

# Stabilization of a Chemostat Model with Haldane Growth Functions and a Delay in the Measurements

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## Abstract

The stabilization of equilibria in chemostats with measurement delays is a complex and challenging problem, and is of significant ongoing interest in bioengineering and population dynamics. In this paper, we solve an output feedback stabilization problem for chemostat models having two species, one limiting substrate, and either Haldane or Monod growth functions. Our feedback stabilizers depend on a given linear combination of the species concentrations, which are both measured with a constant time delay. The values of the delays are unknown. Instead, one only knows an upper bound on the delays, and we allow the upper bound to be arbitrarily large. The stabilizing feedback depends on the known upper bound for the delays as well. Our work is based on a Lyapunov-Krasovskii argument.

*Key words:* Delayed systems, systems biology, uniform global asymptotic stability

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## 1 Introduction

The chemostat model plays a central role in current research in bioengineering, microbial ecology, and population biology (Gouzé & Robledo, 2006; Mazenc, Malisoff, & De Leenheer, 2007; Mazenc, Malisoff, & Harmand, 2009; Robledo, 2009; Wolkowicz & Lu, 1992). For two species chemostats with one limiting substrate, and with the yield coefficients taken equal to one without loss of generality, the model has the form (Smith & Waltman, 1995)

$$\begin{cases} \dot{s} = (s_{\text{in}} - s)D - \sum_{i=1}^2 \mu_i(s)x_i \\ \dot{x}_i = [\mu_i(s) - D]x_i, \quad i = 1, 2, \end{cases} \quad (1)$$

where the  $x_i$ 's are the concentrations of the species,  $s$  is the level of the substrate,  $D$  is an everywhere positive dilution rate controller (which we specify below),  $s_{\text{in}} > 0$  is the constant input nutrient concentration, and the  $\mu_i$ 's are the growth functions. The dynamics (1)

models two competing species growing on one substrate in a well stirred biological reactor. Since the function  $D$  is the ratio of the volumetric flow rate (with units of volume over time) to the constant reactor volume, it is proportional to the speed of the pump that supplies the reactor with fresh nutrient. In classical cases where  $D$  is constant, the well known competitive exclusion principle says that only one species survives in an equilibrium (Smith & Waltman, 1995). In this paper, we wish to stabilize a componentwise positive equilibrium for (1) (i.e., persistence of both species), so we must take  $D$  to be a nonconstant controller.

There are several commonly used expressions for the growth functions. One important choice is the *Haldane* function

$$\mu_i(s) = \frac{K_i s}{L_i + s + g_i s^2}, \quad (2)$$

where  $g_i$  is a nonnegative constant, and where  $K_i$  and  $L_i$  are positive constants for  $i = 1, 2$ . Growth functions of the type (2) for which  $g_i = 0$  are called *Monod* growth functions (Smith & Waltman, 1995). However, it is often useful to allow positive  $g_i$ 's, to model more realistic cases where an excessive level of the substrate  $s$  can impede the growth (Gouzé & Robledo, 2005b; Mailleret, Bernard, & Steyer, 2004).

In our earlier work Mazenc et al. (2009), we designed a controller  $D$ , depending only on a linear combination

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$Y(t) = x_1(t) + ax_2(t)$  of the species levels and with an arbitrary but known constant  $a \in (0, 1)$ , that ensures that (1) is globally asymptotically stable to an appropriate equilibrium  $(s_*, x_{1*}, x_{2*}) \in (0, \infty)^3$ . However, Mazenc et al. (2009) only applied to cases where the  $\mu_i$ 's are Monod, and did not allow any measurement delay. See Karafyllis, Kravaris, Syrou, & Lyberatos (2008) for a thorough stability analysis for one species chemostats with general growth functions but with no delays, and see Karafyllis & Kravaris (2009) and Karafyllis, Kravaris, & Kalogerakis (2009) for significant results for undelayed chemostats under (possibly unbounded) state feedback. It is important to allow delays, because measurements may take time. See Robledo (2009), which makes a strong case for delays only in the dilution rate controller  $D$  and gives a stability analysis for one species chemostats with delays in  $D$ . In this paper, we analyze the more complex case of two species, where  $D$  now depends on a delayed linear combination

$$y(t) = x_1(t - \tau) + ax_2(t - \tau)$$

of the  $x_i$ 's and  $\tau \geq 0$  is an unknown delay.

As in Mazenc, De Leenheer, & Malisoff (2006), Mazenc et al. (2007), Mazenc, Malisoff, & Harmand (2008a), and Mazenc et al. (2009), we design  $D$  to globally asymptotically stabilize a componentwise positive equilibrium triple. However, when there are time delays, the strict Lyapunov function approach in Mazenc et al. (2006), Mazenc et al. (2007), Mazenc et al. (2008a), and Mazenc et al. (2009) does not apply. By contrast, here we present a new Lyapunov-Krasovskii construction. The construction uses the formulas for the  $\mu_i$ 's and is reminiscent of the Lyapunov function under a constant dilution rate that was used in Hsu (1978). See also Ballyk & Barany (2006, 2009) for an alternative feedback linearization method for two species chemostats, and see Remark 4 below for more information on the relationship between this work and our previous results.

Our control design is very different from the one in De Leenheer & Smith (2003). The advantage of De Leenheer & Smith (2003) is that it does not require any specific formula for the  $\mu_i$ 's. However, De Leenheer & Smith (2003) uses a Poincaré-Bendixson approach that does not allow measurement delays and it assumes monotonicity of the  $\mu_i$ 's. The present paper and Mazenc et al. (2007, 2008a, 2009) all use the analytic expression of the growth functions. The advantage of our new work is that for any given upper bound  $\tau_M$  on the delay  $\tau$ , we obtain uniform globally asymptotically stabilizing control laws depending only on the delayed output  $y(t)$  and  $\tau_M$ . By allowing feedbacks that only involve a linear combination of the species concentrations, we cover cases where the measurements are done using photometric methods that may not distinguish between the species; see Gouzé & Robledo (2005a) and Mazenc et al. (2009) for details and more motivation for the important setting where only a linear combination of the  $x_i$ 's can be measured.

## 2 Assumptions, definitions, and notation

Consider the two species chemostat (1) with a constant  $s_{\text{in}} > 0$ , the growth rates (2), and a bounded controller  $D$  to be determined that is bounded from below by a positive constant and dependent on the unique output

$$y(t) = x_1(t - \tau) + ax_2(t - \tau). \quad (3)$$

Here  $a \in (0, 1)$  is any given constant and the constant time delay  $\tau \geq 0$  is unknown. We always assume:

**Assumption 1** *There is a constant  $s_* \in (0, s_{\text{in}})$  such that*

$$\mu_1(s_*) = \mu_2(s_*). \quad (4)$$

Therefore, we are free to fix any constants  $x_{1*} > 0$  and  $x_{2*} > 0$  such that  $s_* + x_{1*} + x_{2*} = s_{\text{in}}$ . Set  $D_* = \mu_1(s_*)$ . However, the constants are assumed to satisfy:

**Assumption 2** *The constants*

$$\delta_i = L_i - g_i s_* s_{\text{in}}, \quad i = 1, 2 \quad (5)$$

are positive. Also,

$$\aleph = \delta_1 D_* \left[ 1 + \frac{1}{s_*} \sum_{i=1}^2 \frac{\delta_i}{L_i + s_{\text{in}} + g_i s_{\text{in}}^2} x_{i*} \right] + K_2 s_* x_{2*} \frac{L_2 g_1 - L_1 g_2}{\delta_2} \quad (6)$$

is positive,  $L_2 g_1 - L_1 g_2 \leq 0$ , and

$$\mathfrak{U} = \frac{s_*}{(1-a)D_*} \left[ -K_1 + \frac{L_1 - g_1 s_*^2}{L_2 - g_2 s_*^2} K_2 \right] \quad (7)$$

is nonzero.

**Assumption 3**  $\min\{\mu_1(s_{\text{in}}), \mu_2(s_{\text{in}})\} > D_*$ .

Therefore, we can also fix a constant  $p \in (0, 1)$  so that

$$\mu_i(s_{\text{in}}) > (1+p)D_* \quad \text{for } i = 1, 2. \quad (8)$$

Finally, we assume that there is a known bound  $\tau_M$  on the unknown constant delay  $\tau$ :

**Assumption 4** *There is a known constant  $\tau_M > 0$  so that  $0 \leq \tau \leq \tau_M$ .*

Since we wish to stabilize the triple  $(s_*, x_{1*}, x_{2*})$  using a delayed feedback, it is also convenient to define

$$z = s + x_1 + x_2, \quad \tilde{z} = z - s_{\text{in}}, \quad (9)$$

$$\tilde{s} = s - s_*, \quad \tilde{x}_i = x_i - x_{i*} \quad (i = 1, 2), \quad (10)$$

$$\tilde{y}(t) = \tilde{x}_1(t - \tau) + a\tilde{x}_2(t - \tau),$$

$$Y(t) = x_1(t) + ax_2(t), \quad \text{and} \quad (11)$$

$$\tilde{Y}(t) = \tilde{x}_1(t) + a\tilde{x}_2(t).$$

Then  $\tilde{z} = z + x_1 + x_2 - (s_* + x_{1*} + x_{2*}) = \tilde{s} + \tilde{x}_1 + \tilde{x}_2$ .

