Stabilization of time-varying nonlinear systems with distributed input delay by feedback of plant's state

Frédéric Mazenc, Silviu-Iulian Niculescu, and Mounir Bekaik,

***Abstract*—We address the problem of stabilizing systems belonging to a family of time-varying nonlinear systems with distributed input delay through state feedbacks without retarded term. The approach we adopt is based on a new technique that is inspired by the reduction model technique. The control laws we obtain are nonlinear and time-varying. They globally uniformly exponentially stabilize the origin of the considered system. We illustrate the construction with a networked control system.**

***Index Terms*—Distributed delay, asymptotic stability, time-varying, nonlinear.**

I. INTRODUCTION

Two recent contributions [2] and [5] are devoted to linear time-invariant systems with distributed inputs of the form

$$\dot{x}(t) = Ax(t) + \int_0^\tau F(\ell)u(t-\ell)d\ell, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, $u(t) \in \mathbb{R}^p$ is the input, $\tau > 0$ is a constant and F is the so-called delay kernel, which is assumed to be a measurable function. The authors of these papers explain their motivations for studying systems with distributed inputs, that can be summarized as follows. (i) Only a few works are devoted to them, although their control is a challenging problem since no direct control action can be applied to them. (ii) They are important from a practical point of view: the study [2] points out that they can appear in population dynamics, in propellant rocket motors as well as in networked control systems. Finally, the contribution [5] explains in details how the problem of stabilizing a linear system

$$\dot{x}(t) = Ax(t) + Bv(t)$$

through a digital network by state feedback leads to the problem of finding a control rendering the origin of the system (1) globally exponentially stable.

It is worth mentioning that the results of [2] and [5] significantly improve the stabilization techniques available for the system (1). Thus, in [2] a new approach provides with a Lyapunov functional for the system (1) in closed-loop with the predictor-based controller of [1] and [5] presents linear matrix inequality (LMI) conditions ensuring that a linear state feedback $u(x(t))$ globally exponentially stabilizes the origin of the system (1) when the delay kernel admits a linear fractional representation. However, these two papers are devoted to linear time-invariant systems only and, to the best of our knowledge, all the works devoted to the stabilization of systems with distributed inputs are focused on linear systems. This fact and the importance of the nonlinear time-varying systems motivate us for aiming at stabilizing nonlinear time-varying systems with distributed inputs. More precisely, we shall consider systems of the form:

$$\begin{aligned}\dot{x}(t) &= f(t,x(t)) + \int_0^\tau g(t,\ell,x(t-\ell))u(t-\ell)d\ell \\ &\quad + w(t,x(t)),\end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^p$ is the input, $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times p}$ are nonlinear functions bounded in norm respectively by a linear function of $|x|$ and a constant, $\tau > 0$ is a constant and $w : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a function which typically represents disturbances or unknown terms. This family is rather general: we notice that it incorporates both "convolution-type" and "summation-type" models (see [20] for the definition of convolution-type and summation-type) and that the delay kernel $g(t, \sigma, x)$ is not supposed to admit a linear fractional representation. To the best of our knowledge, the problem of stabilizing these systems is addressed for the first time. In particular, it is worth noting that it does not seem that any finite spectrum assignment technique can be applied to the system (2) because all of them apply to linear systems only.

Under additional assumptions, we shall design state feedbacks that globally uniformly exponentially stabilize the origin of the system (2), when τ is sufficiently small. Since our control does not incorporate distributed terms, the instability which may arise when is implemented a distributed control (see for instance [4], [19]) is automatically avoided. Our approach is based on an approach of a new type, which has been introduced for the first time in [15], and is inspired by the classical reduction model approach (which originates in the contributions [17], [12], [10] and [1]). It made it possible for systems with pointwise delays of the family

$$\dot{x}(t) = f(t,x(t)) + f_\tau(t-\tau, x(t-\tau))u(t-\tau), \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$ is the input, and $\tau > 0$ is a pointwise delay and where f and f_τ are Lipschitz continuous functions, to design a state feedback that renders their origin globally uniformly exponentially stable. The proof of the stability of the corresponding closed-loop systems is based on the construction of a Lyapunov-Krasovskii functional. Similarly, in the present work, we shall solve the stabilization problem

Frédéric Mazenc is with EPI INRIA DISCO, L2S-CNRS-Supélec, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France (e-mail: Frederic.MAZENC@lss.supelec.fr).

Silviu-Iulian Niculescu is with Laboratory of Signals and Systems (L2S), UMR CNRS 8506, CNRS-Supélec, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France. The author is also with EPI DISCO. (e-mail: Silviu.NICULESCU@lss.supelec.fr).

Mounir Bekaik is with EPI INRIA DISCO, L2S-CNRS-Supélec, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France (e-mail: Mounir.BEKAIK@lss.supelec.fr).

for (2) we have described above by constructing a Lyapunov-Krasovskii functional with the help of an operator of a new type (the definition of Lyapunov-Krasovskii functional we adopt is the one proposed in [11, Chapt. 2], [6]).

In contrast to many contributions that are devoted to systems with distributed delay, our control design does not rely on an LMI condition and the feedbacks we propose are nonlinear. We will illustrate our main result by using a networked control system with a distributed delay transmission channel in the case where the dynamical system to be controlled is nonlinear and time-varying.

