# Asymptotic Stabilization for Feedforward Systems with Delayed Feedbacks

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**Abstract**

A problem of state feedback stabilization of time-varying feedforward systems with a pointwise delay in the input is solved. The approach relies on a time-varying change of coordinates and Lyapunov-Krasovskii functionals. The result applies for any given constant delay, and provides uniformly globally asymptotically stabilizing controllers of arbitrarily small amplitude. Although exact model knowledge and periodicity of the dynamics are required, the closed-loop systems enjoy input-to-state stability properties with respect to additive uncertainty on the controllers. The work is illustrated through a tracking problem for a model for high level formation flight of unmanned air vehicles.

*Key words:* Stabilization, delay compensation, robustness

## 1 Introduction

Time delayed systems are prevalent in aerospace systems, marine robotics, network control, population dynamics, and many other applications (Krstic, 2009). Input delays naturally arise due to transport phenomena, time consuming information processing, and sensor designs (Michiels & Niculescu, 2007). In the present paper, we continue our search (begun in (Mazenc & Malisoff, 2010)-(Mazenc, Niculescu, & Krstic, 2012)) for Lyapunov-Krasovskii functional designs for state feedbacks for systems with delayed inputs. Several approaches to input delays exist for linear systems, including reduction (which is also called finite spectrum assignment) and (modified Smith) prediction (Artstein, 1982), (Jankovic, 2008), (Krstic, 2009). Reduction transforms time-invariant delayed systems into finite dimensional ODEs (Fiagbedzi & Pearson, 1986; Wang, Lee, & Tan, 1998), but it can lead to complex controllers that may not always be suited to time-varying applications. The prediction technique in (Krstic, 2009) is an alternative tool for input delay compensation, where the system is augmented by a PDE that produces the feedback.

Although prediction produces infinite dimensional dynamics, it can lead to explicit Lyapunov-Krasovskii functional constructions and an explicit exponential stability estimate (Krstic & Smyshlaev, 2008). See (Bekiaris-Liberis & Krstic, 2011; Dashkovskiy & Naujok, 2010; Karafyllis & Jiang, 2011; Pepe & Jiang, 2006; Pepe, Jiang, & Fridman, 2008; Pepe & Verriest, 2003; Rasvan & Niculescu, 2002; Zhou, Lin, & Duan, 2010b) for the important advantages of Lyapunov-Krasovskii functionals, e.g., for inverse optimality and for quantifying the robustness of controllers to uncertainty. However, to the best of our knowledge, the preceding results do not extend to time-varying systems with control constraints and uncertainty. Hence, it may be difficult to apply methods such as prediction, e.g., in aerospace applications with thrust limitations (but see (Fischer, Dani, Sharma, & Dixon, 2011) for ultimate boundedness results for an important class of systems under input constraints and uncertainty using prediction). Moreover, (Mazenc, Mondié, & Francisco, 2004) cannot be adapted to the problem we consider here, because it considers time-invariant systems with dimension 1 blocks.

Our work (Mazenc, Malisoff, & Lin, 2008) gave a different method. There we assumed that a uniformly globally asymptotically stabilizing controller, and a corresponding strict Lyapunov function, are available for the closed loop system that is obtained by setting all input delays equal to zero. Then we computed an upper bound on the input delays that can be introduced while still ensuring uniform global asymptotic stability for the closed loop system. The work (Mazenc et al., 2008) covers time-varying nonlinear systems, and gives input-to-state stability (ISS) with respect to additive uncertainty on the control. See (Sontag,

⋆ Corresponding author: F. Mazenc. A summary of this paper excluding proofs is in preparation for possible inclusion in the Proceedings of the 2012 IEEE Conference on Decision and Control. Malisoff was supported by AFOSR Grant FA9550-09-1-0400 and NSF Grant ECCS-1056255.

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2008) for the importance of ISS and the standard definitions of the classes of functions $\mathcal{KL}$ and $\mathcal{K}_\infty$ used in ISS. However, (Mazenc et al., 2008) requires global Lipschitzness conditions for the dynamics, and the maximum allowable delays in (Mazenc et al., 2008; Mazenc, Niculescu, & Bekaik, 2011) depend on the structure of the dynamics.

Our work (Mazenc & Malisoff, 2010) used a different approach, where the uniformly globally asymptotically stabilizing feedback contains a tuning parameter that can make the supremum norm of the delayed term in the controller arbitrarily small. By reducing the tuning parameter, we proved uniform global asymptotic stability for the closed loop system for any given positive constant delay. Related ideas were used in (Zhou, Lin, & Duan, 2010a) to obtain stabilizing control laws without distributed terms for linear time invariant systems that have all poles in the closed-left half plane, using Barbalat's Lemma and linear matrix inequalities, but this earlier work does not lead to robustness in the ISS sense. It is natural to generalize the key idea of (Mazenc & Malisoff, 2010). In this note, we establish an important generalization for a broad family of linear time-varying feedforward systems, using a novel version of the time-varying change of coordinates from (Mazenc & Praly, 2000). This compensates any positive constant input delay using controllers of arbitrarily small amplitude, and leads to a Lyapunov-Krasovskii functional that makes it possible to prove ISS with respect to additive uncertainties on the controller. We illustrate our work using a model for unmanned air vehicles (UAVs).

## 2 Linear Feedforward Systems: Motivation

Consider the class of all feedforward systems of the form

$$\left\{\begin{array}{rcl} \dot{x} &=& h_1(z) + h_2(z)v(t-\tau) \\ \dot{z} &=& f(z) + g(z)v(t-\tau) \end{array}\right. \tag{1}$$

evolving on $\mathbb{R}^n \times \mathbb{R}^p$ for any dimensions $n$ and $p$, where $v$ is the input, $\tau$ is a constant delay, and the nonlinear functions $f$, $g$, $h_1$, and $h_2$ are at least $C^1$. When solving a tracking problem, we typically have the flexibility to choose a reference trajectory $(x_r, z_r)$ and a reference input $v_r$ that have period $\tau$ (but see Section 7 for an application with *non*periodic reference trajectories). Then, $\dot{x}_r(t) = h_1(z_r(t)) + h_2(z_r(t))v_r(t)$ and $\dot{z}_r(t) = f(z_r(t)) + g(z_r(t))v_r(t)$ hold for all times $t \geq 0$, so the error variables $\tilde{x} = x - x_r$ and $\tilde{z} = z - z_r$ satisfy

$$\left\{\begin{array}{rcl} \dot{\tilde{x}} &=& h_1(\tilde{z} + z_r(t)) - h_1(z_r(t)) \\ && +[h_2(\tilde{z} + z_r(t)) - h_2(z_r(t))]v_r(t) \\ && +h_2(\tilde{z} + z_r(t))\tilde{v}(t-\tau) \\ \dot{\tilde{z}} &=& f(\tilde{z} + z_r(t)) - f(z_r(t)) \\ && +[g(\tilde{z} + z_r(t)) - g(z_r(t))]v_r(t) \\ && +g(\tilde{z} + z_r(t))\tilde{v}(t-\tau) \end{array}\right. \tag{2}$$

where $\tilde{v}(t-\tau) = v(t-\tau) - v_r(t)$. In the absence of delay, (Mazenc & Praly, 2000) propose a solution to the problem of stabilizing the origin of the system (2). However, it does not seem that it can be easily extended to the case where there is a delay. The approximation of (2) at its equilibrium $(\tilde{x}, \tilde{z}) = (0, 0)$ has the form

$$\left\{\begin{array}{rcl} \dot{\tilde{x}} &=& C(t)\tilde{z} + D(t)\tilde{v}(t-\tau) \\ \dot{\tilde{z}} &=& A(t)\tilde{z} + B(t)\tilde{v}(t-\tau) \end{array}\right. \tag{3}$$

where $A$, $B$, $C$, and $D$ have period $\tau$. Therefore, if one can exponentially stabilize (3), then one can locally exponentially stabilize the trajectory $(x_r, z_r)$ of (1).