**Remark 1** Assumptions 1-3 automatically hold in the Monod case (where  $g_1 = g_2 = 0$ ) if the  $\mu_i$ 's intersect

at a unique positive level  $s_* \in (0, s_{\text{in}})$ . In the Monod case, nonvanishing of  $\mathcal{U}$  is equivalent to the condition  $K_1/L_1 \neq K_2/L_2$ , which is the case when the growth functions have a unique positive intersection value  $s_*$ . If our conditions  $\aleph > 0$  and  $\mathcal{U} \neq 0$  hold for a given pair of Monod growth functions  $\mu_i(s) = K_i s / (L_i + s)$  having a unique positive intersection, then these two assumptions also hold for the Haldane growth functions (2) when the constants  $g_i$ 's are positive but small enough relative to the other parameters. Therefore, our work can be viewed as a robustness analysis with respect to measurement delays and nonmonotone  $\mu_i$ 's (corresponding to nonzero constants  $g_i$ ).

**Remark 2** The requirement that  $\mathcal{U} \neq 0$  cannot be removed. If all of the requirements in Assumption 1-4 hold except that  $\mathcal{U} = 0$ , and if  $\tau = 0$  and  $\mu_1 \neq \mu_2$ , then it is impossible to render  $(s_*, x_{1*}, x_{2*})$  globally uniformly asymptotically stable, even if we allow a time dependent controller  $D$ ; see Appendix A.4.

Given an open subset  $\mathcal{G}$  of a Euclidean space containing the origin, a function  $\alpha : [0, \infty) \times \mathcal{G} \rightarrow [0, \infty)$  is called *positive definite (on  $\mathcal{G}$ )* provided  $\alpha(t, 0) = 0$  for all  $t \geq 0$  and  $\inf_{t \geq 0} \alpha(t, \zeta) > 0$  for all  $\zeta \in \mathcal{G} \setminus \{0\}$ . A *modulus* with respect to  $\mathcal{G}$  is any continuous function  $\alpha : \mathcal{G} \rightarrow [0, \infty)$  satisfying: (A)  $\lim_{\zeta \rightarrow \zeta^*} \alpha(\zeta) = \infty$  for each point  $\zeta^*$  in the boundary of  $\mathcal{G}$  and (B)  $\lim_{|\zeta| \rightarrow \infty} \alpha(\zeta) = \infty$ , where  $|\cdot|$  is the usual Euclidean norm. Condition (A) holds vacuously if  $\mathcal{G} = \mathbb{R}^n$ , and (B) holds vacuously if  $\mathcal{G}$  is bounded. A function  $V$  for which  $\zeta \mapsto \inf_t V(t, \zeta)$  is a modulus with respect to  $\mathcal{G}$  is called *proper (on  $\mathcal{G}$ )*. Also,  $C(A, B)$  is the set of all continuous functions  $\phi : A \rightarrow B$  for any subsets  $A$  and  $B$  of Euclidean spaces.

### 3 Main result

#### 3.1 Statement of theorem

Our desire to stabilize  $(s_*, x_{1*}, x_{2*}) \in (0, \infty)^3$  leads to our designing a *nonconstant* dilution rate controller

$$D = D_* - \text{sign}(\mathcal{U})\varepsilon_1 \sigma(\varepsilon_2(Y(t - \tau) - x_{1*} - ax_{2*})) \quad (12)$$

for small enough constants  $\varepsilon_i > 0$ , where  $\sigma : \mathbb{R} \rightarrow [-1, +1]$  is the standard saturation defined by  $\sigma(r) = r$  if  $|r| \leq 1$  and  $\sigma(r) = \text{sign}(r)$  otherwise. To specify the bounds on the admissible  $\varepsilon_i$ 's, we use the following constants, whose choices will be explained where they are used in the proof of our theorem:

$$\begin{aligned} \bar{c}_1 &= \frac{(L_1 - g_1 s_*^2)(s_{\text{in}} - s_*)}{s_*} + \frac{1}{1-a} \frac{s_*}{D_*} \left[ -K_1 a + \frac{L_1 - g_1 s_*^2}{L_2 - g_2 s_*^2} K_2 \right], \\ \bar{c}_2 &= 1 + 180(1+a)^2 s_{\text{in}}^2 |\mathcal{U}| \left( \max \left\{ \frac{D_*^2 s_{\text{in}}}{s_*^2}, \frac{\bar{c}_1^2 s_{\text{in}}}{2\aleph} \right\} + 1 \right)^2, \\ \bar{\varepsilon}_1 &= \min \left\{ 0.5pD_*, \frac{\aleph s_*}{10(L_1 + g_1 s_*^2) s_{\text{in}}} \right\}, \quad \bar{\varepsilon}_2 = \frac{10}{11s_{\text{in}}}, \\ \bar{\varepsilon}_3 &= \min \left\{ \frac{\aleph |\mathcal{U}|}{5(\bar{c}_1^2 + 1) s_{\text{in}}}, |\mathcal{U}|, \frac{1}{10\bar{c}_2 \tau_M} \min \left\{ \frac{\aleph}{\tau_M}, \sqrt{|\mathcal{U}|} \right\} \right\}. \end{aligned} \quad (13)$$

Our main result is:

**Theorem 1** *If (1) satisfies Assumptions 1-4, then for any positive constants  $\varepsilon_1$  and  $\varepsilon_2$  such that  $\varepsilon_i \leq \bar{\varepsilon}_i$  for  $i = 1, 2$  and  $\varepsilon_1 \varepsilon_2 \leq \bar{\varepsilon}_3$ , the control law (12) globally asymptotically stabilizes the equilibrium point  $(s_*, x_{1*}, x_{2*})$  of (1) for all initial functions  $(\phi_s, \phi_{x_1}, \phi_{x_2}) \in C([-2\tau_M, 0], (0, \infty)^3)$ .*

**Remark 3** Our proof of Theorem 1 shows that the  $(\tilde{s}, \tilde{x}_1, \tilde{x}_2)$  dynamics, in closed loop with (12), is uniformly globally asymptotically stable (UGAS) to 0 on its invariant set, which gives persistence of both species; see Appendix A.3 for the form for the UGAS estimate. Theorem 1 goes beyond asserting existence of a stabilizing delayed feedback of the form (12). Instead, it computes bounds on the allowable  $\varepsilon_i$ 's. When  $\tau_M \rightarrow \infty$ , we get  $\bar{\varepsilon}_3 \rightarrow 0$  (giving slower convergence to the equilibrium), so in the limiting case, we have a constant controller, giving competitive exclusion. This can be interpreted as saying that the closed loop system has a bifurcation when  $\varepsilon_3$  converges to zero that significantly changes the equilibrium behavior.

#### 3.2 Outline of proof

We first find an appropriate nonstrict control-Lyapunov function  $V$  for the open loop error dynamics for  $(\tilde{s}, \tilde{x}_1, \tilde{x}_2)$  obtained by setting  $D = D_* + v$ , viewing  $v$  as the control. This is done in Section 4. Then in Section 5.1, we transform  $V$  into an appropriate function  $V^\sharp$  that is a strict Lyapunov function for the error dynamics, when we choose the feedback (12) and when the delays are zero; see Malisoff & Mazenc (2009) for the standard definitions of strict and nonstrict Lyapunov functions. Finally, in Section 5.2, we transform  $V^\sharp$  into a functional  $U_1$  that satisfies the usual Lyapunov-Krasovskii functional decay properties for the closed loop error dynamics under delays, provided  $t \geq \tau_M$ . This leads to the desired UGAS estimate; see Remark 4 for a comparison of this transformation process to earlier Lyapunov transformation processes in the literature.

### 4 Main lemmas

In all of what follows, all (in)equalities should be understood to hold globally, unless otherwise indicated, and  $C_{\text{in}} = C([-2\tau_M, 0], (0, \infty)^3)$  is our set of initial functions. Writing (12) as  $D = D_* + v$ , where  $D_* = \mu_1(s_*) = \mu_2(s_*)$  as before, gives  $|v| \leq 0.5pD_*$  and therefore also  $(1 + 0.5p)D_* \geq D \geq (1 - 0.5p)D_*$  everywhere, by our choice of the bound  $\bar{\varepsilon}_1$  in (13). Also,

$$\dot{\tilde{z}} = -D\tilde{z} \quad (14)$$

along all trajectories of (1). Hence, the time derivative of  $Q(\tilde{z}) = \frac{1}{2}\tilde{z}^2$  along all trajectories of (1) satisfies

$$\dot{Q} = -2DQ(\tilde{z}) \leq -2(1 - 0.5p)D_*Q(\tilde{z}). \quad (15)$$

Since  $D_* > 0$  and  $p \in (0, 1)$ , this gives  $\tilde{z} \rightarrow 0$  exponentially. Moreover, forward completeness of (1) in closed

loop with (12) follows from the proof of (Mazenc, Malisoff, & Lin, 2008b, Lemma A.1) and Lipschitzness of the feedback.

From the structure of the dynamics (1) and the positive lower bound on  $D$ , it is clear that  $s(t)$  must converge towards  $(0, s_{\text{in}}]$  when it starts above  $s_{\text{in}}$ , for all initial functions in  $C_{\text{in}}$ . The next lemma implies that  $s(t)$  must in fact enter  $(0, s_{\text{in}})$ , and uses Assumption 3.