Finally, it is worth mentioning that the technique of proof we adopt presents two main advantages. (i) It relies on the representation of the closed-loop system which makes it possible, for a fairly large family of systems, to perform a sign analysis of the solutions as done in [15]. But this study is beyond the scope of the present contribution. (ii) It provides with a strict Lyapunov-Krasovskii functional for the closed-loop system. The knowledge of a functional of this type for a system makes it possible to establish robustness properties of many types. Thus, the importance of Lyapunov functionals is more and more acknowledged by the researchers, as illustrated by the contributions [8], [7], [18], [2], [9], [14], [21], [11, Chapt. 2] and the references therein.

The remainder of this paper is organized as follows. Section II is devoted to the main result of the paper and its proof. An illustrating example is given in Section III. Concluding remarks in Section IV end the work.

Notation and definitions.

• The notation will be simplified whenever no confusion can arise from the context. • We denote by Id the identity matrix in $\mathbb{R}^{n\times n}$. • We let $|\cdot|$ denote the Euclidean norm of matrices and vectors of any dimension. • Given $\phi:\mathcal{I}\to\mathbb{R}^p$ defined on an interval $\mathcal{I}$, let $|\phi|_{\mathcal{I}}$ denote its (essential) supremum over $\mathcal{I}$. • We let $C_{\rm in}=C([-\tau,0],\mathbb{R}^n)$ denote the set of all continuous $\mathbb{R}^n$-valued functions defined on a given interval $[-\tau,0]$. • For a function $x:[-\tau,+\infty)\to\mathbb{R}^k$, for all $t\geq 0$, the function x_t is defined by $x_t(\theta)=x(t+\theta)$ for all $\theta\in[-\tau,0]$.

II. Main result

A. Problem statement and main result

We consider the nonlinear time-varying system (2). We introduce the function $G_\tau:\mathbb{R}^{n\times n}\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^{n\times p}$,

$$G_\tau(M,t,x)=\int_0^\tau e^{-M\ell}g(\ell+t,\ell,x)d\ell \tag{4}$$

and a set of appropriate assumptions:

Assumption 1. *The functions f, g, w are locally Lipschitz. There exist a constant Hurwitz matrix $L\in\mathbb{R}^{n\times n}$, and a function $\mu_{\tau,L}:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^p$ of class C^1 such that*

$$f(t,x)-Lx+G_\tau(L,t,x)\mu_{\tau,L}(t,x)=0, \tag{5}$$

for all $(t,x)\in\mathbb{R}^{1+n}$ and there exists a constant $H_L>0$ such that

$$|g(t_1,t_2,x)\mu_{\tau,L}(t_3,x)|\leq\frac{H_L}{\tau}|x|, \tag{6}$$

for all $(t_1,t_2,t_3,x)\in\mathbb{R}^{3+n}$.

Since the matrix L in Assumption 1 is Hurwitz, there exists a constant symmetric positive definite matrix Q such that the inequality

$$QL+L^\top Q\leq -3Id \tag{7}$$

is satisfied. With this remark, we can give the next assumption:

Assumption 2. *Let Q be a constant symmetric positive definite matrix such that (7) is satisfied. There exists a positive real number w_0 such that the inequality*

$$w_0<\frac{1}{24|Q|} \tag{8}$$

is satisfied and, for all $(t,x)\in\mathbb{R}^{1+n}$ the inequality

$$|w(t,x)|\leq w_0|x| \tag{9}$$

is fulfilled.

We are ready to state and prove the main result:

Theorem 1: Assume that the system (2) satisfies Assumptions 1 and 2 and that $\tau\in(0,\tau_M)$, where $\tau_M>0$ is the constant such that

$$\tau_M e^{|L|\tau_M}=\frac{1}{2H_L}. \tag{10}$$

Then the origin of the system (2) is globally uniformly exponentially stabilized by the state feedback

$$u(t)=\mu_{\tau,L}(t,x(t)). \tag{11}$$

B. Discussion of Theorem 1

1. Assumption 1 implies that

$$|f(t,x)|\leq\left(|L|+e^{|L|\tau}H_L\right)|x|\ ,\ \forall(t,x)\in\mathbb{R}^{1+n}. \tag{12}$$

Therefore it ensures that the system (2) is forward-complete (see [9] for the definition of forward-complete system). The (implicit) assumption that f has a linear growth cannot be removed without being replaced by another assumption ensuring forward completeness: if it does not hold, the system (2) may be not stabilizable by a continuous state feedback $u(t,x(t))$, uniformly bounded with respect to t. We establish this fact in Appendix A via an example similar to the one studied in [15, Appendix C].

2. For the sake of simplicity, we assumed that f,g,w are locally Lipschitz, but slightly less restrictive regularity assumptions may be imposed: for instance we may assume that w is only piecewise continuous with respect to t and locally Lipschitz with respect to x.

3. The feedback in (11) depends only on t and $x(t)$. Hence, it does not require the knowledge of past values. The control laws proposed in [5] have also this feature.

4. Theorem 1, in combination with the main result of [15], can be extended to many systems and in particular to systems of the type

$$\begin{aligned}\dot{x}(t) &= f\left(t,x(t)\right)+\sum_{k=1}^{j}h_k(t-\tau_k)u(t-\tau_k)\\ &\quad+\sum_{k=1}^{j}\int_0^{\tau_k}g_k(\ell,x(t-\ell))u(t-\ell)d\ell,\end{aligned} \tag{13}$$

where j is a positive integer and the g_k, h_k are continuous functions, bounded in norm, which have been studied in a particular linear case in [3]. For the sake of simplicity, we shall not consider in this work systems (13).