This motivates the study of systems of the form

$$\left\{\begin{array}{rcl} \dot{x}(t) &=& C(t)z(t) + D(t)u(t-\tau) \\ \dot{z}(t) &=& A(t)z(t) + B(t)u(t-\tau)\ , \end{array}\right. \tag{4}$$

where the $C^1$ functions $A$, $B$, $C$, and $D$ have period $\tau$, which commonly arise in tracking problems in engineering. We design feedbacks that render (4) uniformly globally asymptotically stable (UGAS). We restrict ourselves to linear systems (4), but see Section 7 where we leverage our results to cover an important nonlinear UAV model, and Section 8 for comments on other possible generalizations.

## 3 Assumptions and Lemmas

The following assumption on (4) will be useful:

**Assumption 1** *The system*

$$\dot{\theta}(t) = A(t)\theta(t) \tag{5}$$

*is UGAS, and the matrices $A, B, C$, and $D$ in (4) are $C^1$ on $\mathbb{R}$ and have some constant period $\tau$.* □

Let $\psi_a$ be defined by

$$\tfrac{\partial \psi_a}{\partial t}(t,m) = -\psi_a(t,m)A(t), \quad \psi_a(m,m) = \mathrm{I} \tag{6}$$

for all real values $t$ and $m$, where I is the identity matrix. We prove the following in Appendix A.1:

**Lemma 1** *Let Assumption 1 hold. Then the function $\mathrm{I} - \psi_a(\ell, \ell-\tau)$ is invertible for all $\ell \in \mathbb{R}$. Moreover, the function $q : \mathbb{R} \to \mathbb{R}^{n\times p}$ defined by*

$$q(t) = -\int_{t-\tau}^{t} C(\ell)[\mathrm{I} - \psi_a(\ell, \ell-\tau)]^{-1}\psi_a(t,\ell)\mathrm{d}\ell \tag{7}$$

*has period $\tau$, and $\dot{q}(t) + q(t)A(t) + C(t) = 0$ for all $t \in \mathbb{R}$.*

We use the $C^1$ function $R(t) = q(t)B(t) + D(t)$, which has period $\tau$. Our main theorem requires one more assumption, in which $M_1 \geq M_2$ for square matrices $M_i$ of the same size means that $M_1 - M_2$ is positive semidefinite:

**Assumption 2** *There exists a constant $c > 0$ such that*

$$\int_{t-\tau}^{t} R(m)R(m)^\top \mathrm{d}m \ \geq\ c\mathrm{I} \tag{8}$$

*for all $t \in \mathbb{R}$.*

The preceding assumption holds in many cases of interest. For example, we prove the following in Appendix A.2:

**Lemma 2** *If $R = (r_1, r_2)^\top : \mathbb{R} \to \mathbb{R}^2$ with $r_i$ having period $\tau$ for $i = 1, 2$, and if there is no constant $h$ such that $r_1(t) = hr_2(t)$ for all $t \in \mathbb{R}$ or such that $r_2(t) = hr_1(t)$ for all $t \in \mathbb{R}$, then Assumption 2 holds.*

## 4 Statement of and Remarks on Main Result

Set $\xi(t) = x(t)+q(t)z(t)$. By Lemma 1, system (4) becomes

$$\begin{cases} \dot{\xi}(t) = R(t)u(t-\tau) \\ \dot{z}(t) = A(t)z(t) + B(t)u(t-\tau)\,. \end{cases} \tag{9}$$

We prove the following, where $||\cdot||$ denotes the sup norm:

**Theorem 1** *Let Assumptions 1-2 hold. Then for all constants $\tau > 0$ and $\epsilon \in (0, \frac{1}{1+4\tau||R||^2})$, the controller*

$$u(t-\tau) = -\epsilon \frac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \tag{10}$$

*renders (9) UGAS.*

**Remark 1** *The controller (10) is bounded by $\epsilon||R||$. Therefore, we can satisfy any given control amplitude constraint $|u(t-\tau)| \leq \bar{c}$ for any constant $\bar{c} > 0$ if $\epsilon$ is sufficiently small. The restriction $0 < \epsilon < 1/(1+4\tau||R||^2)$ is needed for the stabilization. The controller requires knowing $R$ and so also $A$, $B$, $C$, and $D$, but see Section 6 for an extension that covers ISS with respect to small additive uncertainties on the control.*

**Remark 2** *A key feature of our proof of Theorem 1 is our Lyapunov-Krasovskii functional, which makes it possible to prove ISS with respect to actuator errors. Our Lyapunov-Krasovskii functional shares some features with the ones in (Mazenc et al., 2008, 2011). However, Theorem 1 is totally different from (Mazenc et al., 2008, 2011), because the earlier results have finite maximum allowable input delays when the system has drift, while $\tau$ in Theorem 1 can be as large as desired.*

## 5 Proof of Theorem 1

Let $\tau$ and $\epsilon$ be as in the statement of the theorem. We show that (9)-(10) admits the Lyapunov-Krasovskii functional

$$V^\sharp(t, \xi_t, z(t)) = V(t, z(t)) + \beta_1 W_3(t, \xi_t), \tag{11}$$

where $V(t, \theta) = \theta^\top P(t)\theta$ is a quadratic Lyapunov function for (5) such that $\dot{V} \leq -|\theta|^2$ along all trajectories of (5) and such that the $C^1$ positive definite valued function $P$ has period $\tau$ (which exists, e.g., by (Khalil, 2002, Chapter 4)),

$$\begin{aligned} &W_3(t, \xi_t) = W_2(t, \xi_t) + k\left[(1+2U(\xi_t))^{3/2} - 1\right], \\ &W_2(t, \xi_t) = W_1(t, \xi_t) + \beta_0 \int_{t-\tau}^{t} \left| \frac{R(m)^\top \xi(m)}{\sqrt{1+|\xi(m)|^2}} \right|^2 \mathrm{d}m, \\ &W_1(t, \xi_t) = \xi(t)^\top \left[\int_{t-\tau}^{t}\int_{m}^{t} R(\ell)R(\ell)^\top \mathrm{d}\ell\mathrm{d}m\right] \xi(t), \\ &U(\xi_t) = Q(\xi(t)) + \frac{1}{4\tau}\int_{t-2\tau}^{t}\int_{m}^{t} \frac{\epsilon|R(\ell)^\top \xi(\ell)|^2}{2\sqrt{2}\sqrt{1+|\xi(\ell)|^2}}\,\mathrm{d}\ell\mathrm{d}m, \end{aligned} \tag{12}$$

and $Q(\xi) = \frac{1}{2}|\xi|^2$, and $\xi_t$ is defined by $\xi_t(r) = \xi(t+r)$ for $-2\tau \leq r \leq 0$, $\beta_0 = ||R||^6\tau^4\epsilon^2/\{2c\}$, $c$ is from Assumption 2, $\beta_1 = \max\{(2/c)[4||P||^2\,||B||^2\,||R||^2 + 1], (16\sqrt{2}\tau/\{3\epsilon k\})(1 + 8\tau||P||^2\,||B||^2\,||R||^4)\}$, and $k = 4\sqrt{2}(\tau+\beta_0)/\{3\epsilon\}$. We often use the triangle inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ for values of $a$ and $b$ that we will specify.