**Lemma 1** *For all initial functions in  $C_{\text{in}}$ , the set*

$$S = \{(s, x_1, x_2) \in (0, s_{\text{in}}) \times (0, \infty)^2 : z \leq 1.1s_{\text{in}}\}$$

*is positively invariant and globally attractive for the system (1) in closed loop with  $D$  defined in (12).*

See Appendix A.1 for the proof of Lemma 1. From the definition of  $D_*$  and the relations

$$\mu_i(s) - D_* = \frac{K_i s}{L_i + s + g_i s^2} - \frac{K_i s_*}{L_i + s_* + g_i s_*^2}, \quad (16)$$

one can readily check that:

**Lemma 2** *For  $i = 1, 2$  and all  $s > 0$ , we have*

$$\mu_i(s) - D_* = \frac{D_* [L_i - g_i s_* s]}{s_* (L_i + s + g_i s^2)} \tilde{s}. \quad (17)$$

Moreover, since  $\dot{s} = \dot{\tilde{s}}$  and  $\mu_i(s_*) = D_*$  for  $i = 1, 2$ ,

$$\begin{aligned} \dot{\tilde{s}} &= v(s_{\text{in}} - s) + (s_{\text{in}} - \tilde{s} - s_*) D_* - \sum_{i=1}^2 D_* x_{i*} \\ &\quad - \sum_{i=1}^2 (\mu_i(s) - D_*) x_{i*} - \sum_{i=1}^2 \mu_i(s) \tilde{x}_i. \end{aligned} \quad (18)$$

Hence, the fact that  $s_{\text{in}} = s_* + x_{1*} + x_{2*}$  and (17) give

$$\begin{aligned} \dot{\tilde{s}} &= -D_* \tilde{s} - \sum_{i=1}^2 \frac{D_* [L_i - g_i s_* s]}{s_* (L_i + s + g_i s^2)} \tilde{s} x_{i*} \\ &\quad - \sum_{i=1}^2 \frac{K_i s}{L_i + s + g_i s^2} \tilde{x}_i + (s_{\text{in}} - s)v. \end{aligned} \quad (19)$$

We readily conclude from our choices of the  $\delta_i$ 's in Assumption 2 that:

**Lemma 3** *The controller (12) guarantees that along all trajectories of (1), we have*

$$\dot{\tilde{s}} = -H(s) \tilde{s} - \sum_{i=1}^2 \frac{K_i s}{L_i + s + g_i s^2} \tilde{x}_i + (s_{\text{in}} - s)v, \quad (20)$$

where

$$H(s) = D_* + \sum_{i=1}^2 \frac{D_* [L_i - g_i s_* s]}{s_* (L_i + s + g_i s^2)} x_{i*}. \quad (21)$$

Moreover,

$$H(s) \geq D_* \left[ 1 + \frac{1}{s_*} \sum_{i=1}^2 \frac{\delta_i x_{i*}}{L_i + s_{\text{in}} + g_i s_{\text{in}}^2} \right] \quad (22)$$

for all  $s \in [0, s_{\text{in}}]$ .

Combining Lemmas 1-3 shows that the error dynamics

$$\begin{cases} \dot{\tilde{s}} = -H(s) \tilde{s} - \sum_{i=1}^2 \frac{K_i s}{L_i + s + g_i s^2} \tilde{x}_i + (s_{\text{in}} - s)v \\ \dot{\tilde{x}}_i = D_* \frac{L_i - g_i s_* s}{s_* (L_i + s + g_i s^2)} \tilde{s} x_i - v x_i, \quad i = 1, 2 \end{cases} \quad (23)$$

has the globally attractive and positively invariant set  $\mathcal{E} = \{(\tilde{s}, \tilde{x}_1, \tilde{x}_2) \in (-s_*, s_{\text{in}} - s_*) \times (-x_{1*}, \infty) \times$

$(-x_{2*}, \infty) : \tilde{z} \leq 0.1s_{\text{in}}\}$  for all initial functions in  $C_{\text{in}}$ . To simplify the exposition, we set

$$C_1 = K_1 \frac{s_*}{D_*} \quad \text{and} \quad C_2 = \frac{s_*}{D_*} \frac{K_2 (L_1 - g_1 s_*^2)}{L_2 - g_2 s_*^2} \quad (24)$$

and we use the function

$$\begin{aligned} M(\tilde{s}, \tilde{x}_1, \tilde{x}_2) &= \\ &\tilde{s} \frac{L_1 - g_1 s_* s}{s} (s_{\text{in}} - s) - K_1 \frac{s_*}{D_*} \tilde{x}_1 - K_2 \frac{s_*}{D_*} \frac{L_1 - g_1 s_*^2}{L_2 - g_2 s_*^2} \tilde{x}_2. \end{aligned} \quad (25)$$

Our nonstrict control-Lyapunov construction is:

**Lemma 4** *The function*

$$\begin{aligned} V(\tilde{s}, \tilde{x}_1, \tilde{x}_2) &= C_1 \int_0^{\tilde{x}_1} \frac{\ell}{\ell + x_{1*}} d\ell + C_2 \int_0^{\tilde{x}_2} \frac{\ell}{\ell + x_{2*}} d\ell \\ &\quad + \int_0^{\tilde{s}} \frac{|L_1 - g_1 s_* (s_* + m)|}{s_* + m} m dm \end{aligned} \quad (26)$$

*is positive definite on  $\mathcal{E}$ , and is positive definite and proper on  $\mathcal{X} = (-s_*, \infty) \times (-x_{1*}, \infty) \times (-x_{2*}, \infty)$ . Also,*

$$\dot{V} \leq -\aleph \frac{\tilde{s}^2}{s} + M(\tilde{s}, \tilde{x}_1, \tilde{x}_2)v \quad (27)$$

*along all trajectories of (23) in  $\mathcal{E}$ .*

For the proof of Lemma 4, see Appendix A.2. The function  $V$  is nonstrict because when  $v = 0$ , the right side of its decay estimate is negative semidefinite as a function of  $(\tilde{s}, \tilde{x}_1, \tilde{x}_2)$  but not negative definite.

## 5 Completing the proof of Theorem 1

### 5.1 Transforming $V$ into $V^\sharp$

We first transform  $V$  into an appropriate function  $V^\sharp$  that is a strict Lyapunov functions when the delays are set to zero; this is a key step in our Lyapunov-Krasovskii construction. We refer to the trajectories of (23) with

$$v = -\text{sign}(\mathcal{U}) \varepsilon_1 \sigma(\varepsilon_2 \tilde{Y}(t - \tau)) \quad (28)$$

and with initial functions in  $C_{\text{in}}$  as the *closed loop trajectories of (23)*. Since  $\mathcal{E}$  is forward invariant and globally attractive for this closed loop system, there is no loss in generality in assuming that the solution satisfies  $(\tilde{s}(t), \tilde{x}_1(t), \tilde{x}_2(t)) \in \mathcal{E}$  for all  $t \geq -2\tau_M$ . Since  $a \in (0, 1)$ ,

$$\begin{aligned} \tilde{x}_1 &= \frac{-a\tilde{z} + a\tilde{s} + \tilde{Y}}{1-a}, \quad \tilde{x}_2 = \frac{\tilde{z} - \tilde{s} - \tilde{Y}}{1-a}, \quad \text{and} \\ \frac{\tilde{s}}{s} &= \frac{-\tilde{s}^2}{ss_*} + \frac{\tilde{s}}{s_*}. \end{aligned} \quad (29)$$

Plugging (29) into (25) and collecting coefficients gives

$$\begin{aligned} M(\tilde{s}, \tilde{x}_1, \tilde{x}_2) &= L_1 (s_{\text{in}} - s_*) \left[ \frac{-\tilde{s}^2}{ss_*} + \frac{\tilde{s}}{s_*} \right] - \frac{L_1}{s} \tilde{s}^2 \\ &\quad + \{C_3 - g_1 s_* (s_{\text{in}} - s_*)\} \tilde{s} \\ &\quad + \mathcal{U} \tilde{Y}(t) - C_3 \tilde{z} + g_1 s_* \tilde{s}^2 \\ &= -\frac{L_1 s_{\text{in}}}{s_*} \frac{\tilde{s}^2}{s} + g_1 s_* \tilde{s}^2 + \bar{c}_1 \tilde{s} + \mathcal{U} \tilde{Y}(t) - C_3 \tilde{z}, \end{aligned}$$

where  $\mathcal{U}$  is defined in (7) and

$$C_3 = \frac{1}{1-a} \frac{s_*}{D_*} \left[ -K_1 a + \frac{L_1 - g_1 s_*^2}{L_2 - g_2 s_*^2} K_2 \right]. \quad (30)$$

Substituting into (27) and recalling that  $s \leq s_{\text{in}}$  on  $\mathcal{E}$ ,

$$\begin{aligned} \dot{V} &\leq -\aleph \frac{\tilde{s}^2}{s} + \left[ -\frac{L_1 s_{\text{in}}}{s_*} + g_1 s_* s \right] \frac{\tilde{s}^2}{s} v + \bar{c}_1 \tilde{s} v \\ &\quad + \mathcal{U} \tilde{Y}(t) v - C_3 \tilde{z} v \\ &\leq -\frac{9}{10} \aleph \frac{\tilde{s}^2}{s} + \bar{c}_1 \tilde{s} v + \mathcal{U} \tilde{Y}(t) v - C_3 \tilde{z} v, \end{aligned} \quad (31)$$

where the last inequality is from the second argument in the min defining our upper bound  $\bar{\varepsilon}_1$  from (13). Here and in the rest of the proof, dots indicate time derivatives along the closed loop trajectories of (23), and we omit the arguments of our functions when they are clear.