5. It does not always exist a matrix L and a function $\mu_{\tau,L}$ such that the equality (5) is satisfied. Moreover, finding a matrix L and a function $\mu_{\tau,L}$ such that this equality is satisfied may be difficult. However, this obstacle can be overcome in some important cases. When there is a Hurwitz matrix L such that the matrix $\int_0^{\tau} e^{-Lm} g(m+t,m,x)dm$ is invertible for all $(t,x) \in \mathbb{R}^{1+n}$, (5) is satisfied with $\mu_{\tau,L}(t,x) = \left(\int_0^{\tau} e^{-Lm} g(m+t,m,x)dm\right)^{-1} [Lx - f(t,x)]$. Moreover, since there is some flexibility in the choice of the functions f and w (only $f+w$ is imposed by the system), the robustness with respect to $w(t,x(t))$ of the stabilization result of Theorem 1 can be used in some cases to make it possible to find a matrix L and a functions $\mu_{\tau,L}$ such that (5) is satisfied. To understand how, we consider the basic case where $f(t,x) = Rx$, $g(t,\sigma,x) = S$, where R and S are constant matrices such that the pair (R,S) is stabilizable i.e. there exists a constant matrix K such that the matrix $R+SK$ is Hurwitz. Then the system (2) admits the representation

$$\begin{aligned} \dot{x}(t) &= Rx(t) + \int_0^{\tau} Su(t-\ell)d\ell \\ &= f(x(t)) + \int_0^{\tau} Su(t-\ell)d\ell + w(x(t)), \end{aligned} \quad (14)$$

with

$$f(x) = (R+SK)x - \int_0^{\tau} e^{-(R+SK)\ell} S d\ell \frac{1}{\tau} Kx$$

and $w(x) = Rx - f(x)$. Thus the matrix $L = R+SK$ is Hurwitz and such that

$$f(x) - Lx + \int_0^{\tau} e^{-L\ell} S d\ell \frac{1}{\tau} Kx = 0 \ , \ \forall x \in \mathbb{R}^n.$$

It follows that the equality (5) is satisfied with $\mu_{\tau,L}(x) = \frac{1}{\tau}Kx$. Finally, one can check that

$$|w(x)| \leq |Rx - f(x)| \leq w_0|x|,$$

with $w_0 = \left| SK - \frac{1}{\tau}\int_0^{\tau} e^{-(R+SK)\ell} SK d\ell \right|$. We deduce easily that, for a given matrix Q such that (7) is satisfied, one can determine a constant $\overline{\tau} > 0$ such that if $\tau \in (0,\overline{\tau})$, then $w_0 < \frac{1}{24|Q|}$. It follows that, if $\tau \in (0,\overline{\tau})$, Assumptions 1 to 3 are satisfied by the system (14).

6. From the forthcoming proof, one can deduce that when Theorem 1 applies, it leads to systems in closed-loop which are ISS relative to additive disturbances. However, due to length limitation, we do not prove this result.

C. Proof of Theorem 1

To begin with, we observe that the system (2) in closed-loop with $\mu_{\tau,L}(t,x)$ can be written as

$$\dot{x}(t) = f(t,x(t)) + \int_{t-\tau}^{t} \lambda(t,\ell,x(\ell))d\ell + w(t,x(t)), \quad (15)$$

with $\lambda(t,\ell,x) = g(t,t-\ell,x)\mu_{\tau,L}(\ell,x)$. Now, to ease the stability analysis of this system, we introduce the operator $\alpha : \mathbb{R} \times C_{\text{in}} \to \mathbb{R}^n$,

$$\alpha(t,\phi) = \int_{-\tau}^{0} e^{Lr} \int_{-\tau-r}^{0} \lambda(t-r,t+v,\phi(v))dvdr. \quad (16)$$

Along the trajectories of (15), it satisfies

$$\alpha(t,x_t) = \int_{t-\tau}^{t} e^{L(t-m-\tau)} \int_{m}^{t} \lambda(m+\tau,\ell,x(\ell))d\ell dm. \quad (17)$$

Elementary calculations give:

$$\begin{aligned} \dot{\alpha}(t) &= L\alpha(t,x_t) - \int_{t-\tau}^{t} \lambda(t,\ell,x(\ell))d\ell \\ &\quad + \int_{t-\tau}^{t} e^{L(t-m-\tau)}\lambda(m+\tau,t,x(t))dm \\ &= L\alpha(t,x_t) - \int_{t-\tau}^{t} g(t,t-\ell,x(\ell))\mu_{\tau,L}(\ell,x(\ell))d\ell \\ &\quad + \left(\int_0^{\tau} e^{-L\ell} g(\ell+t,\ell,x(t))d\ell\right)\mu_{\tau,L}(t,x(t)). \end{aligned} \quad (18)$$