The $\xi$ subsystem of (9), in closed loop with (10), becomes

$$\dot{\xi}(t) = -\epsilon\frac{R(t)R(t)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}} \tag{13}$$

since $R$ has period $\tau$. Observe for later use that

$$\xi(t-\tau) = \xi(t) + \epsilon\int_{t-\tau}^{t} \frac{R(m)R(m)^\top \xi(m-\tau)}{\sqrt{1+|\xi(m-\tau)|^2}}\mathrm{d}m\,. \tag{14}$$

Here and in the sequel, all relations should be understood to hold for all $t \geq \tau$, and dots indicate time derivatives along all trajectories of (9). Substituting (14) into (13) gives

$$\begin{aligned} \dot{Q} = &-\epsilon\frac{|R(t)^\top \xi(t)|^2}{\sqrt{1+|\xi(t-\tau)|^2}} \\ &-\epsilon^2\frac{\xi(t)^\top R(t)R(t)^\top}{\sqrt{1+|\xi(t-\tau)|^2}}\int_{t-\tau}^{t}\frac{R(m)R(m)^\top \xi(m-\tau)}{\sqrt{1+|\xi(m-\tau)|^2}}\mathrm{d}m. \end{aligned} \tag{15}$$

The triangle inequality (with $a = \sqrt{\epsilon}|R^\top(t)\xi(t)|$ and $b = \epsilon^{3/2}|R(t)||\mathcal{J}|$, where $\mathcal{J}$ is the integral in (15)), Jensen's inequality, and the periodicity of $R$ give

$$\begin{aligned} \dot{Q} \leq &-\epsilon\frac{|R(t)^\top \xi(t)|^2}{2\sqrt{1+|\xi(t-\tau)|^2}} \\ &+\frac{\epsilon^3||R||^2}{2\sqrt{1+|\xi(t-\tau)|^2}}\left|\int_{t-\tau}^{t}\frac{R(m)R(m)^\top \xi(m-\tau)}{\sqrt{1+|\xi(m-\tau)|^2}}\mathrm{d}m\right|^2 \\ \leq &-\epsilon\frac{|R(t)^\top \xi(t)|^2}{2\sqrt{1+|\xi(t-\tau)|^2}} \\ &+\frac{\epsilon^3||R||^2\tau}{2\sqrt{1+|\xi(t-\tau)|^2}}\left(\int_{t-\tau}^{t}\frac{||R||^2|R(m)^\top \xi(m-\tau)|^2}{1+|\xi(m-\tau)|^2}\mathrm{d}m\right). \end{aligned}$$

Using (14) and our upper bound on $\epsilon$, we have $|\xi(t-\tau)|^2 \leq (|\xi(t)| + \epsilon||R||^2\tau)^2 \leq 2|\xi(t)|^2 + 2\epsilon^2||R||^4\tau^2 \leq 2|\xi(t)|^2 + 1$ and so $-1/\sqrt{1+|\xi(t-\tau)|^2} \leq -1/(\sqrt{2}\sqrt{1+|\xi(t)|^2})$ and

$$\dot{Q} \leq -\frac{\epsilon|R(t)^\top \xi(t)|^2}{2\sqrt{2}\sqrt{1+|\xi(t)|^2}} + \frac{\epsilon^3||R||^4\tau}{2}\int_{t-2\tau}^{t-\tau}\frac{|R(m)^\top \xi(m)|^2}{1+|\xi(m)|^2}\mathrm{d}m\,.$$

We next find decay estimates on $U$ and $W_1$. We have

$$\begin{aligned} \dot{U} \leq &-\frac{\epsilon|R(t)^\top \xi(t)|^2}{4\sqrt{2}\sqrt{1+|\xi(t)|^2}} + \frac{\epsilon^3||R||^4\tau}{2}\int_{t-2\tau}^{t-\tau}\frac{|R(m)^\top \xi(m)|^2}{1+|\xi(m)|^2}\mathrm{d}m \\ &-\frac{\epsilon}{8\sqrt{2}\tau}\int_{t-2\tau}^{t}\frac{|R(\ell)^\top \xi(\ell)|^2}{\sqrt{1+|\xi(\ell)|^2}}\mathrm{d}\ell \\ \leq &-\frac{\epsilon|R(t)^\top \xi(t)|^2}{4\sqrt{2}\sqrt{1+|\xi(t)|^2}} - \frac{\epsilon}{16\sqrt{2}\tau}\int_{t-2\tau}^{t}\frac{|R(\ell)^\top \xi(\ell)|^2}{\sqrt{1+|\xi(\ell)|^2}}\mathrm{d}\ell, \end{aligned}$$

where the last inequality used our upper bound on $\epsilon$.

Moreover, Assumption 2 gives

$$\begin{aligned} \dot{W}_1 = &-\xi(t)^\top\left[\int_{t-\tau}^{t}R(\ell)R(\ell)^\top \mathrm{d}\ell\right]\xi(t) \\ &+\tau\xi(t)^\top R(t)R(t)^\top \xi(t) \\ &+2\xi(t)^\top\left[\int_{t-\tau}^{t}\int_{m}^{t}R(\ell)R(\ell)^\top \mathrm{d}\ell\mathrm{d}m\right]\dot{\xi}(t) \\ \leq &-c|\xi(t)|^2 + \tau|R(t)^\top \xi(t)|^2 \\ &+||R||^2\tau^2|\dot{\xi}(t)||\xi(t)|\,. \end{aligned} \tag{16}$$

The triangle inequality with $a = ||R||^2\tau^2|\dot{\xi}(t)|/\sqrt{c}$ and $b = \sqrt{c}|\xi(t)|$ applied to the last term in (16), (13), and the periodicity of $R$ then give

$$\begin{aligned}\dot{W}_1 \leq & -\tfrac{c}{2}|\xi(t)|^2 + \tau|R(t)^\top\xi(t)|^2 \\ & + \tfrac{||R||^6\tau^4\epsilon^2}{2c}\left|\tfrac{R(t-\tau)^\top\xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}}\right|^2 .\end{aligned}$$

Therefore, our choices of $W_2$ and $\beta_0$ from (12) give

$$\dot{W}_2 \leq -\tfrac{c}{2}|\xi(t)|^2 + (\tau+\beta_0)\,|R(t)^\top\xi(t)|^2. \quad (17)$$

