Reduction Model Approach for Linear Time-Varying Systems with Delays

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***Abstract*—We study stabilization problems for time-varying linear systems with constant input delays. Our reduction method ensures input-to-state stability with respect to additive uncertainties, under arbitrarily long delays. It applies to rapidly time-varying systems, and gives a lower bound on the admissible rapidness parameters. We also cover slowly time-varying systems, including upper bounds on the allowable slowness parameters. We illustrate our work using a pendulum model.**

***Index Terms*—delay, time-varying, reduction, stabilization.**

I. Introduction

STABILIZING equilibria under input delays is a challenging problem that has been studied using many methods [1], [2], [3], [4], [5], [6]. Input delays arise from measurement and transport phenomena in engineering. There are two major streams of research. One involves finding a uniformly globally asymptotically stabilizing nominal controller with the input delay set equal to zero and a corresponding strict Lyapunov function for the closed loop undelayed system, and then finding an upper bound on the delay that can be introduced in the control without destroying the stability. Two benefits of this approach are that (a) it does not require a new control to compensate for the delay and (b) the Lyapunov function can often be transformed into a Lyapunov-Krasovskii functional to quantify the effects of actuator errors [7], [8]. However, this approach usually produces finite bounds on the allowable delays when the system has drift [8], [9] (but see [7], [10], [11] for classes of systems where changing parameter values makes it possible to compensate arbitrarily long input delays). A different approach involves finding a controller u whose value at each time depends on values of u at smaller times. See, e.g., the reduction approach, which seems to have been introduced in [12] (with significant further developments in [5], [13], [14]) for stabilizing linear time-invariant systems or systems with finite bounds on the delays. See also [15] for an H_∞ approach, and the observer predictor structure in [16].

The problem of asymptotically stabilizing the origin of linear time-varying systems naturally arises when one wishes to locally asymptotically track a trajectory of a nonlinear system [17], [18]. However, the methods from [5], [12], [13], [14] do not apply to this problem, because the nominal delay-free designs are for linear time-invariant systems. Linear time-varying systems with delays have been studied in only a few works, and in most of them, the knowledge of time-varying Lyapunov functionals is assumed [19], [20], [21]. This can be a significant limitation, since determining a Lyapunov function is often difficult. Control free linear time-varying systems with delay have been studied in [6], using tools that are very different from those we use in this paper. The study [7] covers systems with a special feedforward structure. The work [1] gives a reduction approach that establishes uniform global asymptotic stability for time-varying systems, using the fundamental matrices for the systems obtained by setting the controls equal to zero. However, fundamental solutions cannot always be obtained in a usable closed form.

These remarks motivate the present work. We adapt the reduction model approach to a family of time-varying linear systems with input delay of the form

$$\dot{x}(t) = F(t)x(t) + G(t)u(t-\tau) + \delta(t) \ , \tag{1}$$

where $\tau > 0$ is any constant delay and δ is an unknown measurable essentially bounded function representing uncertainty. The control u will be a solution of an integral equation involving x, and x and u are valued in $\mathbb{R}^n$ and $\mathbb{R}^p$ for any n and p, respectively; see below for our exact assumptions on F and G. We derive a formula for u to ensure input-to-state stability (ISS) properties for the closed loop system. For the special case where F is periodic, we obtain a convenient closed form expression for u that does not require an expression for the fundamental matrix. The ISS property generalizes uniform global asymptotic stability (UGAS) by adding an overshoot term in the UGAS estimate to quantify the effects of the perturbation; see [22] for the standard ISS definition for systems without time delays. Since our controllers are dynamic ones, our ISS estimates quantify the decay of both the norm $|x(t)|$ of the state and the sup norm $|u|_{[t-\tau,t]}$ of the control over $[t-\tau, t]$, with an overshoot depending on the sup norm of the perturbation; see below for the precise definitions. To the best of our knowledge, no ISS results for input delayed time-varying systems stabilized through the reduction approach are available in the literature. As special cases, we study systems of the form $\dot{x}(t) = A(\omega t)x(t) + B(\omega t)u(t-\tau) + \delta(t)$, where $\omega > 0$ is a constant and A is periodic of some period T. We refer to ω as the slowness (resp., rapidness) parameter, when we wish to design stabilizing controllers for all values of ω that are below (resp., above) a given constant value $\omega_* > 0$, in which case we call the dynamics a slowly (resp., rapidly) time-varying system. Using our reduction model approach, we can find upper bounds on the slowness parameter, and lower bounds on the rapidness parameter, that ensure ISS.