It follows that the operator $\beta : \mathbb{R} \times C_{\text{in}} \to \mathbb{R}^n$,

$$\beta(t,\phi) = \phi(0) + \alpha(t,\phi), \quad (19)$$

satisfies, along the trajectories of (15),

$$\beta(t,x_t) = x(t) + \alpha(t,x_t), \quad (20)$$

and

$$\begin{aligned} \dot{\beta}(t) &= f(t,x(t)) + L\alpha(t,x(t)) \\ &+ G_{\tau}(L,t,x(t))\mu_{\tau,L}(t,x(t)) + w(t,x(t)) \\ &= L\beta(t,x_t) + f(t,x(t)) - Lx(t) \\ &+ G_{\tau}(L,t,x(t))\mu_{\tau,L}(t,x(t)) + w(t,x(t)), \end{aligned} \quad (21)$$

where G_{τ} is the function defined in (4). It follows from (5) in Assumption 1 that

$$\dot{\beta}(t) = L\beta(t,x_t) + w(t,x(t)). \quad (22)$$

Moreover, from the expression of $\alpha(t,x_t)$ in (17) and the definition of β in (19), it follows that

$$\begin{aligned} x(t) &= \beta(t,x_t) \\ &\quad - \int_{t-\tau}^{t} e^{L(t-m-\tau)} \int_{m}^{t} \lambda(m+\tau,\ell,x(\ell))d\ell dm. \end{aligned} \quad (23)$$

Next, we use (22), (23) to prove that the system (15) admits the origin as a uniformly exponentially stable equilibrium point when $\tau \in (0,\tau_M)$. If the integral equation (23) was affine with respect to x, then one could design a Lyapunov functional for it by borrowing some ideas from [16]. However, the equation is not affine with respect of x so that an alternative construction has to be performed. We start our analysis by observing that (23) implies that

$$\begin{aligned} |x(t)| &\leq |\beta(t,x_t)| \\ &\quad + \int_{t-\tau}^{t} \left|e^{L(t-m-\tau)}\right| \int_{m}^{t} |\lambda(m+\tau,\ell,x(\ell))| d\ell dm. \end{aligned} \quad (24)$$

From (6) in Assumption 1, we deduce that

$$\begin{array}{rcl} |x(t)| & \leq & |\beta(t,x_t)| \\ & + & \displaystyle\int_{t-\tau}^{t} e^{|L|(-t+m+\tau)} \int_{m}^{t} \frac{H_L}{\tau} |x(\ell)| d\ell dm \\ & \leq & |\beta(t,x_t)| + e^{|L|\tau} H_L \displaystyle\int_{t-\tau}^{t} |x(\ell)| d\ell. \end{array} \tag{25}$$

This inequality leads us to consider the functional $V_1 : C_{\text{in}} \to [0,+\infty)$,

$$V_1(\phi) = \int_{-\tau}^{0} \left[\frac{1}{2}\tau|\phi(m)| + \int_{m}^{0} |\phi(\ell)| d\ell \right] dm \tag{26}$$

whose derivative, along the trajectories of (15), satisfies

$$\dot{V}_1(t) = \frac{3}{2}\tau|x(t)| - \frac{1}{2}\tau|x(t-\tau)| - \int_{t-\tau}^{t} |x(\ell)| d\ell. \tag{27}$$

This equality, in combination with (25), gives

$$\begin{array}{rcl} \dot{V}_1(t) & \leq & \frac{3}{2}\tau \left[|\beta(t,x_t)| + e^{|L|\tau} H_L \displaystyle\int_{t-\tau}^{t} |x(\ell)| d\ell \right] \\ & & - \displaystyle\int_{t-\tau}^{t} |x(\ell)| d\ell. \end{array} \tag{28}$$

Since $\tau \leq \tau_M$, the equality (10) implies that

$$\dot{V}_1(t) \quad \leq \quad \tfrac{3}{2}\tau|\beta(t,x_t)| - \tfrac{1}{4} \int_{t-\tau}^{t} |x(\ell)| d\ell. \tag{29}$$

We observe that

$$\frac{1}{2}\tau \int_{-\tau}^{0} |\phi(\ell)| d\ell \leq V_1(\phi) \leq \frac{3}{2}\tau \int_{-\tau}^{0} |\phi(\ell)| d\ell, \tag{30}$$

for all $\phi \in C_{\text{in}}$. It follows that

$$\dot{V}_1(t) \quad \leq \quad \tfrac{3}{2}\tau|\beta(t,x_t)| - \tfrac{1}{6\tau} V_1(x_t). \tag{31}$$

It follows that the derivative of the functional $V_2 : C_{\text{in}} \to [0,+\infty)$,

$$V_2(\phi) = \frac{1}{2} V_1(\phi)^2 \tag{32}$$

along the trajectories of (15) satisfies

$$\dot{V}_2(t) \quad \leq \quad \tfrac{3}{2}\tau|\beta(t,x_t)| V_1(x_t) - \tfrac{1}{6\tau} V_1(x_t)^2. \tag{33}$$

Therefore the functional $V_3 : \mathbb{R} \times C_{\text{in}} \to [0,+\infty)$,

$$V_3(t,\phi) = V_2(\phi) + c_1 \beta(t,\phi)^\top Q \beta(t,\phi), \tag{34}$$

where $c_1 > 0$ is a constant to be chosen later and Q is the matrix defined in (7), has a time derivative along the trajectories of (15) that satisfies

$$\begin{array}{l} \dot{V}_3(t) \leq \frac{3}{2}\tau|\beta(t,x_t)| V_1(x_t) - \frac{1}{6\tau} V_1(x_t)^2 \\ \quad - 3c_1|\beta(t,x_t)|^2 + 2c_1 \beta(t,x_t)^\top Q w(t,x(t)). \end{array} \tag{35}$$