Since we are assuming that  $(\tilde{s}(t), \tilde{x}_1(t), \tilde{x}_2(t)) \in \mathcal{E}$  for all  $t \geq -\tau_M$ , we have

$$|\tilde{Y}(t - \tau)| < 1.1 s_{\text{in}} \quad \forall t \geq 0. \quad (32)$$

Therefore, our upper bound  $\bar{\varepsilon}_2$  for  $\varepsilon_2$  from (13) gives

$$v = -\text{sign}(\mathcal{U}) \varepsilon_3 \tilde{Y}(t - \tau) \quad \forall t \geq 0, \quad (33)$$

where  $\varepsilon_3 = \varepsilon_1 \varepsilon_2$ . Substituting into (31) and then using

$$\tilde{Y}(t - \tau) = \tilde{Y}(t) - \mathcal{I}_t(\tilde{Y}),$$

where  $\mathcal{I}_t(\tilde{Y}) = \int_{t-\tau}^t \dot{\tilde{Y}}(m) dm$ , shows that for all  $t \geq 0$ ,

$$\begin{aligned} \dot{V} &\leq -\frac{9}{10} \aleph \frac{\tilde{s}^2}{s} - |\mathcal{U}| \varepsilon_3 \tilde{Y}(t)^2 - \text{sign}(\mathcal{U}) \bar{c}_1 \varepsilon_3 \tilde{s} \tilde{Y}(t) \\ &\quad + \text{sign}(\mathcal{U}) C_3 \varepsilon_3 \tilde{z} \tilde{Y}(t) - C_3 \text{sign}(\mathcal{U}) \varepsilon_3 \tilde{z} \mathcal{I}_t(\tilde{Y}) \\ &\quad + \left[ \bar{c}_1 \text{sign}(\mathcal{U}) \varepsilon_3 \tilde{s} + |\mathcal{U}| \varepsilon_3 \tilde{Y}(t) \right] \mathcal{I}_t(\tilde{Y}). \end{aligned} \quad (34)$$

Applying Hölder's Inequality  $ab \leq a^2/4 + b^2$  to the last four terms in (34) gives

$$\begin{aligned} -\text{sign}(\mathcal{U}) \bar{c}_1 \varepsilon_3 \tilde{s} \tilde{Y} &\leq \frac{1}{4} \varepsilon_3 |\mathcal{U}| \tilde{Y}^2 + \varepsilon_3 \frac{\bar{c}_1^2}{|\mathcal{U}|} \tilde{s}^2 \quad \text{and} \\ \text{sign}(\mathcal{U}) C_3 \varepsilon_3 \tilde{z} \tilde{Y} &\leq \frac{1}{16} \varepsilon_3 |\mathcal{U}| \tilde{Y}^2 + 4 \varepsilon_3 \frac{C_3^2}{|\mathcal{U}|} \tilde{z}^2 \end{aligned} \quad (35)$$

and

$$\begin{aligned} -C_3 \text{sign}(\mathcal{U}) \varepsilon_3 \tilde{z} \mathcal{I}_t(\tilde{Y}) &\leq \frac{1}{4} \varepsilon_3^2 \mathcal{I}_t^2(\tilde{Y}) + C_3^2 \tilde{z}^2, \\ \bar{c}_1 \text{sign}(\mathcal{U}) \varepsilon_3 \tilde{s} \mathcal{I}_t(\tilde{Y}) &\leq \frac{\aleph}{10} \frac{\tilde{s}^2}{s} + \frac{5 \bar{c}_1^2 \varepsilon_3^2 s}{2 \aleph} \mathcal{I}_t^2(\tilde{Y}), \\ |\mathcal{U}| \varepsilon_3 \tilde{Y} \mathcal{I}_t(\tilde{Y}) &\leq \frac{\bar{\varepsilon}_3}{16} |\mathcal{U}| \tilde{Y}^2 + 4 \varepsilon_3 |\mathcal{U}| \mathcal{I}_t^2(\tilde{Y}). \end{aligned} \quad (36)$$

Summing the inequalities (35)-(36) and collecting like terms, and then substituting the result into (34), gives

$$\begin{aligned} \dot{V} &\leq \left[ -\frac{4}{5} \frac{\aleph}{s} + \varepsilon_3 \frac{\bar{c}_1^2}{|\mathcal{U}|} \right] \tilde{s}^2 - \frac{5}{8} \varepsilon_3 |\mathcal{U}| \tilde{Y}^2 \\ &\quad + C_3^2 \left( \frac{4 \varepsilon_3}{|\mathcal{U}|} + 1 \right) \tilde{z}^2 + \left[ \frac{5 \bar{c}_1^2 \varepsilon_3^2 s}{2 \aleph} + 4 \varepsilon_3 |\mathcal{U}| + \frac{1}{4} \varepsilon_3^2 \right] \mathcal{I}_t^2(\tilde{Y}). \end{aligned} \quad (37)$$

Therefore, since  $s \leq s_{\text{in}}$  on  $\mathcal{E}$ , we get

$$\begin{aligned} \dot{V} &\leq \left[ -\frac{4}{5} \aleph + \varepsilon_3 \frac{\bar{c}_1^2 s_{\text{in}}}{|\mathcal{U}|} \right] \frac{\tilde{s}^2}{s} - \frac{5}{8} \varepsilon_3 |\mathcal{U}| \tilde{Y}^2 \\ &\quad + C_3^2 \left( \frac{4 \varepsilon_3}{|\mathcal{U}|} + 1 \right) \tilde{z}^2 + C_4 \varepsilon_3 \mathcal{I}_t^2(\tilde{Y}) \\ &\leq -\frac{3}{5} \aleph \frac{\tilde{s}^2}{s} - \frac{5}{8} \varepsilon_3 |\mathcal{U}| \tilde{Y}^2 + C_5 \tilde{z}^2 + C_4 \varepsilon_3 \mathcal{I}_t^2(\tilde{Y}), \end{aligned} \quad (38)$$

where

$$C_4 = \frac{5 \bar{c}_1^2 \varepsilon_3 s_{\text{in}}}{2 \aleph} + 4 |\mathcal{U}| + \frac{1}{4} \varepsilon_3, \quad C_5 = C_3^2 \left( \frac{4 \varepsilon_3}{|\mathcal{U}|} + 1 \right),$$

and the last inequality in (38) is by the first argument in the min in our bound  $\bar{\varepsilon}_3$  for  $\varepsilon_3$  from (13). Hence, the function  $V^\sharp$  defined by

$$V^\sharp = V + \frac{C_5 + 1}{D_*(1 - 0.5p)} Q(\tilde{z}), \quad (39)$$

where  $Q(\tilde{z}) = \frac{1}{2} \tilde{z}^2$  is defined in Section 4, is a strict Lyapunov function when the delays are set to zero, because it satisfies

$$\dot{V}^\sharp \leq -\frac{3}{5} \aleph \frac{\tilde{s}^2}{s} - \frac{5}{8} \varepsilon_3 |\mathcal{U}| \tilde{Y}^2 - \tilde{z}^2 + C_4 \varepsilon_3 \mathcal{I}_t^2(\tilde{Y}), \quad (40)$$

by (15).

## 5.2 Transforming $V^\sharp$ into $U_1$

The next key step is to transform  $V^\sharp$  into a functional  $U_1$  that satisfies the Lyapunov-Krasovskii conditions when  $t \geq \tau_M$ . By (23),

$$\dot{\tilde{x}}_i = \left[ \frac{D_* [L_i - g_i s_* s]}{s_* (L_i + s + g_i s^2)} \tilde{s} + \text{sign}(\mathcal{U}) \varepsilon_3 \tilde{Y}(t - \tau) \right] x_i \quad (41)$$

for  $i = 1, 2$  for all  $t \geq 0$ . The preceding observations give

$$\begin{aligned} |\mathcal{I}_t(\tilde{Y})| &\leq \varepsilon_3 \int_{t-\tau}^t |\tilde{Y}(r - \tau)| |Y(r)| dr \\ &\quad + \int_{t-\tau}^t \left[ \sum_{i=1}^2 \frac{D_* [L_i - g_i s_* s(r)]}{s_* (L_i + s(r) + g_i s(r)^2)} |\tilde{s}(r)| \right] |Y(r)| dr \end{aligned} \quad (42)$$

for all  $t \geq \tau_M$ , since  $\tilde{Y} = \tilde{x}_1 + a \tilde{x}_2$ .