Rapidly and slowly time-varying systems without input

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delays have been studied using partial averaging, limiting dynamics, strict Lyapunov functions, and other methods [23], [24], [25], [26], [27], [28]. While slowly time-varying systems can be transformed into rapidly time-varying systems using a time scaling, the scaling produces rapidly time-varying systems that are beyond the scope of the limiting dynamics approach to rapidly time-varying systems [24]. Instead, see, e.g., the partial averaging results in [27] for slowly time-varying control systems, which generalize the corresponding averaging results in [29, pp.190-195] for uncontrolled systems. Extending these results for undelayed rapidly or slowly time-varying control systems to the input-delayed situation we consider here appears to be nontrivial. More generally, as mentioned in [30, Chapter 8], a necessary and sufficient condition for stabilizability is missing for linear time-varying systems with input delays. The stabilization results for delayed rapidly varying systems in [6, Chapter 6] do not use a reduction model approach, do not construct Lyapunov-Krasovskii functionals, and do not find bounds on the allowable rapidness parameters, nor do they produce ISS results, so they do not intersect with the results we present here. Instead, our work owes a lot to the recent work [31] which introduced a new approach to finding Lyapunov-Krasovskii functionals for the case of time-invariant vector fields. For the rapidly time-varying case, our ISS result follows as a special case of our work for (1) with the choice $F(t) = A(\omega t)$, which has the period $\overline{T} = T/\omega$ when A has period T. However, our analysis for slowly time-varying input delayed systems uses a different method.

Our work is primarily a methodological and mathematical development, rather than a specific real-world application. However, rapidly and slowly time-varying systems are important in engineering, e.g., to model pendula with nonlinear friction [27], mechanical systems where different components of the system have different rates of wear and tear [25], and much more. Therefore, our work is strongly motivated from both the control and the mathematical perspectives. For rapidly and slowly time-varying nonlinear systems, we can use our results to prove ISS under input delays for the linearizations around a reference trajectory, and to compute bounds on the allowable parameters ω without assuming that Lyapunov functions are known. Our Lyapunov-Krasovskii arguments produce closed form comparison functions in the ISS estimate, including an exponential decay estimate when the perturbations are zero. Therefore, we believe that our work is a significant new theoretical contribution with many potential applications.

The next section provides all of the required definitions, to make our work self contained. In Section III, we give a fundamental result on the existence of dynamic controllers that ensure ISS for (1). While of independent interest and applicable without any periodicity assumptions on the coefficient matrices, this fundamental result may not always lend itself to applications, because its controller uses a fundamental matrix solution that may be difficult to compute. Therefore, in Section IV, we give a general ISS result for (1) that applies when F is periodic with a small enough period, including a recursive method for computing the controller. Sections V and VI cover the special cases of rapidly and slowly time-varying systems, respectively. We illustrate our work in examples, in Section VII. In Section VIII, we summarize our contributions and suggest future research directions.

II. Definitions and Notation

Let $n \in \mathbb{N}$ be arbitrary. Let I_n denote the identity matrix in $\mathbb{R}^{n\times n}$, and $|\cdot|$ be the usual Euclidean norm of matrices and vectors. For square matrices M and N of the same size, $M \geq N$ means that $M - N$ is nonnegative definite. For any measurable essentially bounded matrix valued function ϕ defined on any interval I and any interval $J \subseteq I$, we use $|\phi|_J$ to denote its essential supremum over J, and $|\phi|_\infty$ when $J = \mathbb{R}$. Given any constant $\tau > 0$, we let $C([-\tau, 0], \mathbb{R}^n)$ denote the set of all continuous $\mathbb{R}^n$-valued functions that are defined on $[-\tau, 0]$. We call it the set of (all) initial functions. For any continuous function $\varphi : [-\tau, +\infty) \to \mathbb{R}^n$ and any $t \geq 0$, we define the function φ_t by $\varphi_t(\theta) = \varphi(t + \theta)$ for all $\theta \in [-\tau, 0]$. Let C^1 denote the set of all continuously differentiable functions, and C^0 denote the set of all continuous functions, when the domains and ranges of the functions are clear from the context. Let $\mathcal{K}_\infty$ be the set of all C^0 functions $\gamma : [0, +\infty) \to [0, +\infty)$ such that $\gamma(0) = 0$, γ is strictly increasing, and $\lim_{s\to+\infty} \gamma(s) = +\infty$. A C^0 function $\beta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is of class $\mathcal{KL}$ provided (a) for each $s \geq 0$, the function $\beta(\cdot, s)$ is in $\mathcal{K}_\infty$ and (b) for each $r \geq 0$, the function $\beta(r, \cdot)$ is non-increasing and satisfies $\lim_{s\to+\infty} \beta(r, s) = 0$. We use $M_{i,j}$ to denote the (i, j) entry of any matrix M for all i and j.