From Assumption 2, we deduce that

$$\begin{array}{l} \dot{V}_3(t) \leq \frac{3}{2}\tau|\beta(t,x_t)| V_1(x_t) - \frac{1}{6\tau} V_1(x_t)^2 \\ \quad - 3c_1|\beta(t,x_t)|^2 + 2c_1 w_0 |Q| |\beta(t,x_t)| |x(t)|. \end{array} \tag{36}$$

Now, from (25) and (30) we deduce that

$$|x(t)| \leq |\beta(t,x_t)| + e^{|L|\tau} H_L \frac{2}{\tau} V_1(x_t). \tag{37}$$

It follows that

$$\begin{array}{l} \dot{V}_3(t) \leq \frac{3\tau}{2}|\beta(t,x_t)| V_1(x_t) - \frac{1}{6\tau} V_1(x_t)^2 - 3c_1|\beta(t,x_t)|^2 \\ \quad + 2c_1 w_0 |Q| |\beta(t,x_t)| \left(|\beta(t,x_t)| + e^{|L|\tau} H_L \frac{2}{\tau} V_1(x_t) \right). \end{array} \tag{38}$$

By grouping the terms, we obtain

$$\begin{array}{rcl} \dot{V}_3(t) & \leq & -\frac{1}{6\tau} V_1(x_t)^2 + c_1 \left(-3 + 2w_0|Q|\right) |\beta(t,x_t)|^2 \\ & + & \left(\frac{3}{2}\tau + \frac{4}{\tau} c_1 w_0 |Q| e^{|L|\tau} H_L \right) |\beta(t,x_t)| V_1(x_t). \end{array} \tag{39}$$

Let us choose $c_1 = 6\tau^3$. Then, using Assumption 2 in combination with the equality (10), we obtain $w_0 < \frac{3}{4|Q|}$, $w_0 < \frac{1}{|Q|} \frac{1}{48 H_L \tau_M e^{|L|\tau_M}}$,

$$c_1 \left(-3 + 2w_0|Q|\right) \leq 6\tau^3 \left(-3 + 2\frac{3}{4|Q|}|Q| \right) = -9\tau^3$$

and

$$\begin{array}{l} \frac{3}{2}\tau + \frac{4}{\tau} c_1 w_0 |Q| e^{|L|\tau} H_L \\ \qquad \leq \frac{3}{2}\tau + \frac{4}{\tau} 6\tau^3 \frac{1}{48 H_L \tau_M e^{|L|\tau_M}} e^{|L|\tau} H_L \\ \qquad \leq \frac{3}{2}\tau + \tau^2 \frac{1}{2 H_L \tau_M e^{|L|\tau_M}} e^{|L|\tau} H_L. \end{array}$$

It follows that

$$\begin{array}{l} \dot{V}_3(t) \leq -\frac{1}{6\tau} V_1(x_t)^2 - 9\tau^3 |\beta(t,x_t)|^2 \\ \qquad + \left(\frac{3}{2}\tau + \tau^2 \frac{1}{2\tau_M e^{|L|\tau_M}} e^{|L|\tau} \right) |\beta(t,x_t)| V_1(x_t). \end{array} \tag{40}$$

Since $\tau \in (0, \tau_M)$, the inequality

$$\begin{array}{rcl} \dot{V}_3(t) & \leq & -\frac{1}{6\tau} V_1(x_t)^2 - 9\tau^3 |\beta(t,x_t)|^2 \\ & & + 2\tau |\beta(t,x_t)| V_1(x_t) \end{array} \tag{41}$$

holds. With the usual completion of squares, we get

$$\begin{array}{rcl} \dot{V}_3(t) & \leq & -\frac{1}{24\tau} V_1(x_t)^2 - \tau^3 |\beta(t,x_t)|^2 \\ & \leq & -\frac{1}{12\tau} V_2(x_t) - \frac{\tau^3}{|Q|} \beta(t,x_t)^\top Q \beta(t,x_t). \end{array} \tag{42}$$

It follows that

$$\dot{V}_3(t) \quad \leq \quad -\min\left\{ \tfrac{1}{12\tau}, \tfrac{1}{6|Q|} \right\} V_3(t,x_t). \tag{43}$$

This inequality implies that V_3 decays exponentially to zero. However, V_3 is a not a Lyapunov-Krasovskii functional because this function is not lower bounded by a function of class $\mathcal{K}_\infty$ of $|\phi(0)|$. Therefore, to establish the uniform exponential stability of the closed-loop system, we need to modify V_3 to obtain a Lyapunov-Krasovskii functional. To complete our construction of a Lyapunov-Krasovskii functional, we consider first the positive definite quadratic functional $V_4 : C_{\text{in}} \to [0,+\infty)$,

$$V_4(\phi) = \frac{1}{2}|\phi(0)|^2. \tag{44}$$

Its time derivative along the solutions of (15) satisfies

$$\begin{array}{l} \dot{V}_4(t) = x(t)^\top f(t,x(t)) \\ \qquad + x(t)^\top \displaystyle\int_{t-\tau}^{t} g(t,t-\ell,x(\ell)) \mu_{\tau,L}(\ell,x(\ell)) d\ell \\ \qquad + x(t)^\top w(t,x(t)). \end{array} \tag{45}$$