Setting

$$\mathcal{M}(\xi_t) = \tfrac{3k\epsilon}{16\sqrt{2}\tau}[1+2U(\xi_t)]^{1/2}\int_{t-2\tau}^{t}\tfrac{|R(\ell)^\top\xi(\ell)|^2}{\sqrt{1+|\xi(\ell)|^2}}\mathrm{d}\ell, \quad (18)$$

our choice of $k$ that is present in $W_3$ from (12) gives

$$\begin{aligned}\dot{W}_3 \leq & -\tfrac{c}{2}|\xi(t)|^2 + (\tau+\beta_0)\,|R(t)^\top\xi(t)|^2 \\ & -3k(1+2U(\xi_t))^{1/2}\tfrac{\epsilon|R(t)^\top\xi(t)|^2}{4\sqrt{2}\sqrt{1+|\xi(t)|^2}} - \mathcal{M}(\xi_t) \\ \leq & -\tfrac{c}{2}|\xi(t)|^2 + (\tau+\beta_0)\,|R(t)^\top\xi(t)|^2 \\ & -3k\tfrac{\epsilon|R(t)^\top\xi(t)|^2}{4\sqrt{2}} - \mathcal{M}(\xi_t) \\ \leq & -\tfrac{c}{2}|\xi(t)|^2 - \tfrac{3k\epsilon}{16\sqrt{2}\tau}\int_{t-2\tau}^{t}\tfrac{|R(\ell)^\top\xi(\ell)|^2}{\sqrt{1+|\xi(\ell)|^2}}\,\mathrm{d}\ell\ ,\end{aligned}$$

since $U(\xi_t) \geq \frac{1}{2}\xi^2(t)$ for all $t$. Notice that (14), Jensen's inequality, and the general relation $(a+b)^2 \leq 2a^2+2b^2$ give

$$\begin{aligned}& |\xi(t-\tau)|^2 \\ & \leq 2|\xi(t)|^2 + 4\tau||R||^2\int_{t-2\tau}^{t}\tfrac{|R^\top(\ell)\xi(\ell)|^2}{\sqrt{1+|\xi(\ell)|^2}}\,\mathrm{d}\ell,\end{aligned} \quad (19)$$

and $|u(t-\tau)| \leq ||R||\,|\xi(t-\tau)|$. Hence, the triangle inequality with $a = |z(t)|$ and $b = 2||P||\,||B||\,|u(t-\tau)|$ gives

$$\begin{aligned}\dot{V} \leq & -|z(t)|^2 + \{|z(t)|\}\{2||P||\,||B||\,|u(t-\tau)|\} \\ \leq & -\tfrac{1}{2}|z(t)|^2 + 2||P||^2\,||B||^2|u(t-\tau)|^2 \\ \leq & -\tfrac{1}{2}|z(t)|^2 + 4||P||^2\,||B||^2\,||R||^2|\xi(t)|^2 \\ & + 8||P||^2\,||B||^2\tau||R||^4\int_{t-2\tau}^{t}\tfrac{|R^\top(\ell)\xi(\ell)|^2}{\sqrt{1+|\xi(\ell)|^2}}\mathrm{d}\ell.\end{aligned} \quad (20)$$

Hence, $V^\sharp$ as defined in (11) satisfies $\dot{V}^\sharp \leq -0.5|z(t)|^2 - |\xi(t)|^2$ along all trajectories of (9), by our choice of $\beta_1$. Also, $V^\sharp$ admits $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ such that

$$\underline{\alpha}(|(\phi_\xi,\phi_z)(0)|) \leq V^\sharp(t,\phi_\xi,\phi_z(0)) \leq \overline{\alpha}(||(\phi_\xi,\phi_z)||) \quad (21)$$

for all continuous functions $(\phi_\xi,\phi_z) : [-2\tau,0] \to \mathbb{R}^n \times \mathbb{R}^p$. It follows from the decay condition on $V^\sharp$ and (21) that $V^\sharp$ is a Lyapunov-Krasovskii functional for the closed loop system (9)-(10), which is therefore UGAS.

## 6 Robustness to Actuator Errors

Under our Assumptions 1-2, we can also prove ISS results for the perturbed version

$$\begin{cases}\dot{\xi}(t) = R(t)\big[u(t-\tau)+\delta(t)\big] \\ \dot{z}(t) = A(t)z(t) + B(t)\big[u(t-\tau)+\delta(t)\big]\end{cases} \quad (22)$$

of (9) where the unknown disturbance $\delta$ is valued in some set $\mathcal{D}$. The result to follow differs from existing nonpredictive ISS results under input delays (e.g., (Mazenc et al., 2008)) because it allows arbitrarily long input delays and only requires the value of $u$ at times $t-\tau$. By ISS of a delayed system with respect to $\mathcal{D}$, we mean that there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that $|q(t)| \leq \beta(|q|_{[t_0-\tau,t_0]}, t-t_0) + \gamma(|\delta|_{[t_0,t]})$ holds along all of its trajectories for all initial times $t_0 \geq 0$, all $t \geq t_0$, all initial functions defined on $[t_0-\tau,t_0]$, and all measurable essentially bounded functions $\delta : [0,\infty) \to \mathcal{D}$, where the subscripts denote essential suprema over the indicated intervals. Set

$$\begin{aligned}& \overline{\delta} = \tfrac{c}{9k||R||(1+2\overline{u})^{1/2}}, \quad \text{where} \\ & \overline{u} = \max\left\{\tfrac{1}{2} + \tfrac{\epsilon||R||^2\tau}{4\sqrt{2}},\ \tfrac{\epsilon||R||^2\tau}{4\sqrt{2}}\left(1+2\epsilon||R||^2\tau\right)\right\},\end{aligned} \quad (23)$$

and $\epsilon$, $k$, and $c$ satisfy the requirements from Section 5. It follows from (13) that $U(\xi_t) \leq \overline{u}(1+|\xi(t)|^2)$ for all $t$ along all trajectories $\xi : \mathbb{R} \to \mathbb{R}^n$. We prove the following ISS result in Appendix A.3, where for any constant $\mu > 0$, $\mu\mathcal{B}$ is the closed ball of radius $\mu$ centered at 0:

**Theorem 2** *If Assumptions 1-2 hold, then (22) in closed loop with (10) is ISS with respect to $\mathcal{D} = \overline{\delta}\mathcal{B}$.* □

There is a constant $\delta_*$ that is independent of $\epsilon$ such that $\overline{\delta} \leq \delta_*\epsilon$ for all $\epsilon$ (by our choice of $k$), so $\overline{\delta}$ converges to zero as $\epsilon$ converges to 0, but this is expected because $|u(t-\tau)| \leq ||R||\epsilon$ along all trajectories, and the disturbance must have smaller magnitude than the controller.

## 7 Application to UAV Dynamics

Consider a UAV with standard autopilots, which is first order for heading and Mach hold and second order for altitude hold. This yields the key model (Gruszka, Malisoff, & Mazenc, 2012; Ren & Beard, 2004)

$$\begin{cases}\dot{x} = v\cos(\theta) \\ \dot{y} = v\sin(\theta) \\ \dot{\theta} = \alpha_\theta(\theta_c(t-\tau)-\theta) \\ \dot{v} = \alpha_v(v_c(t-\tau)-v),\end{cases} \quad (24)$$

where we omit the second order altitude subdynamics since altitude controllers $h^c$ are already available (Ren & Beard, 2004). In (24), $(x,y)$ is the position of the UAV with respect to an inertial coordinate system, $\theta$ is the heading angle, $v$ is the inertial velocity, $\alpha_\theta$ and $\alpha_v$ are positive constants associated with the autopilot, the controllers $\theta_c$ and $v_c$ are to be determined, and $\tau > 0$ is any constant delay. This underactuated kino-dynamic representation is justifiable for high-level formation flight control; see (Ren & Beard, 2004) for realistic UAV simulations for transitioning through several targets under multiple dynamic threats using models of this type and (Gruszka et al., 2012; Ren & Beard, 2004) for numerous references for work on UAV models with uncertainty. However, to the best of our knowledge, there are no non-predictive finite dimensional tracking controllers that compensate arbitrarily long delays in (24) under the mild assumptions we give in this section, so our work is a significant improvement over the existing results.