We also have  $x_i(t) \leq \frac{11}{10} s_{\text{in}} < \frac{3}{2} s_{\text{in}}$  for  $i = 1, 2$ . Hence, (42) and Assumption 2 imply that for all  $t \geq \tau_M$ ,

$$\begin{aligned} \mathcal{I}_t^2(\tilde{Y}) &\leq 9(1+a)^2 s_{\text{in}}^2 \left( \varepsilon_3 \int_{t-\tau}^t |\tilde{Y}(r - \tau)| dr \right)^2 \\ &\quad + 9 s_{\text{in}}^2 \frac{D_*^2}{s_*^2} (1+a)^2 \left( \int_{t-\tau}^t \left[ \sum_{i=1}^2 \frac{L_i - g_i s_* s(r)}{L_i + s(r) + g_i s(r)^2} |\tilde{s}(r)| \right] dr \right)^2. \end{aligned}$$

Since  $0 < s \leq s_{\text{in}}$  on  $\mathcal{E}$ , Assumption 2 gives  $0 < L_i - g_i s_* s(r) \leq L_i + s(r) + g_i s^2(r)$  on  $\mathcal{E}$  for  $i = 1, 2$ , so we can replace  $(L_i - g_i s_* s(r))/(L_i + s(r) + g_i s(r)^2)$  by its upper bound 1 in the second integral. Hence,

$$\begin{aligned} \mathcal{I}_t^2(\tilde{Y}) &\leq C_6 \left( \int_{t-\tau}^t |\tilde{s}(r)| dr \right)^2 + C_7 \varepsilon_3^2 \left( \int_{t-\tau}^t |\tilde{Y}(r - \tau)| dr \right)^2 \\ &\leq C_6 s_{\text{in}} \tau \int_{t-\tau}^t \frac{\tilde{s}(r)^2}{s(r)} dr + C_7 \varepsilon_3^2 \tau \int_{t-\tau}^t \tilde{Y}(r - \tau)^2 dr \end{aligned} \quad (43)$$

for all  $t \geq \tau_M$ , where

$$C_6 = 36 s_{\text{in}}^2 \frac{D_*^2}{s_*^2} (1+a)^2 \quad \text{and} \quad C_7 = 9(1+a)^2 s_{\text{in}}^2, \quad (44)$$

by Hölder's Inequality  $(\int_{t-\tau}^t P(r) dr)^2 \leq \tau \int_{t-\tau}^t P^2(r) dr$  for continuous functions  $P$ , and then multiplying by  $s_{\text{in}}/s(r) \geq 1$  inside the first integral and taking  $s_{\text{in}}$  outside the first integral. Combining (43) with (40) gives

$$\begin{aligned} \dot{V}^\sharp &\leq -\frac{3}{5} \aleph \frac{\tilde{s}^2}{s} - \frac{5}{8} \varepsilon_3 |\mathcal{U}| \tilde{Y}^2 - \tilde{z}^2 \\ &\quad + \varepsilon_3 C_8 \tau_M \int_{t-\tau}^t \{ \tilde{s}(r)^2 / s(r) \} dr \\ &\quad + C_8 \varepsilon_3^3 \tau_M \int_{t-2\tau}^t \tilde{Y}(r)^2 dr \end{aligned} \quad (45)$$

for all  $t \geq \tau_M$ , where  $C_8 = C_4(C_6 s_{\text{in}} + C_7)$ .

Then (40) implies that along the closed loop trajectories,

$$\begin{aligned} U_1 &= V^\sharp(\tilde{s}, \tilde{x}_1, \tilde{x}_2) \\ &+ 2\varepsilon_3 C_8 \tau_M \int_{t-2\tau_M}^t \left( \int_\ell^t \{ \tilde{s}(r)^2 / s(r) \} dr \right) d\ell \quad (46) \\ &+ 2C_8 \varepsilon_3^3 \tau_M \int_{t-2\tau_M}^t \left( \int_\ell^t \tilde{Y}(r)^2 dr \right) d\ell \end{aligned}$$

satisfies

$$\begin{aligned} \dot{U}_1 &\leq -\tilde{z}^2 - \left[ \frac{3}{5} \aleph - 4\varepsilon_3 C_8 \tau_M^2 \right] \frac{\tilde{s}^2}{s} \\ &- \varepsilon_3 \left[ \frac{5}{8} |\tilde{U}| - 4C_8 \varepsilon_3^2 \tau_M^2 \right] \tilde{Y}^2 \end{aligned} \quad (47)$$

for all  $t \geq \tau_M$ , by (45). Our choice of  $\bar{\varepsilon}_3$  in (13) gives  $\bar{\varepsilon}_3 \leq |\tilde{U}|$ , and  $\bar{c}_2 \geq 1$ , so simple calculations show that the constant  $\bar{c}_2$  in (13) satisfies  $C_8 \leq \bar{c}_2$ . It follows from our choice of the bound  $\bar{\varepsilon}_3$  for  $\varepsilon_3$  in (13) that

$$\begin{aligned} \dot{U}_1 &\leq -\tilde{z}^2 - \frac{\aleph}{5} \frac{\tilde{s}^2}{s} - \frac{\varepsilon_3 |\tilde{U}|}{8} \tilde{Y}^2 \\ &= -(\tilde{s} + \tilde{x}_1 + \tilde{x}_2)^2 - \frac{\aleph}{5} \frac{\tilde{s}^2}{s} - \frac{\varepsilon_3 |\tilde{U}|}{8} (\tilde{x}_1 + a\tilde{x}_2)^2 \end{aligned} \quad (48)$$

for all  $t \geq \tau_M$ . Since  $a \neq 1$ , the right side of (48) is negative definite in  $(\tilde{s}, \tilde{x}_1, \tilde{x}_2)$ . The function  $U_1$  is not a Lyapunov-Krasovskii functional in the usual sense because it only characterizes the behavior after time  $\tau_M$ , but it leads to the desired UGAS estimate, by standard Lyapunov-Krasovskii arguments; see Appendix A.3.  $\square$

**Remark 4** The same transformation we used in Section 5.1 works if  $V$  is any function satisfying the conclusions of Lemma 4. Similarly, the transformation from Section 5.2 leads to the appropriate Lyapunov-Krasovskii decay estimate (48) for any proper and positive definite function  $V^\sharp$  satisfying the estimate (40). The preceding stabilization problem cannot be solved using Mazenc et al. (2008b). The paper Mazenc et al. (2008b) analyzed a class of systems  $\dot{x} = f(t, x) + g(t, x)u_s(t, x)$  that were known to be UGAS to 0 under a known feedback  $u_s$ , and then used a Lyapunov-Krasovskii argument to compute a constant  $\bar{\tau} > 0$  such that

$$\dot{x}(t) = f(t, x(t)) + g(t, x(t))u_s(t, x(t - \tau))$$

is UGAS to 0 for all constant delays  $\tau \in [0, \bar{\tau}]$ . This result from Mazenc et al. (2008b) does not apply in our case because (a) we wish to construct *output* feedbacks that stabilize under *arbitrarily large* delay bounds and (b) the error dynamics (23) we are stabilizing has a drift term. See Remarks 4 and A.4 in Mazenc et al. (2008b). While Malisoff & Mazenc (2009) and our other chemostat-related works in the references below also use strictification approaches to build strict Lyapunov functions, they do not apply in the situation we consider here, because the systems they cover have different forms, and because our earlier chemostat Lyapunov constructions are only for undelayed systems with *Monod* uptake functions.

**Remark 5** Our only requirements on  $\bar{c}_2$  are that it is at least  $C_8 = C_4(C_6 s_{\text{in}} + C_7)$  and at least 1, so there are many possible choices of  $\bar{c}_2$ . Another choice is as follows. Assume that  $\bar{\varepsilon}_3 < 2 \min\{1, \aleph / (10\bar{c}_1^2 s_{\text{in}})\}$ , and

that  $s_{\text{in}} \geq 1$ . Then,  $C_4 \leq 4|\tilde{U}| + 1$ , so we can take

$$\bar{c}_2 = (4|\tilde{U}| + 1) \left\{ 36s_{\text{in}}^3 \frac{D_*^2}{s_*^2} (1+a)^2 + 9(1+a)^2 s_{\text{in}}^2 \right\}.$$

We use this alternative formula for  $\bar{c}_2$  in the next section.

## 6 Example

Consider the system (1) with the Haldane growth functions

$$\mu_1(s) = \frac{6s}{8+s+0.12s^2} \quad \text{and} \quad \mu_2(s) = \frac{2s}{1+s+0.04s^2}. \quad (49)$$

In our earlier notation,  $K_1 = 6$ ,  $L_1 = 8$ ,  $g_1 = 0.12$ ,  $K_2 = 2$ ,  $L_2 = 1$ , and  $g_2 = 0.04$ . See Fig. 1. Assumption 1 holds with  $s_* = 2.5$ ,  $s_{\text{in}} = 2.75$ , and  $D_* = 1.33$ . To check Assumption 2, notice that  $L_1 - g_1 s_* s_{\text{in}} = 7.18 = \delta_1 > 0$  and  $L_2 - g_2 s_* s_{\text{in}} = 0.73 = \delta_2 > 0$ . Also,  $L_2 g_1 - L_1 g_2 = \frac{3}{25} - \frac{8}{25} < 0$ . We must now choose the  $x_{i*}$ 's so that Assumption 2 is satisfied and so that  $s_* + x_{1*} + x_{2*} = s_{\text{in}} = 11/4$ , which we do by selecting  $x_{1*} = 0.19$  and  $x_{2*} = 0.06$ ; in particular, this gives the necessary condition  $\aleph > 0$ . Moreover,

$$-K_1 + \frac{L_1 - g_1 s_*^2}{L_2 - g_2 s_*^2} K_2 = -6 + \frac{8 - \frac{3}{4}}{1 - \frac{3}{4}} 2 = \frac{40}{3} > 0,$$

so Assumption 2 holds for any constant  $a \in (0, 1)$ . Also,  $\mu_1(s_{\text{in}}) = 1.42$  and  $\mu_2(s_{\text{in}}) = 1.36$  are strictly larger than  $D_* = 1.33$  so Assumption 3 is also satisfied. Hence, we can take  $(s_*, x_{1*}, x_{2*}) = (2.5, 0.19, 0.06)$ , and then Theorem 1 applies for any given delay bound  $\tau_M > 0$  and any constant  $a \in (0, 1)$ .