From Assumptions 1 and 2, we deduce that

$$\begin{array}{rcl} \dot{V}_4(t) & \leq & (f_m + w_0)|x(t)|^2 \\ & & + \frac{H_L}{\tau}|x(t)| \displaystyle\int_{t-\tau}^{t} |x(\ell)| d\ell \\ & \leq & -\frac{1}{2}|x(t)|^2 + \left(f_m + w_0 + \frac{1}{2}\right)|x(t)|^2 \\ & & + \frac{H_L}{\tau}|x(t)| \displaystyle\int_{t-\tau}^{t} |x(\ell)| d\ell, \end{array} \tag{46}$$

with $f_m = |L| + e^{|L|\tau} H_L$. Using (37), we obtain

$$\begin{aligned}\dot{V}_4(t) &\leq -\tfrac{1}{2}|x(t)|^2 \\ &+ \left(f_m + w_0 + \tfrac{1}{2}\right)\left(|\beta(t,x_t)| + \tfrac{2e^{|L|\tau}H_L}{\tau}V_1(x_t)\right)^2 \\ &+ \tfrac{H_L}{\tau}\left(|\beta(t,x_t)| + \tfrac{2e^{|L|\tau}H_L}{\tau}V_1(x_t)\right)\int_{t-\tau}^{t}|x(\ell)|d\ell.\end{aligned} \tag{47}$$

From this inequality, one can determine, through lengthy but simple calculations, a constant $c_2 > 0$ such that the inequality

$$\dot{V}_4(t) \quad \leq \quad -V_4(x_t) + c_2 V_3(t,x_t) \tag{48}$$

holds. We deduce that the time derivative along the trajectories of (15) of the functional $V_5 : \mathbb{R} \times C_{\text{in}} \to [0,+\infty)$,

$$V_5(t,\phi) = V_4(\phi) + 2\frac{c_2}{\min\left\{\frac{1}{12\tau}, \frac{1}{6|Q|}\right\}}V_3(t,\phi), \tag{49}$$

satisfies

$$\begin{aligned}\dot{V}_5(t) &\leq -V_4(x_t) - c_2 V_3(t,x_t) \\ &\leq -\min\left\{1, \tfrac{1}{24\tau}, \tfrac{1}{12|Q|}\right\}V_5(t,x_t).\end{aligned} \tag{50}$$

Finally, through lengthy, but simple calculations, one can prove that there exists a constant $c_3 > 0$ such that

$$\frac{1}{2}|\phi(0)|^2 \leq V_5(t,\phi) \leq c_3|\phi|^2_{[-\tau,0]}, \tag{51}$$

for all $t \in \mathbb{R}$ and $\phi \in C_{\text{in}}$. From (50) and (51), it follows that the origin of the system (15) is globally uniformly exponentially stable.

III. Example

The example we consider is similar to the one given in [5]. Instead of the one-dimensional system

$$\begin{aligned}\dot{x}(t) &= 0.36x(t) + v(t), \\ v(t) &= \int_0^{\tau} F(\ell)u(\ell)d\ell,\end{aligned} \tag{52}$$

with $\tau = 0.045$ and $F(\ell) = \frac{3542\ell(1-21\ell)}{1-21\ell+35^2\ell^2}$, we consider the one dimensional system

$$\begin{aligned}\dot{x}(t) &= \tfrac{1}{2}[\cos(t)+1]\sin(x(t))x(t) \\ &+ \int_0^{\tau} 3542\ell e^{-\frac{1771}{120}\ell}u(t-\ell)d\ell + w(t,x(t),\end{aligned} \tag{53}$$

with $\tau \in (0, 0.045)$ and w such that, for all $(t,x) \in \mathbb{R}^2$,

$$|w(t,x)| \leq \frac{1}{4320}|x|. \tag{54}$$

Then, with the notation of the previous section,

$$G_\tau(L,t,x) \quad = \quad \int_0^{\tau} e^{-L\ell}3542\ell e^{-\frac{1771}{120}\ell}d\ell. \tag{55}$$

It follows that the choice $L = -\frac{1}{120}$ gives

$$\begin{aligned}G_\tau(L,t,x) &= 3542\int_0^{\tau}\ell e^{-\left(-\frac{1}{120}+\frac{1771}{120}\right)\ell}d\ell \\ &= 3542\left[\frac{12^2(1-e^{-\frac{177}{12}\tau})}{177^2} - \frac{0.54e^{-\frac{177}{12}\tau}}{177}\right].\end{aligned} \tag{56}$$

Moreover, one can choose $Q = 180$ and then $w_0 < \frac{1}{24|Q|}$, with $w_0 = \frac{1}{4320}$. This leads us to consider

$$\mu_\tau(t,x) = \frac{177\left[(\cos(t)+1)\sin(x) + \frac{1}{60}\right]x}{85008\left[\frac{12}{177} - (0.045 + \frac{12}{177})e^{-\frac{177}{12}0.045}\right]}. \tag{57}$$

Since $g(t,\ell,x) = 3542\ell e^{-\frac{1771}{120}\ell}$, we deduce that one can take

$$H_L \geq \frac{16107.177}{75274584.e\left[12 - 19.965e^{-\frac{7.965}{12}}\right]}. \tag{58}$$

So we can choose $H_L = \frac{1}{123}$.