Fix any $C^1$ reference trajectory $(x_r, y_r, \theta_r, v_r)$ for (24), i.e., there is a $C^1$ function $(\theta_{cr}, v_{cr}) : \mathbb{R} \to \mathbb{R}^2$ such that

$$\left\{\begin{array}{rcl} \dot{x}_r(t) &=& v_r(t)\cos(\theta_r(t)) \\ \dot{y}_r(t) &=& v_r(t)\sin(\theta_r(t)) \\ \dot{\theta}_r(t) &=& \alpha_\theta(\theta_{cr}(t) - \theta_r(t)) \\ \dot{v}_r(t) &=& \alpha_v(v_{cr}(t) - v_r(t)) \end{array}\right. \tag{25}$$

holds for all $t \in \mathbb{R}$. We also assume:

**Assumption 3** *The functions $\cos(\theta_r(t))$ and $\sin(\theta_r(t))$ are periodic of period $\tau$, there exists a constant $t_c \in [0, \tau]$ such that $\dot{\theta}_r(t_c) \neq 0$, and $v_r$ is bounded.* □

**Remark 3** *The reference trajectory is not required to be periodic. For example Assumption 3 holds for cases where $\theta_r(t) = t$ for all $t$ where the UAV is to orbit a point (e.g., circles, where $v_r = v_{cr} = 1$ and $\theta_{cr} = t + 1/\alpha_\theta$). More generally, if $\theta_r : \mathbb{R} \to \mathbb{R}$ is any $C^2$ function that admits integers $k$ and $\ell$ such that $\dot{\theta}_r$ has period $\tau = 2\pi\ell$ and $\theta_r(\tau) = \theta_r(0) + k\tau$, then $\cos(\theta_r(t))$ and $\sin(\theta_r(t))$ have period $\tau$. This is because $\phi_1(t) = (\cos(\theta_r(t)), \sin(\theta_r(t)))$ and $\phi_2(t) = (\cos(\theta_r(t+\tau)), \sin(\theta_r(t+\tau)))$ are both solutions of the system $\dot{p}_1 = -\dot{\theta}_r(t)p_2$ and $\dot{p}_2 = \dot{\theta}_r(t)p_1$ with the same value at $t = 0$. Then our assumptions hold for any bounded $C^1$ function $v_r : \mathbb{R} \to \mathbb{R}$ using $\theta_{cr}(t) = \theta_r(t) + \dot{\theta}_r(t)/\alpha_\theta$ and $v_{cr}(t) = v_r(t) + \dot{v}_r(t)/\alpha_v$ if $\dot{\theta}_r$ is not the zero function on $[0, \tau]$. This allows $(x_r(t), y_r(t))$ to be unbounded, e.g., by taking $v_r(t) = \cos(\theta_r(t))$.*

Choose the error variables $\overline{x} = x - x_r(t)$, $\overline{y} = y - y_r(t)$, $\overline{\theta} = \theta - \theta_r(t)$, and $\overline{v} = v - v_r(t)$ and the controllers $\theta_c(t - \tau) = \theta_{cr}(t)$ and $v_c(t - \tau) = v_{cr}(t) + u(t - \tau)/\alpha_v$ for all $t$ where $u$ is to be determined. This gives the tracking dynamics

$$\left\{\begin{array}{rl} \dot{\overline{x}} =& \cos(\theta_r(t))\overline{v} \\ & + [\overline{v} + v_r(t)][\cos(\overline{\theta} + \theta_r(t)) - \cos(\theta_r(t))] \\ \dot{\overline{y}} =& \sin(\theta_r(t))\overline{v} \\ & + [\overline{v} + v_r(t)][\sin(\overline{\theta} + \theta_r(t)) - \sin(\theta_r(t))] \\ \dot{\overline{v}} =& -\alpha_v\overline{v} + u(t - \tau), \\ \dot{\overline{\theta}} =& -\alpha_\theta\overline{\theta}\ . \end{array}\right. \tag{26}$$

Our stabilization analysis for (26) has two steps. In the first step, we apply Theorem 1 to the simpler system

$$\left\{\begin{array}{rcl} \dot{\overline{x}} &=& \cos(\theta_r(t))\overline{v}, \\ \dot{\overline{y}} &=& \sin(\theta_r(t))\overline{v} \\ \dot{\overline{v}} &=& -\alpha_v\overline{v} + u(t - \tau) \end{array}\right. \tag{27}$$

obtained by setting $\bar{\theta} = 0$, to find a control $u$. Then, we prove that (26) with our controller for (27) is UGAS.

*First Step.* To apply Theorem 1 to (27), note that in terms of our previous notation, we have $n = 2$, $p = 1$, $A(t) = -\alpha_v$, $B(t) = 1$, $C(t) = (\cos(\theta_r(t)), \sin(\theta_r(t)))^\top$, $D(t) = 0$, $\psi_a(t, \ell) = e^{\alpha_v(t-\ell)}$, $q(t) = R(t)$, $x = (\overline{x}, \overline{y})$, $z = \overline{v}$,

$$\begin{array}{l} R(t) = \dfrac{1}{e^{\alpha_v\tau} - 1} \displaystyle\int_{t-\tau}^{t} \begin{pmatrix} \cos(\theta_r(\ell)) \\ \sin(\theta_r(\ell)) \end{pmatrix} e^{\alpha_v(t-\ell)} \mathrm{d}\ell\ , \\ \text{and } \xi = (\xi_1, \xi_2)^\top = (\overline{x}, \overline{y})^\top + q(t)\overline{v}\ . \end{array} \tag{28}$$

We prove:

**Lemma 3** *If Assumption 3 holds and $R = (r_1, r_2)^\top$ is defined by (28), then there does not exist a constant $h$ such that $r_1(t) = hr_2(t)$ for all $t \in \mathbb{R}$ or such that $r_2(t) = hr_1(t)$ for all $t \in \mathbb{R}$. Therefore, $R$ satisfies Assumption 2.* □

*Proof.* First suppose that there were an $h \in \mathbb{R}$ such that $r_1(t) = hr_2(t)$ for all $t$. Then $h \neq 0$, because otherwise Assumption 3 would give $(d/dt)\int_{t-\tau}^{t} \cos(\theta_r(\ell))e^{-\alpha_v\ell}\mathrm{d}\ell = 0$ and $\cos(\theta_r(t)) = 0$ for all $t$, so $\dot{\theta}_r(t)\sin(\theta_r(t)) = 0$ for all $t$, which would give the contradiction $\cos(\theta_r(t_c)) = \sin(\theta_r(t_c)) = 0$. Using the formulas for the $r_i$ and differentiating both sides of equality $r_1(t) = hr_2(t)$ gives

$$\begin{array}{l} \alpha_\nu \int_{t-\tau}^{t} \cos(\theta_r(\ell))e^{\alpha_\nu(t-\ell)}\mathrm{d}\ell \\ + \cos(\theta_r(t)) - \cos(\theta_r(t - \tau))e^{\alpha_\nu\tau} \\ = h\alpha_\nu \int_{t-\tau}^{t} \sin(\theta_r(\ell))e^{\alpha_\nu(t-\ell)}\mathrm{d}\ell \\ + h\sin(\theta_r(t)) - h\sin(\theta_r(t - \tau))e^{\alpha_\nu\tau}\ . \end{array}$$

Therefore, $\cos(\theta_r(t)) - \cos(\theta_r(t - \tau))e^{\alpha_\nu\tau} = h\sin(\theta_r(t)) - h\sin(\theta_r(t - \tau))e^{\alpha_\nu\tau}$. Since $\cos(\theta_r(t))$ and $\sin(\theta_r(t))$ are periodic of period $\tau$, and since $\alpha_v\tau > 0$, it follows that

$$\cos(\theta_r(t)) = h\sin(\theta_r(t)) \quad \text{for all } t \in \mathbb{R}\ . \tag{29}$$

Differentiating both sides of (29) gives the equality $-\dot{\theta}_r(t_c)\sin(\theta_r(t_c)) = h\dot{\theta}_r(t_c)\cos(\theta_r(t_c))$, and so also $-\sin(\theta_r(t_c)) = h\cos(\theta_r(t_c))$, which when combined with (29) gives $\cos(\theta_r(t_c)) = -h^2\cos(\theta_r(t_c))$. Hence, $\cos(\theta_r(t_c)) = 0$. This is a contradiction, because then (29) would also give $\sin(\theta_r(t_c)) = 0$, because $h \neq 0$. The same reasoning applies with the roles of $r_1$ and $r_2$ reversed, so the lemma now follows from Lemma 2. □

It follows from Theorem 1 that for any $\tau > 0$, the controller (10), with the above choices of $R$ and $\xi$, renders (27) UGAS.