We simulated (1) using the growth functions (49),  $(s_*, x_{1*}, x_{2*}) = (2.5, 0.19, 0.06)$ , and the feedback (12) with  $\tilde{U}$  from (7). Rounding our bounds  $\bar{\varepsilon}_i$  to the second decimal places and using the alternative formula for  $\bar{c}_2$  from Remark 5, we are able to choose  $a = 0.1$ ,  $\varepsilon_1 = 0.01$ ,  $\varepsilon_2 = 0.01$ , and the delay  $\tau = 0.5$ . We initialized the dynamics with  $(s(t), x_1(t), x_2(t)) \equiv (2.5, 1, 0.1)$  on  $[-0.5, 0]$ . See Figs. 2-4. Our simulation shows the rapid convergence of the state trajectories toward the equilibrium triple  $(s_*, x_{1*}, x_{2*})$ , and so validates our stabilization theorem. In particular, our feedback design keeps the species levels from becoming very small, which would be dangerous in practice because small species levels can lead to extinction. Note too that if we had used the constant dilution rate  $D = D_*$  as the controller, then Assumption 1 would imply that  $D = \mu_1(s_*) = \mu_2(s_*)$  and therefore all points of the form  $(s_*, x_{1n}, x_{2n}) \in (0, \infty)^3$  with  $x_{1n} + x_{2n} = s_{\text{in}} - s_*$  would be equilibria of (1), so  $(s_*, x_{1*}, x_{2*})$  would not be an isolated equilibrium and so would not even be locally asymptotically stable with this constant controller.

## 7 Conclusion

The output feedback stabilization of chemostats with measurement delays in the dilution rate controller is a challenging problem that is of considerable interest. For

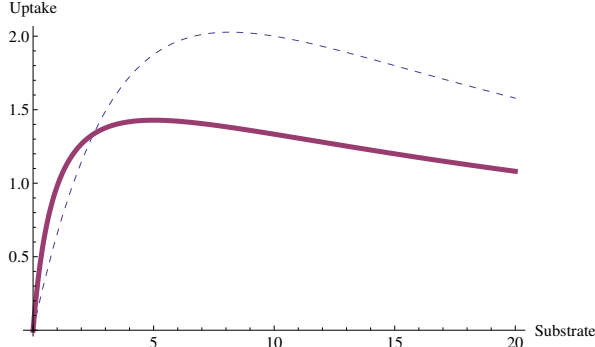


Fig. 1.  $\mu_1$  (Dashed) and  $\mu_2$  (Solid) from (49).

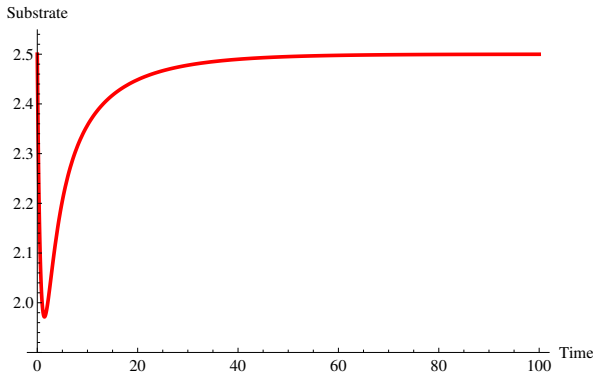


Fig. 2. Convergence of  $s$  towards  $s_* = 2.5$ .

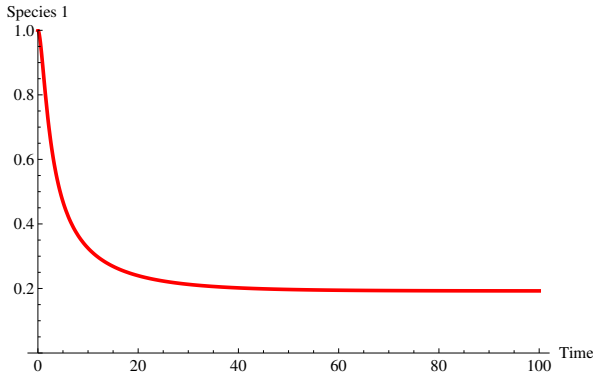


Fig. 3. Convergence of  $x_1$  towards  $x_{1*} = 0.19$ .

one species chemostats, this problem was analyzed in Robledo (2009). Here we solved the problem for two species chemostats. Our stabilizing feedbacks depend on a given linear combination of the species concentrations, which are measured with an unknown time delay. One only knows an upper bound on the possible delays, and we allow the upper bound to be arbitrarily large. The stabilizing feedback also depends on the known upper bound for the delay. When delays are present, the Poincaré-Bendixson and strict Lyapunov function approaches do not apply. Instead, we used a Lyapunov-Krasovskii approach. Our results cover cases where the growth functions are either Monod or Haldane. Our new Lyapunov-Krasovskii functional ap-

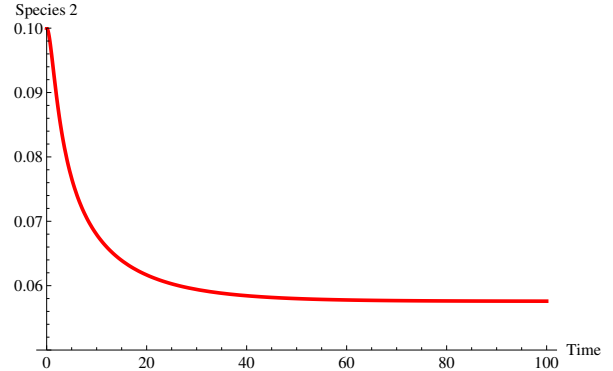


Fig. 4. Convergence of  $x_2$  towards  $x_{2*} = 0.06$ .

proach makes it possible to adapt the input-to-state stability arguments from Mazenc et al. (2009) to show robustness to actuator errors. Due to space constraints, we omit this robustness analysis.

## Appendices

### A.1 Proof of Lemma 1

Let  $(s(t), x_1(t), x_2(t))$  be any solution of (1) with the feedback (12), and with any initial function in  $C_{in}$ . One easily checks that each component of the solution stays positive. Also, the exponential stability of (14) provides a constant  $T_a > 0$  such that  $\tilde{z}(t) \leq 0.1s_{in}$ , hence also  $z(t) \leq \frac{11}{10}s_{in}$ , for all  $t \geq T_a$ . We prove by contradiction that  $s(t)$  enters  $(0, s_{in})$  in finite time. Suppose that  $s(t) \geq s_{in}$  for all  $t \geq 0$ . Then  $\dot{s} \leq (1 - 0.5p)D_*(s_{in} - s) \leq 0$  for all  $t \geq 0$ . This provides a constant  $s_L \geq s_{in}$  such that  $\lim_{t \rightarrow \infty} s(t) = s_L$ . In fact,  $s_L = s_{in}$ , since otherwise  $s$  would go to  $-\infty$ . We would then be able to find a time  $T_b \geq T_a$  such that  $\mu_i(s(t)) > \mu_i(s_{in}) - 0.25pD_*$  for all  $t \geq T_b$  and  $i = 1, 2$ , so (8) gives  $\mu_i(s(t)) - D \geq \mu_i(s_{in}) - 0.25pD_* - (1 + 0.5p)D_* \geq 0.25pD_*$  for all  $t \geq T_b$  and  $i = 1, 2$ . This would imply that  $x_1(t)$  and  $x_2(t)$  go to  $\infty$ , contradicting the fact that  $z(t) = s(t) + x_1(t) + x_2(t)$  is bounded. Hence, there exists a time  $T_c \geq T_a$  such that  $s(T_c) \in (0, s_{in})$ . This shows that  $S$  is globally attractive for (1). Positive invariance follows because (I)  $\dot{s} < 0$  when  $s = s_{in}$  and (II)  $(-s_{in}, 0.1s_{in}]$  is positively invariant for (14), so  $z(t) = \tilde{z}(t) + s_{in} \leq 1.1s_{in}$  for all  $t \geq 0$  if  $z$  starts in  $(0, 1.1s_{in}]$ .

### A.2 Proof of Lemma 4

Assumption 2 ensures that the function  $V$  defined in (26) is positive definite on  $\mathcal{E}$ . To see why, notice that if  $m \in (-s_*, s_{in} - s_*)$ , then  $L_1 - g_1s_*(s_* + m) \geq L_1 - g_1s_*s_{in} = \delta_1 > 0$  and  $m + s_* > 0$ , which give

$$\int_0^{\tilde{s}} \frac{L_1 - g_1s_*(s_* + m)}{s_* + m} m dm \geq \int_0^{\tilde{s}} \frac{\delta_1}{s_* + m} m dm.$$

Moreover,  $V$  is positive definite on  $\mathcal{X} = (-s_*, \infty) \times (-x_{1*}, \infty) \times (-x_{2*}, \infty)$ , and  $V(\tilde{s}, \tilde{x}_1, \tilde{x}_2) \rightarrow \infty$  as

$(\tilde{s}, \tilde{x}_1, \tilde{x}_2) \in \mathcal{X}$  approaches any boundary point of  $\mathcal{X}$  or as  $|(\tilde{s}, \tilde{x}_1, \tilde{x}_2)| \rightarrow \infty$ , which is the properness condition. However,  $V(\tilde{s}, \tilde{x}_1, \tilde{x}_2)$  does not go to the infinity when  $\tilde{s}$  approaches  $s_{\text{in}} - s_*$ . Nevertheless, this Lyapunov-like function, combined with the positive invariance of  $\mathcal{E}$ , will allow us to establish global asymptotic stability.

