^{t-t_i}Kz(t_i-\tau) \\ &\quad + \int_{t-\tau}^{t} e^{A(t-\ell)}BKz(\ell)d\ell. \end{aligned} \quad (56)$$

III. Conclusion

By adapting the reduction model approach, we solved the problem of stabilizing through piecewise constant control laws linear time-invariant systems with an arbitrary pointwise constant delay in the inputs. We demonstrated the main result by using a Lyapunov functional which made it possible to establish a robustness property with respect to small terms that depend on the inputs. Much remains to be done. Other types of robustness properties such as the celebrated Input to State Stability property with respect to additive terms can be established by combining the ideas of the construction of a Lyapunov-Krasovskii functional in [18] with the construction presented in our paper. Extensions to systems with several delays or distributed delays are expected as well as the treatment of nonlinear dynamics by taking advantage of the techniques of [17] and [21], [22], [9]. The problem of relaxing the restriction on the size of the maximal sampling rate we imposed may be investigated by borrowing the tools used in [19] and [20].

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