*Second Step.* In Appendix A.4, we prove the following, which implies trackability of our reference trajectories:

**Theorem 3** *If Assumption 3 holds, then for any constant delay $\tau > 0$, the controller (10) renders (26) UGAS.*

The complete expressions for our controllers are therefore

$$\begin{array}{l} \theta_c(t - \tau) = \theta_{cr}(t) \ \text{ and} \\ v_c(t - \tau) = v_{cr}(t) - \frac{\epsilon}{\alpha_v} \frac{R(t-\tau)^\top \xi(t-\tau)}{\sqrt{1+|\xi(t-\tau)|^2}}, \end{array} \tag{30}$$

where $R$ and $\xi$ are from (28), $(\theta_{cr}, v_{cr})$ is the reference input, $\epsilon \in (0, 1/(1 + 4\tau||R||^2))$ is any constant, and $\tau > 0$ is any constant delay. Moreover, given any velocity control set $[v_a, v_b]$ with positive constants $v_a$ and $v_b$, and any constant $\mu > 0$ such that $v_a + \mu \le v_{cr}(t) \le v_b - \mu$ for all $t \in \mathbb{R}$, we can choose $\epsilon$ small enough so that $v_a \le v_c(t) \le v_b$ for all $t \in \mathbb{R}$ to satisfy input constraints, and similarly for $\theta_c$. We can also apply the ISS result from the preceding section.

To illustrate our ISS result, we simulated (24) with (30), the constants $\alpha_v = 0.192$ and $\alpha_\theta = 0.55$ from (Ren & Beard, 2004), $\tau = 2$, the period 2 reference trajectory $(x_r(t), y_r(t), \theta_r(t), v_r(t)) = (20 + 10\sin(\pi t)/\pi, 20 - 10\cos(\pi t)/\pi, \pi t, 10)$ for reference control $(\theta_{cr}(t), v_{cr}(t)) = (\pi(t + 1/\alpha_\theta), 10)$, the controls (30) with $\epsilon = 0.257732$, the disturbance $\delta(t) = 0.1\sin(t)$ added to $v_c$, and the initial function $(x_0, y_0, \theta_0, v_0) = (17, 22, -0.5, 8)$ defined on the interval $[-2, 0]$, which satisfy our requirements. In Fig. 1, we plot $(x(t), y(t))$ on the time interval $[480, 1000]$, and the short and long term behavior of the tracking errors and the closed loop control. The units are radians, seconds, and meters. Our simulation shows good tracking performance and therefore helps validate our theory.

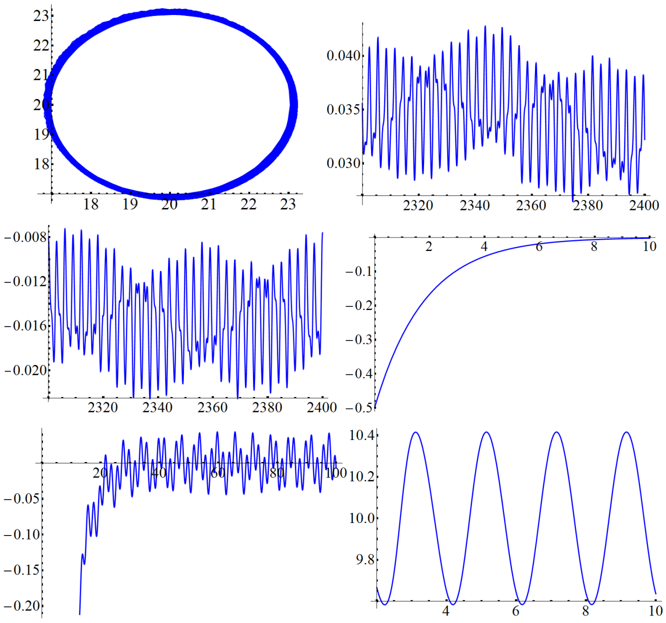


Fig. 1. Short and Long Term Trajectory $(x(t), y(t))$ (Top Left), $x(t) - x_r(t)$ (Top Right), $y(t) - y_r(t)$ (Middle Left), $\theta(t) - \theta_r(t)$ (Middle Right), $v(t) - v_r(t)$ (Bottom Left), and Closed Loop Control $v_c(t - \tau)$ as Function of Time (Bottom Right) for UAV System (24). Initial Function: $(x_0, y_0, \theta_0, v_0) = (17, 22, -0.5, 8)$. Units for Simulation: Meters, Radians, and Seconds.

## 8 Conclusions and Generalizations

Feedback stabilization under delayed inputs has been studied by several authors, using many methods. Our approach ensures UGAS for time-varying feedforward systems with arbitrarily long input delays, based on a tuning parameter that yields controllers of arbitrarily small amplitude and a time varying change of coordinates. Our new Lyapunov-Krasovskii functional construction was key to proving ISS with respect to additive uncertainty on the controllers. We applied our work to a key input delayed UAV model.

Due to the delays, Theorem 1 cannot be applied repeatedly. However, since our feedbacks can be chosen to have arbitrarily small gains and our proof of Theorem 1 relies on a Lyapunov-Krasovskii functional, we conjecture that extensions to linear feedforward systems with an arbitrary number of blocks are possible. Indeed, an appropriate selection of small gains makes it possible to decompose $u(t - \tau)$ as $u(t)$ plus extra terms, which can be neglected, as explained in (Zhou et al., 2010a) in a time-invariant context. Upper bounds for the gains can be derived using a Lyapunov-Krasovskii functional. The special structure of our system (4) including the periodicity of the coefficient matrices was required in the proofs of Lemmas 1-2, but we conjecture that analogues of the lemmas can be found to prove our results without periodicity assumptions. Due to space constraints, we leave these extensions to future studies.