Since  $\tilde{s} = s - s_*$ , the expression (23) for the error dynamics and our choices (24) of  $C_1$  and  $C_2$  readily give

$$\begin{aligned} \frac{L_1 - g_1 s_* s}{s} \tilde{s} \dot{\tilde{s}} &= -\frac{L_1 - g_1 s_* s}{s} H(s) \tilde{s}^2 - \frac{K_1(L_1 - g_1 s_* s)}{L_1 + s + g_1 s_*^2} \tilde{x}_1 \tilde{s} \\ &\quad + \frac{K_2 g_1 s_*}{L_2 + s + g_2 s_*^2} \tilde{x}_2 \tilde{s}^2 - \frac{K_2(L_1 - g_1 s_*^2)}{L_2 + s + g_2 s_*^2} \tilde{x}_2 \tilde{s} \\ &\quad + \tilde{s} \frac{L_1 - g_1 s_* s}{s} (s_{\text{in}} - s) v, \\ C_1 \frac{\tilde{x}_1 \dot{\tilde{x}}_1}{\tilde{x}_1} &= K_1 \frac{L_1 - g_1 s_* s}{L_1 + s + g_1 s_*^2} \tilde{x}_1 \tilde{s} - \frac{K_1 s_*}{D_*} v \tilde{x}_1, \text{ and} \\ C_2 \frac{\tilde{x}_2 \dot{\tilde{x}}_2}{\tilde{x}_2} &= -\frac{(L_1 - g_1 s_*^2) K_2 g_2 s_*}{(L_2 - g_2 s_*^2)(L_2 + s + g_2 s_*^2)} \tilde{x}_2 \tilde{s}^2 \\ &\quad + \frac{K_2(L_1 - g_1 s_*^2)}{L_2 + s + g_2 s_*^2} \tilde{x}_2 \tilde{s} - \frac{L_1 - g_1 s_*^2}{L_2 - g_2 s_*^2} \frac{K_2 s_*}{D_*} \tilde{x}_2 v, \end{aligned}$$

where we used the relation  $L_i - g_i s_* s = -g_i s_* \tilde{s} + (L_i - g_i s_*^2)$  for  $i = 1$  to get the first equality and for  $i = 2$  to get the third equality. On  $\mathcal{E}$ , the sum of the preceding three equalities is  $\dot{V}$  by definition. It follows from our choice (25) of the function  $M$  and by canceling terms that on the positively invariant attractive set  $\mathcal{E}$ , we have

$$\begin{aligned} \dot{V} &= -\frac{L_1 - g_1 s_* s}{s} H(s) \tilde{s}^2 + \frac{K_2 g_1 s_*}{L_2 + s + g_2 s_*^2} \tilde{x}_2 \tilde{s}^2 \\ &\quad - \frac{(L_1 - g_1 s_*^2) K_2 g_2 s_*}{(L_2 - g_2 s_*^2)(L_2 + s + g_2 s_*^2)} \tilde{x}_2 \tilde{s}^2 \\ &\quad + M(\tilde{s}, \tilde{x}_1, \tilde{x}_2) v \\ &\leq -\frac{\delta_1}{s} H(s) \tilde{s}^2 + M(\tilde{s}, \tilde{x}_1, \tilde{x}_2) v \\ &\quad + K_2 s_* \left[ \frac{L_2 g_1 - L_1 g_2}{L_2 - g_2 s_*^2} \right] \frac{(x_2 - x_{2*}) \tilde{s}^2}{L_2 + s + g_2 s_*^2}, \end{aligned} \quad (\text{A.1})$$

where the inequality in (A.1) followed from Assumption 2 and the positivity of the lower bound (22).

By Assumption 2,  $L_2 g_1 - L_1 g_2 \leq 0$  and  $L_2 - g_2 s_*^2 \geq \delta_2 > 0$ , since  $s_* \in (0, s_{\text{in}})$ . We deduce that

$$\begin{aligned} K_2 s_* \left[ \frac{L_2 g_1 - L_1 g_2}{L_2 - g_2 s_*^2} \right] \frac{(x_2 - x_{2*}) \tilde{s}^2}{L_2 + s + g_2 s_*^2} &\leq \\ -K_2 s_* \left[ \frac{L_2 g_1 - L_1 g_2}{\delta_2} \right] \frac{x_{2*} \tilde{s}^2}{L_2 + s + g_2 s_*^2}. \end{aligned} \quad (\text{A.2})$$

Also,  $L_2 + s + g_2 s_*^2 \geq s$  everywhere. Therefore, along all closed loop trajectories of (23) in  $\mathcal{E}$ , Assumption 2 and (22) give the nonstrict Lyapunov decay condition (27).

### A.3 UGAS estimate

Our decay estimate (48) makes it possible to construct the function  $\beta \in \mathcal{KL}$  in the uniform global asymptotic stability estimate for (23) on  $\mathcal{E}$ , thereby quantifying the convergence of the trajectories to the equilibrium; we construct  $\beta$  in this section.<sup>1</sup> This can be done by a vari-

<sup>1</sup> We say that a continuous function  $\gamma : [0, \infty) \rightarrow [0, \infty)$  belongs to class  $\mathcal{K}_\infty$  and write  $\gamma \in \mathcal{K}_\infty$  provided it is strictly increasing and unbounded and  $\gamma(0) = 0$ . We say that a continuous function  $\beta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  is of class  $\mathcal{KL}$  provided (a) for each fixed  $s \geq 0$ , the function  $\beta(\cdot, s)$

ant of the very last part of the proof of the main result in Mazenc et al. (2008b). Here is a sketch. Our choice (46) of  $U_1$  actually guarantees that along all of the closed loop trajectories of (23) in  $\mathcal{E}$ , we have

$$\begin{aligned} \dot{U}_1 &\leq -\tilde{z}^2 - \frac{\varepsilon_3}{5} \frac{\tilde{s}^2}{s} - \frac{\varepsilon_3 |\bar{U}|}{8} \tilde{Y}^2 \\ &\quad - c_o \left( \int_{t-2\tau_M}^t \frac{\tilde{s}(r)^2}{s(r)} dr + \int_{t-2\tau_M}^t \tilde{Y}^2(r) dr \right) \end{aligned} \quad (\text{A.3})$$

for all  $t \geq \tau_M$ , where  $c_o = \varepsilon_3 C_8 \tau_M \min\{1, \varepsilon_3^2\}$ . By reducing  $c_o > 0$  without relabeling, we can assume that

$$\begin{aligned} \dot{U}_1 &\leq -c_0 \left\{ \tilde{z}^2(t) + \tilde{Y}^2(t) + \frac{\tilde{s}^2(t)}{s(t)} \right. \\ &\quad \left. + \int_{t-2\tau_M}^t \left( \int_\ell^t \frac{\tilde{s}(r)^2}{s(r)} dr \right) d\ell \right. \\ &\quad \left. + \int_{t-2\tau_M}^t \left( \int_\ell^t \tilde{Y}^2(r) dr \right) d\ell \right\} \end{aligned} \quad (\text{A.4})$$

along the closed loop trajectories of (23) in  $\mathcal{E}$  for all  $t \geq \tau_M$  (by bounding the double integrals in (A.4) with a constant times the single integrals in (A.3)). Let  $L$  denote the quantity in braces in (A.4), and set  $\mathcal{E}^o = \mathcal{E} \cap \{(-0.5s_*, 0.5[s_{\text{in}} - s_*]) \times (-0.5x_{1*}, 0.5x_{1*}) \times (-0.5x_{2*}, 0.5x_{2*})\}$ . Since  $a \in (0, 1)$ , we can find a constant  $c_1$  so that  $U_1 \leq c_1 L$  whenever  $(\tilde{s}(t), \tilde{x}_1(t), \tilde{x}_2(t)) \in \mathcal{E}^o$ . Set  $U_2 = e^{U_1} - 1$ .

We can then find a constant  $c_2 > 0$  so that  $\dot{U}_1(t, \zeta) \leq -c_2 \min\{1, U_1(t, \zeta)\}$  and so also

$$\dot{U}_2(t, \zeta) \leq -c_2 U_2(t, \zeta) \quad (\text{A.5})$$

along all closed loop trajectories  $\zeta$  of (23) for all  $t \geq 2\tau_M$ ; this follows by separately considering the cases  $(\tilde{s}(t), \tilde{x}_1(t), \tilde{x}_2(t)) \in \mathcal{E}^o$  and  $(\tilde{s}(t), \tilde{x}_1(t), \tilde{x}_2(t)) \in \mathcal{E} \setminus \mathcal{E}^o$ , and then using the fact that  $e^u u \geq e^u - 1$  for all  $u \geq 0$ . We can find a function  $\alpha_0 \in \mathcal{K}_\infty$  and a modulus  $\alpha_1$  with respect to  $\mathcal{E}$  so that (i)  $\alpha_0(|\zeta(t)|) \leq U_2(t, \zeta) \leq |\alpha_1(\zeta(r))|_{[t-2\tau_M, t]}$  for all  $t \geq 0$  and (ii)  $|\zeta(r)|_{[0, 2\tau_M]} \leq |\alpha_1(\zeta(r))|_{[-2\tau_M, 0]}$  for all closed loop trajectories  $\zeta(t) = (\tilde{s}(t), \tilde{x}_1(t), \tilde{x}_2(t))$  of (23) in  $\mathcal{E}$ .<sup>2</sup> Therefore,

$$|\zeta(t)| \leq \beta(|\alpha_1(\zeta(r))|_{[-2\tau_M, 0]}(t))$$

along all closed loop trajectories  $\zeta$  of (23) in  $\mathcal{E}$  for all  $t \geq 0$ , where  $\beta \in \mathcal{KL}$  is defined by  $\beta(r, t) = \alpha_0^{-1}(\sup\{\alpha_1(q) : |q| \leq r, q \in \mathcal{E}\} \exp(c_2(2\tau_M - t))) + \text{rexp}(c_2(2\tau_M - t))$ .