Now, let us check that $\tau \in (0, \tau_M)$ with $\tau_M > 0$ defined by the equality $\tau_M e^{\frac{\tau_M}{120}} = \frac{123}{2}$.

Observe that

$$\tau e^{\frac{\tau}{120}} = 0.045e^{\frac{0.045}{120}} < 0.046 < \frac{123}{2} = \tau_M e^{\frac{\tau_M}{120}}.$$

It follows that $\tau \in (0, \tau_M)$. Therefore Theorem 1 applies to the system (53), which implies that the origin of the system (53) is globally uniformly exponentially stabilized by the feedback defined in (57).

IV. Conclusion

For nonlinear systems with distributed input delay, we have proposed a new stabilization technique. It provides with state feedbacks which globally uniformly exponentially stabilize the origin of the systems. The proof of stability we have adopted relies on the construction of a Lyapunov-Krasovskii functional. Much remains to be done. We plan to perform robustness analyses with respect to uncertainties on the delay and to additive disturbances on the inputs in the spirit of what is done in [14]. We plan to use our new approach to perform sign analyses of the solutions of the system (2) as done in [15]. We plan to adapt the result to establish a result of global practical stabilization for the (2) with state-dependent sampling, as done in [13] for systems with pointwise delay in the inputs.

Appendix A
Finite escape time

In this appendix, we present a variant of the result of the appendix of [15] to prove that, for any delay $\tau > 0$ and for any continuous time-varying state feedback $u(t,x)$ the system

$$\dot{x}(t) = \int_{t-\tau}^{t} g_p(t-\ell)u(\ell,x(\ell))d\ell + x(t)^4 + x(t) \tag{59}$$

with $x \in \mathbb{R}$ and where g_p is a continuous nondecreasing function such that $g_p(s) = 0$ when $s \in \left[0, \frac{3\tau}{5}\right]$ and $g_p(s) = 1$ when $s \in \left[\frac{4\tau}{5}, \tau\right]$ admits solutions which go to the infinity in finite time.

Let $u(t,x)$ be a continuous feedback, uniformly bounded with respect to t. Then the constant $u_m = \sup_{\ell \in \left[-\tau, -\frac{2\tau}{3}\right]} |u(\ell, 0)|$ is well-defined. Consider a solution $x(t)$ of (59) in closed-loop with the selected control and with an initial condition $\phi_x \in C_{\text{in}}$ such that $\phi_x(\ell) = 0$ for all $\ell \in \left[-\tau, -\frac{4\tau}{5}\right]$ and $\phi_x(\ell) = \phi_0$ for all $\ell \in \left[-\frac{9\tau}{10}, 0\right]$ where $\phi_0 = \left(\frac{2\tau}{5}+1\right)u_m + \left(\frac{20}{\tau}\right)^{\frac{1}{3}}$. We proceed by contradiction: we assume that the finite escape time phenomenon does not occur, which implies that $x(t)$ is defined over $[0, +\infty)$. Then we have, for all $t \geq 0$,

$$\begin{aligned}\dot{x}(t) &= \int_{t-\tau}^{t-\frac{3\tau}{5}} g_p(t-\ell)u(\ell, x(\ell))d\ell \\ &+ \int_{t-\frac{3\tau}{5}}^{t} g_p(t-\ell)u(\ell, x(\ell))d\ell + x(t)^4 + x(t) \\ &= \int_{t-\tau}^{t-\frac{3\tau}{5}} g_p(t-\ell)u(\ell, x(\ell))d\ell + x(t)^4 + x(t) \\ &\geq - \int_{t-\tau}^{t-\frac{3\tau}{5}} |u(\ell, x(\ell))|d\ell + x(t)^4 + x(t) \ .\end{aligned} \tag{60}$$

Then, for all $t \in \left[0, \frac{\tau}{10}\right]$, we have

$$\begin{aligned}\dot{x}(t) &\geq - \int_{t-\tau}^{t-\frac{3\tau}{5}} |u(\ell, \phi_x(\ell))|d\ell + x(t)^4 + x(t) \\ &= - \int_{t-\tau}^{t-\frac{3\tau}{5}} |u(\ell, 0)|d\ell + x(t)^4 + x(t) \\ &\geq -\frac{2\tau}{5}u_m + x(t)^4 + x(t) \ .\end{aligned} \tag{61}$$

Next, we prove by contradiction that, $x(t) > u_m$, for all $t \in \left[0, \frac{\tau}{20}\right]$. First, observe that $x(0) = \phi_0 > u_m$. Next, assume there exists $t_a \in \left(0, \frac{\tau}{20}\right]$ such that $x(t) > u_m$, for all $t \in [0, t_a)$ and $x(t_a) = u_m$. Then, we deduce from (61) and the definition of u_m that for all $t \in [0, t_a)$, $\dot{x}(t) \geq x(t)^4 \geq 0$. It follows that $x(t_a) \geq \phi_0 > u_m$. This yields a contradiction with the definition of t_a. Therefore $x(t) > u_m$, for all $t \in \left[0, \frac{\tau}{20}\right]$. We deduce that, for all $t \in \left[0, \frac{\tau}{20}\right]$,

$$\dot{x}(t) > x(t)^4 \ . \tag{62}$$

By integrating this inequality, we deduce that, for all $t \in \left[0, \frac{\tau}{20}\right]$, we have $x(t) \geq \frac{\phi_0}{\left[1-3\phi_0^3 t\right]^{\frac{1}{3}}}$. Since we assumed that $x(t)$ is defined over $[0, +\infty)$, if follows that $1 - 3\phi_0^3 \frac{\tau}{20} > 0$. But the definition of ϕ_0 implies that this inequality does not hold.