## Appendix A.1: Proof of Lemma 1

Let $\phi_a(t, m)$ be the fundamental matrix for $A$, i.e.,

$$\frac{\partial \phi_a}{\partial t}(t, m) = A(t)\phi_a(t, m), \quad \phi_a(m, m) = \mathrm{I} \tag{A.1}$$

for all $t$ and $m$ in $\mathbb{R}$. Then $\phi_a(t, m) = \psi_a(t, m)^{-1}$ for all $t$ and $m$ in $\mathbb{R}$. For each integer $k > 1$ and all $t \in \mathbb{R}$, we can use the semigroup property of the flow map and $\phi_a(t, t - \tau) = \phi_a(t + \tau, t)$ and induction to get $\phi_a(t + k\tau, t - \tau) = [\phi_a(t, t - \tau)]^{k+1}$. Hence, Assumption 1 implies that $[\phi_a(t, t - \tau)]^k$ converges to zero as $k \to \infty$ for all $t \in \mathbb{R}$. Therefore, for each $t \in \mathbb{R}$, 1 cannot be an eigenvalue of $\phi_a(t, t - \tau)$, nor can it be one for $\psi_a(t, t - \tau)$. Hence, $\mathrm{I} - \psi_a(t, t - \tau)$ is invertible for each $t \in \mathbb{R}$. Next take the function $\Lambda(t) = -C(t)[\mathrm{I} - \psi_a(t, t - \tau)]^{-1}$. Then the proof that $q(t) = \int_{t-\tau}^{t} \Lambda(\ell)\psi_a(t, \ell)\mathrm{d}\ell$ is periodic of period $\tau$ follows by changing variables and noting that $\Lambda(\ell + \tau) = \Lambda(\ell)$ and $\psi_a(t + \tau, \ell + \tau) = \psi_a(t, \ell)$ for all $t$ and $\ell$ in $\mathbb{R}$. Next notice that $\dot{q}(t) = \int_{t-\tau}^{t} \Lambda(\ell)\frac{\partial \psi_a}{\partial t}(t, \ell)\mathrm{d}\ell + \Lambda(t) - \Lambda(t - \tau)\psi_a(t, t - \tau)$ for all $t \in \mathbb{R}$. Then our choice of $\Lambda$ gives $\dot{q}(t) = -q(t)A(t) + \Lambda(t) - \Lambda(t)\psi_a(t, t - \tau) = -q(t)A(t) + \Lambda(t)[\mathrm{I} - \psi_a(t, t - \tau)] = -q(t)A(t) - C(t)$ for all $t \in \mathbb{R}$. This proves the lemma.

## Appendix A.2: Proof of Lemma 2

Since $R$ has period $\tau$, the result will follow once we show that $\bar{M} = \int_0^\tau R(m)R(m)^\top \mathrm{d}m$ is positive definite. First consider the case where $r_2$ is not identically equal to zero. Since $r_1$ and $r_2$ are periodic of period $\tau$, the trace of $\bar{M}$ is positive. Hence, $\bar{M}$ is positive definite provided

$$\int_0^\tau r_1^2(m)\mathrm{d}m \int_0^\tau r_2^2(m)\mathrm{d}m > \left(\int_0^\tau r_1(m)r_2(m)\mathrm{d}m\right)^2. \tag{A.2}$$

Therefore, Lemma 2 follows if (A.2) holds. If (A.2) does not hold, then the Cauchy-Schwarz inequality gives

$$\int_0^\tau r_1^2(m)\mathrm{d}m \int_0^\tau r_2^2(m)\mathrm{d}m = \left(\int_0^\tau r_1(m)r_2(m)\mathrm{d}m\right)^2.$$

Since we assumed that $r_2$ is not identically equal to zero, we can define $g$ by $g = \int_0^\tau r_1(m)r_2(m)\mathrm{d}m / \int_0^\tau r_2^2(m)\mathrm{d}m$.

Define the function $s$ by $s(t) = r_1(t) - gr_2(t)$. Then

$$\begin{array}{l}\int_0^\tau [s(m) + gr_2(m)]^2 \mathrm{d}m \int_0^\tau r_2^2(m)\mathrm{d}m = \\ \left(\int_0^\tau [s(m) + gr_2(m)]r_2(m)\mathrm{d}m\right)^2 .\end{array} \tag{A.3}$$

It follows that

$$\begin{array}{l}\int_0^\tau [s^2(m) + 2gs(m)r_2(m) + g^2 r_2^2(m)]\mathrm{d}m \int_0^\tau r_2^2(m)\mathrm{d}m \\ = \left( \int_0^\tau [s(m)r_2(m) + gr_2^2(m)]\mathrm{d}m\right)^2 .\end{array}$$

The definition of $g$ gives $\int_0^\tau s(m)r_2(m)\mathrm{d}m = 0$. Hence,

$$\begin{array}{l}\int_0^\tau [s^2(m) + g^2 r_2^2(m)]\mathrm{d}m \int_0^\tau r_2^2(m)\mathrm{d}m \\ = g^2 \left(\int_0^\tau r_2^2(m)\mathrm{d}m\right)^2 .\end{array} \tag{A.4}$$

Canceling $\int_0^\tau r_2^2(m)dm > 0$ from both sides of (A.4) gives $\int_0^\tau s^2(m)\mathrm{d}m = 0$, so $r_1(t) = gr_2(t)$ for all $t$. Analogous arguments apply when $r_1$ is not identically equal to 0.

## Appendix A.3: Proof of Theorem 2

We indicate the changes necessary in the proof of Theorem 1 to prove Theorem 2. We have

$$\dot W_2 \le -\tfrac{c}{2}|\xi(t)|^2 + (\tau + \beta_0)\,|R(t)^\top \xi(t)|^2 + \overline{c}|\xi(t)|\,|\delta|,$$

where $\overline{c} = \tau^2||R||^3$. Throughout this appendix, the time derivatives are along all trajectories of (22). By the triangle inequality and our choices of $\overline{u}$ and $\overline{\delta}$ in (23) and $\mathcal{M}$ in (18),

$$\begin{aligned}\dot W_3 \le\ & -\tfrac{c}{2}|\xi(t)|^2 + \overline{c}|\xi(t)|\,|\delta| \\ & + 3k(1 + 2U(\xi_t))^{1/2}||R||\,|\xi(t)|\,|\delta| - \mathcal{M}(\xi_t) \\ \le\ & -\tfrac{c}{2}|\xi(t)|^2 + \overline{c}|\xi(t)|\,|\delta| \\ & + 3k||R||(1 + 2\overline{u})^{1/2}(1 + |\xi(t)|)|\xi(t)||\delta| - \mathcal{M}(\xi_t) \\ \le\ & -\tfrac{c}{2}|\xi(t)|^2 + c^\sharp|\xi(t)|\,|\delta| \\ & + 3k||R||(1 + 2\overline{u})^{1/2}|\xi(t)|^2|\delta| - \mathcal{M}(\xi_t) \\ \le\ & -\tfrac{c}{6}|\xi(t)|^2 + c^\sharp|\xi(t)|\,|\delta| - \mathcal{M}(\xi_t) \\ \le\ & -\tfrac{c}{42}|\xi(t)|^2 + 7(c^\sharp)^2|\delta|^2/c - \mathcal{M}(\xi_t),\end{aligned}$$

where $c^\sharp = \overline{c}+3k||R||(1+2\overline{u})^{1/2}$. It follows from $|u(t-\tau)| \le ||R||\,|\xi(t-\tau)|$, the triangle inequality, and (19) that

$$\begin{aligned}\dot V \le\ & -|z(t)|^2 + 2||P||\,||B||\,|z(t)|\big[|u(t-\tau)| + |\delta|\big] \\ \le\ & -\tfrac{1}{4}|z(t)|^2 + 4||P||^2\,||B||^2\ \big[||R||^2|\xi(t)|^2 + |\delta|^2\big] \\ & + 8||P||^2\,||B||^2\tau||R||^4\int_{t-2\tau}^t \tfrac{|R^\top(\ell)\xi(\ell)|^2}{\sqrt{1+|\xi(\ell)|^2}}\mathrm{d}\ell .\end{aligned}$$

Hence, $V^\sharp = V + 21\beta_1 W_3$ satisfies

$$\begin{aligned}\dot V^\sharp \le\ & -\tfrac{1}{4}|z(t)|^2 - |\xi(t)|^2 - \int_{t-2\tau}^t \tfrac{|R(\ell)^\top\xi(\ell)|^2}{\sqrt{1+|\xi(\ell)|^2}}\mathrm{d}\ell \\ & + \overline{J}|\delta|^2,\end{aligned} \tag{A.5}$$

where $\overline{J} = 4||P||^2||B||^2 + 147\beta_1(c^\sharp)^2/c$.