### A.4 $\bar{U} = 0$ Case

We show that if all of the assumptions of Theorem 1 hold except that  $\bar{U} = 0$ , and if  $\mu_1$  and  $\mu_2$  are different functions and  $\tau = 0$ , then there does not exist a  $C^1$  feedback  $D$  that stabilizes the chemostat dynamics to the componentwise positive equilibrium  $(s_*, x_{1*}, x_{2*})$ .

belongs to class  $\mathcal{K}_\infty$ , and (b) for each fixed  $r \geq 0$ , the function  $\beta(r, \cdot)$  is non-increasing and  $\beta(r, s) \rightarrow 0$  as  $s \rightarrow \infty$ .

<sup>2</sup> The existence of  $\alpha_1$  follows from a slight variant of the proof of (Mazenc et al., 2008b, Lemma A.1), applied on the invariant set  $\mathcal{E}$ . We use the sup norm notation  $|\xi(r)|_J = \sup\{|\xi(w)| : w \in J\}$  for the supremum of any continuous function on any closed interval  $J$ .



To this end, first note that  $\mathcal{U} = 0$  is equivalent to

$$K_1(L_2 - g_2 s_*^2) = K_2(L_1 - g_1 s_*^2). \quad (\text{A.6})$$

Since  $\mu_1(s_*) = \mu_2(s_*)$  implies that

$$\frac{1}{(L_1 + s_* + g_1 s_*^2)^2} = \frac{K_2^2}{K_1^2(L_2 + s_* + g_2 s_*^2)^2},$$

we get

$$\begin{aligned} \mu'_1(s_*) &= K_1 \frac{L_1 - g_1 s_*^2}{(L_1 + s_* + g_1 s_*^2)^2} = \frac{K_1(L_1 - g_1 s_*^2) K_2^2}{K_1^2(L_2 + s_* + g_2 s_*^2)^2} \\ &= \frac{K_2(L_1 - g_1 s_*^2)}{K_1(L_2 + s_* + g_2 s_*^2)} \mu'_2(s_*) = \mu'_2(s_*), \end{aligned} \quad (\text{A.7})$$

where the last equality used (A.6). On the other hand, simple calculations give

$$\begin{aligned} &K_1 K_2 \frac{s_*^2}{2} \mu''_i(s) \\ &= -K_j g_i \mu_i(s)^2 s + \mu_i(s) \left[ \mu'_i(s) - \frac{\mu_i(s)}{s} \right] K_j (L_i - g_i s^2) \end{aligned}$$

for  $j = 1, 2$  and  $i = 1, 2$  with  $i \neq j$ . We deduce that

$$\begin{aligned} &K_1 K_2 \frac{s_*^2}{2} \mu''_1(s_*) = \\ &D_* \left[ \mu'_1(s_*) - \frac{D_*}{s_*} \right] K_2 (L_1 - g_1 s_*^2) - K_2 g_1 D_*^2 s_* \end{aligned} \quad (\text{A.8})$$

and

$$\begin{aligned} &K_1 K_2 \frac{s_*^2}{2} \mu''_2(s_*) = \\ &D_* \left[ \mu'_2(s_*) - \frac{D_*}{s_*} \right] K_1 (L_2 - g_2 s_*^2) - K_1 g_2 D_*^2 s_* \end{aligned} \quad (\text{A.9})$$

Since  $\mu'_2(s_*) = \mu'_1(s_*)$ , (A.6) and (A.9) give

$$\begin{aligned} &K_1 K_2 \frac{s_*^2}{2} \mu''_2(s_*) = \\ &D_* \left[ \mu'_1(s_*) - \frac{D_*}{s_*} \right] K_2 (L_1 - g_1 s_*^2) - K_1 g_2 D_*^2 s_* \end{aligned} \quad (\text{A.10})$$

By subtracting (A.10) from (A.8), we obtain

$$\begin{aligned} &K_1 K_2 \frac{s_*^2}{2} [\mu''_1(s_*) - \mu''_2(s_*)] \\ &= [-K_2 g_1 + K_1 g_2] s_* D_*^2 \end{aligned} \quad (\text{A.11})$$

Therefore,  $\mu''_1(s_*) - \mu''_2(s_*) = 0$  if and only if  $-K_2 g_1 + K_1 g_2 = 0$ . Assume that  $-K_2 g_1 + K_1 g_2 = 0$ . Then we deduce from (A.6) that  $K_1 L_2 = K_2 L_1$ . Since  $\mu_1(s_*) = \mu_2(s_*)$ , we have  $K_1(L_2 + s_* + g_2 s_*^2) = K_2(L_1 + s_* + g_1 s_*^2)$ . Using  $K_2 g_1 = K_1 g_2$  and  $K_1 L_2 = K_2 L_1$ , we now get  $K_1 = K_2$ . It follows that  $g_1 = g_2$  and  $L_1 = L_2$ . Hence  $\mu_1 = \mu_2$ . We conclude that when  $\mathcal{U} = 0$ , we have  $\mu''_1(s_*) - \mu''_2(s_*) = 0$  if and only if  $\mu_1 = \mu_2$ .

Thus, we have established that if  $\mu_1 \neq \mu_2$  and  $\mathcal{U} = 0$ , then  $\mu'_1(s_*) = \mu'_2(s_*)$  and  $\mu''_1(s_*) \neq \mu''_2(s_*)$ . Assume that  $\mathcal{U} = 0$  and  $\mu_1 \neq \mu_2$ , and let  $\xi_i = \ln(x_i)$  and  $\chi = \frac{\xi_1 - \xi_2}{\mu''_1(s_*) - \mu''_2(s_*)}$  for all  $x_1 > 0$  and  $x_2 > 0$ . Then

$$\begin{aligned} \dot{\chi} &= \frac{[\mu_1(s) - \mu_1(s_*)] - [\mu_2(s) - \mu_2(s_*)]}{\mu''_1(s_*) - \mu''_2(s_*)} \\ &= \frac{\int_{s_*}^s \left[ \int_{s_*}^r \mu''_1(m) dm - \int_{s_*}^r \mu''_2(m) dm \right] dr}{\mu''_1(s_*) - \mu''_2(s_*)} \end{aligned} \quad (\text{A.12})$$

We deduce that there is a constant  $\varepsilon > 0$  such that

$$\frac{\int_{s_*}^s \int_{s_*}^r (\mu''_1(m) - \mu''_2(m)) dm dr}{\mu''_1(s_*) - \mu''_2(s_*)} \geq 0$$

for all  $s \in [s_* - \varepsilon, s_* + \varepsilon]$ . The fact that the equilibrium point  $(s_*, x_{1*}, x_{2*})$  cannot be locally uniformly asymptotically stabilized by a  $C^1$  control (which necessarily would have  $D = \mu_1(s_*)$  at the equilibrium, because  $x_{1*} > 0$ ) now follows from the following lemma, applied to the  $(\chi, s, \xi_1)$  dynamics in closed loop with any  $C^1$  feedback and the equilibrium  $(\chi_*, s_*, \xi_{1*}) = (\{\ln(x_{1*}) - \ln(x_{2*})\} / \{\mu''_1(s_*) - \mu''_2(s_*)\}, s_*, \ln(x_{1*}))$ :

**Lemma A.1** *Let  $\mathcal{O} \subseteq \mathbb{R}^2$  be any open set and*

$$\begin{cases} \dot{X} = f_1(t, X, Z) \\ \dot{Z} = f_2(t, X, Z), \quad (X, Z) \in \mathbb{R} \times \mathcal{O} \end{cases} \quad (\text{A.13})$$

*be any  $C^1$  system. Let  $(X_e, Z_e) \in \mathbb{R} \times \mathcal{O}$  be an equilibrium point of (A.13) and assume that there is a constant  $\delta > 0$  such that  $\inf_{t \geq 0} f_1(t, X, Z) \geq 0$  for all  $(X, Z) \in \mathbb{R} \times \mathcal{O}$  satisfying  $|(X - X_e, Z - Z_e)| \leq \delta$ . Then  $(X_e, Z_e)$  is not a locally asymptotically stable equilibrium point of (A.13).*

*Proof:* We prove the result by contradiction. Assume that  $(X_e, Z_e)$  is a locally asymptotically stable equilibrium point of (A.13). Then we could find two positive constants  $\varepsilon_0 \in (0, \delta)$  and  $\varepsilon_1 \in (0, \delta)$  such that if  $|(X_0 - X_e, Z_0 - Z_e)| \leq \varepsilon_0$ , then the solution  $(X(t), Z(t))$  of (A.13) with  $(X_0, Z_0)$  as initial state satisfies  $|(X(t) - X_e, Z(t) - Z_e)| \leq \varepsilon_1$  for all  $t \geq 0$  and  $\lim_{t \rightarrow \infty} (X(t), Z(t)) = (X_e, Z_e)$ . We choose the particular initial state  $(X_0, Z_0) = (\varepsilon_0 + X_e, Z_e)$ . Then  $|(X_0 - X_e, Z_0 - Z_e)| \leq \varepsilon_0$ , so  $f_1(t, X(t), Z(t)) \geq 0$  for all  $t \geq 0$ . Hence,  $X(t) \geq \varepsilon_0 + X_e$  for all  $t \geq 0$ , so  $X(t)$  does not converge to  $X_e$ . We deduce that  $(X_e, Z_e)$  cannot be a locally asymptotically stable equilibrium point of (A.13).  $\square$

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