Acknowledgment

The authors thank the associate editor and the anonymous reviewers for their useful remarks to improve the overall quality of our work.

References

[1] Z. Artstein, *Linear systems with delayed controls: A reduction*. IEEE Trans. Aut. Contr., AC-27, pp. 869-879, 1982.

[2] N. Bekiaris-Liberis, M. Krstic, *Lyapunov Stability of Linear Predictor Feedback for Distributed Input Delays.* IEEE Trans. Aut. Contr., vol. 56, no.3, pp. 655-660, 2011.

[3] K. A. Choon, *Stabilization of Linear Systems With Distributed Input Delay Using Reduction Transformation.* Chinese Science Bulletin, vol. 56, no. 13, pp. 1413-1416, May 2011.

[4] K. Engelborghs, M. Dambrine, D. Roose, *Limitations of a class of stabilization methods for delay systems.* IEEE Trans. Aut. Contr. vol. 46, pp. 336-339, Feb. 2001.

[5] G. Goebel, U. Munz, F. Allgower, *Stabilization of Linear Systems with Distributed Input Delay.* 2010 American Control Conference, June 30-July 02, pp. 5800-5805, 2010.

[6] K. Gu, S.-I. Niculescu, *Stability analysis of time-delay systems: A Lyapunov approach.* In A. Loria and F. Lamnhabi-Lagarrigue, editors, Advanced Topics in Control Systems Theory. Lectures Notes from FAP 2005, volume 328 of Lectures Notes in Control and Information Sciences, pp. 139-170. Springer Berlin, Heidelberg, 2006.

[7] H. Ito, P. Pepe, Z.-P. Jiang, *A small-gain condition for iISS of interconnected retarded systems based on Lyapunov-Krasovskii functionals.* Automatica, vol. 46 Issue 10, pp. 1646-1656, Oct., 2010.

[8] I. Karafyllis, Z.-P. Jiang, *Stability and Stabilization of Nonlinear Systems.* Communications and Control Engineering Series, Springer-Verlag London Ltd., London, UK, 2011.

[9] M. Krstic, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems.* Birkhäuser, Boston, 2009.

[10] W.H. Kwon, A.E. Pearson, *Feedback stabilization of linear systems with delayed control.* IEEE Trans. Aut. Contr., AC-25 (2), pp. 266-269, 1980.

[11] M. Malisoff, F. Mazenc, *Constructions of Strict Lyapunov Functions.* Communications and Control Engineering Series, Springer-Verlag London Ltd., London, ISBN: 978-1-84882-534-5 (Print), UK, 2009.

[12] A. Manitius, A. Olbrot, *Finite spectrum assignment problem for systems with delays.* IEEE Trans. Aut. Contr., vol. 24, pp. 541-553, 1979.

[13] F. Mazenc, C. De Persis, *Practical stabilization of nonlinear systems with state-dependent sampling and retarded inputs.* 2012 American Control Conference, Montreal, Canada 2012, to appear.

[14] F. Mazenc, M. Malisoff, Z. Lin, *Further results on input-to-state stability for nonlineal systems with delayed feedbacks.* Automatica, vol. 44, Issue 9, pp. 2415-2421, Sept. 2008.

[15] F. Mazenc, S.-I. Niculescu, *Generating Positive and Stable Solutions through Delayed State Feedback.* Automatica, vol. 47, pp. 525-533, 2011.

[16] D. Melchor-Aguilar, *On stability of integral delay systems* Applied Mathematics and Computation 217, pp. 3578-3584, 2010.

[17] A. Olbrot, *Stabilizability, detectability and spectrum assignment for linear autonomous systems with general time delays.* IEEE Trans. Aut. Contr., vol. 23, pp. 605-618, 1978.

[18] P. Pepe, E.I. Verriest, *On the Stability of Coupled Delay Differential and Continuous Time Difference Equations.* IEEE Trans. Aut. Contr., vol. 48, No. 8 pp. 1422-1427, Aug. 2003.

[19] V. Van Assche, M. Dambrine, J.-F. Lafay, J.-P. Richard, *Some problems arising in the implementation of distributed-delay control laws.* 38th Conference on Decision and Control, pp.4668-4672, Phoenix, Arizona, USA, 1999.

[20] L. Xie, E. Fridman, U. Shaked, *Robust H_∞ Control of Distributed Delay Systems With Application to Combustion Control.* IEEE Trans. Aut. Contr., vol. 46, no. 12, pp. 1930-1935, 2001.

[21] B. Zhou, Z. Lin, G. Duan, *Stabilization of linear systems with input delay and saturation - A parametric Lyapunov equation approach.* International Journal of Robust and Nonlinear Control vol. 20, Issue 13, pp. 1502-1519, 10 Sept. 2010.