Set $\tilde c = (\overline{J}\,\overline{\delta}^2 + 1)^{1/2}$. Using the formula for $\dot\xi$, we can find a constant $\tilde c^\sharp > 0$ such that $W_3(t,\xi_t) \le \tilde c^\sharp$ along all trajectories of (22) at all times $t$ such that $|\xi(t)| \le \tilde c$. Hence, $W_3(t,\xi_t) \ge \tilde c^\sharp$ implies that $|\xi(t)| \ge \tilde c$. Set $J_1 = 21\tilde c^\sharp\beta_1 + 4||P||(\overline{J}\,\overline{\delta}^2 + 1)$. It follows that for each $t \ge 0$, $V^\sharp(t,\xi_t,z(t)) \ge J_1$ implies that either $W_3 \ge \tilde c^\sharp$ or $V \ge 4||P||(\overline{J}\,\overline{\delta}^2 + 1)$, and in the latter case we have $0.25|z(t)|^2 \ge \overline{J}\,\overline{\delta}^2 + 1$, since $V(t,z) \le ||P||\,|z|^2$ for all $t \in \mathbb{R}$ and $z \in \mathbb{R}^p$. We conclude from (A.5) that if $V^\sharp(t,\xi_t,z(t)) \ge J_1$, then $\dot V^\sharp \le -1$. We can also find a constant $d_1 > 0$ such that

$$W_3(t,\xi_t,z(t)) \le d_1\left(|\xi(t)|^2 + \int_{t-2\tau}^t \tfrac{|R(\ell)^\top\xi(\ell)|^2}{\sqrt{1+|\xi(\ell)|^2}}\mathrm{d}\ell\right) \tag{A.6}$$

along all trajectories of (22) for all $t$ that satisfy $V^\sharp(t,\xi_t,z(t)) \le J_1$. Hence, (A.5) provides a constant $c_0 > 0$ such that if $V^\sharp(t,\xi_t,z(t)) \le J_1$, then

$$\dot V^\sharp \le -\tfrac{1}{4}|z|^2 - \tfrac{1}{d_1}W_3 + \overline{J}|\delta|^2 \le -c_0 V^\sharp + \overline{J}|\delta|^2.$$

Therefore, along all trajectories with disturbances valued in $\mathcal{D}$, (a) $\dot V^\sharp \le -1$ at all points where $V^\sharp \ge J_1$ and (b) $\dot V^\sharp \le -c_o V^\sharp + \overline{J}|\delta|^2$ when $V^\sharp \le J_1$. This readily gives an ISS decay estimate on $V^\sharp$, and then the final ISS estimate follows from the lower bound (21) on $V^\sharp$; see (Gruszka et al., 2012) for this part of the argument in the undelayed case, which carries over to the delayed case.

## Appendix A.4: Proof of Theorem 3

In the new variables $(\xi, \overline{v}, \overline{\theta})$, (26) becomes

$$\left\{\begin{aligned}\dot\xi_1 &= r_1(t)u(t-\tau) \\ &\quad + [\overline{v} + v_r(t)][\cos(\overline{\theta}+\theta_r(t)) - \cos(\theta_r(t))] \\ \dot\xi_2 &= r_2(t)u(t-\tau) \\ &\quad + [\overline{v} + v_r(t)][\sin(\overline{\theta}+\theta_r(t)) - \sin(\theta_r(t))] \\ \dot{\overline{v}} &= -\alpha_v\overline{v} + u(t-\tau) \\ \dot{\overline{\theta}} &= -\alpha_\theta\overline{\theta} .\end{aligned}\right. \tag{A.7}$$

Set $\Omega(\overline{v},\xi) = \frac{1}{2}\overline{v}^2 + |\xi|^2$. We continue the notation from Section 5, with $z = \overline{v}$. Recalling the proof of Theorem 1 and the boundedness of $v_r$ and noting that $|\xi(t)|^2 \le V^\sharp(t,\xi_t,\overline{v}(t))/(\beta_1 k)$ along all trajectories of (A.7) yields a constant $\bar d > 0$ such that along all trajectories of (A.7),

$$\begin{aligned}\dot V^\sharp \le\ & -\Omega(\overline{v}(t),\xi(t)) - \mathcal{M}(\xi_t) \\ & + \left(\tau^2||R||^2 + 3k(1 + 2U(\xi_t))^{1/2}\right) \\ & \times 2\beta_1|\xi(t)|(|\overline{v}(t)| + ||v_r||)\,|\overline{\theta}(t)| \\ \le\ & -\Omega(\overline{v}(t),\xi(t)) - \mathcal{M}(\xi_t) \\ & + \bar d(1 + V^\sharp)(|\overline{v}(t)| + ||v_r||)|\overline{\theta}(t)| .\end{aligned} \tag{A.8}$$

Hence, the function $L^\sharp = \ln(1 + V^\sharp)$ satisfies

$$\dot L^\sharp \le -\tfrac{\Omega(\overline{v}(t),\xi(t)) + \mathcal{M}(\xi_t)}{1+V^\sharp} + \bar d(|\overline{v}(t)| + ||v_r||)|\overline{\theta}(t)| \tag{A.9}$$

where we omit the dependence of $V^\sharp$ on $(t,\xi_t,\overline{v}(t))$. Since $\overline{\theta}(t) = e^{-\alpha_\theta(t-t_0)}\overline{\theta}(t_0)$ and $|\overline{v}(t)| \le |\overline{v}(t_0)| + \epsilon||R||/\alpha_v$ hold along all trajectories of (A.7), we can integrate (A.9) on any interval $[t_0,t]$ to get $L^\sharp(t,\xi_t,\overline{v}(t)) \le \bar d|\overline{\theta}(t_0)|\{||v_r|| + |\overline{v}(t_0)| + \epsilon||R||/\alpha_v\}/\alpha_\theta + L^\sharp(t_0,\xi_{t_0},\overline{v}(t_0))$. Therefore, for each compact subset $\mathcal{S}$ of the state space for (A.7), we can use the first inequality in (A.8), the formula for $V^\sharp$, and the same reasoning that led to (A.6), and then the triangle inequality to find positive constants $J_s$ and $L_s$ such that

$$\dot{V}^\sharp \leq -J_s V^\sharp + L_s\sqrt{V^\sharp}|\overline{\theta}| \leq -\frac{J_s}{2}V^\sharp + \frac{1}{2J_s}L_s^2\overline{\theta}^2$$

along all trajectories of (A.7) starting in $\mathcal{S}$. Hence, along all such trajectories, $L_1 = V^\sharp + L_s^2/\{2J_s\alpha_\theta\}\overline{\theta}^2$ satisfies

$$\dot{L}_1 \leq -\frac{J_s}{2}V^\sharp - \frac{1}{2J_s}L_s^2\overline{\theta}^2 \leq -c_s L_1, \tag{A.10}$$

where $c_s = \min\{J_s/2, \alpha_\theta\}$. Since $\mathcal{S}$ was arbitrary, this gives the UGAS property for (A.7).

## References


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