Since our dynamic control u in (1) will need initial functions for u and x, we use the following variant of the definition of input-to-state stable (ISS); see [22] for the standard ISS definition. We let $t_0 \geq 0$ denote the initial time for the trajectory under consideration. We say that (1) in closed loop with a control u is ISS provided there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all initial times $t_0 \geq 0$, all admissible initial functions $u : [t_0 - \tau, t_0] \to \mathbb{R}^p$ and $x : [t_0 - \tau, t_0] \to \mathbb{R}^n$, all measurable essentially bounded functions $\delta : [t_0, +\infty) \to \mathbb{R}^n$, and all $t \geq t_0$, the (unique) corresponding trajectory-control pair satisfies

$$|x(t)| + |u|_{[t-\tau,t]} \leq \beta(|(x, u)|_{[t_0-\tau,t_0]}, t - t_0) + \gamma(|\delta|_{[t_0,t]}). \quad (2)$$

If, in addition, there are constants a_1 and a_2 such that β can be written as $\beta(s, t) = a_1 s e^{-a_2 t}$ for all $s \geq 0$ and $t \geq 0$, then we say that the system is exponentially ISS, and that the system with $\delta = 0$ is uniformly globally exponentially stable (UGES). The special case of ISS where $\delta = 0$ is called uniform global asymptotic stability (UGAS). We also use UGAS to mean uniformly globally asymptotically stable, and similarly for ISS and UGES. Throughout our work, all (in)equalities should be understood to hold for all $t \geq t_0$, and all dimensions are arbitrary, unless otherwise indicated.

III. Fundamental Result

This section studies general linear time-varying control systems with input delay. Using the fundamental matrices for the systems obtained by setting the controls to zero, we show how to adapt the reduction model approach to prove ISS of the closed loop control systems. The fundamental solution cannot always be obtained in a usable closed form. This motivates our later sections, which solve stabilization problems when the fundamental solution is not available. For the special case where there are no disturbances, the result of this section reduces to a special case of Artstein's work [1] on UGAS. However, our proof is new and more direct and leads to the stronger ISS condition, which was not shown in [1]. The structure of our controller for the state $x(t)$ is $u(t) = K(t)x(t)$ for a suitable matrix K, plus an integral that is zero when $\tau = 0$, which generalizes standard linear controllers; see (9).

A. Assumptions and Statement of the Result

Consider the system

$$\dot{x}(t) = M(t)x(t) + N(t)u(t-\tau) + \delta(t), \tag{3}$$

where x is valued in $\mathbb{R}^n$, the control u is valued in $\mathbb{R}^p$ and is to be specified, $\tau \geq 0$ is a known constant delay, δ is an unknown measurable essentially bounded function, and M and N are continuous and bounded (but not necessarily periodic). Let $\lambda : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ be the fundamental solution associated with M. Then, for all real numbers t and t_0, we have

$$\tfrac{\partial \lambda}{\partial t}(t, t_0) = M(t)\lambda(t, t_0), \quad \lambda(t_0, t_0) = I_n. \tag{4}$$

Since M and N are bounded, Gronwall's inequality provides a constant $c > 0$ such that

$$|\lambda(t, \ell)N(\ell)| \leq c \ \ \forall \ell \in [t, t+\tau] \tag{5}$$

for all $t \geq 0$. We assume:

Assumption 1: There is a bounded continuous function $K : \mathbb{R} \to \mathbb{R}^{p\times n}$ such that the system

$$\dot{z}(t) \ = \ \big[M(t) + \lambda(t, t+\tau)N(t+\tau)K(t)\big]z(t) \tag{6}$$

is UGAS. □

Since (6) is a linear system, Assumption 1 is equivalent to the exponential ISS property for the undelayed system

$$\dot{z}(t) = [M(t) + \lambda(t, t+\tau)N(t+\tau)K(t)]\, z(t) + \delta(t) \tag{7}$$

with the perturbation δ, so we can find functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all $t_0 \geq 0$ and all $t \geq t_0$, we have

$$|z(t)| \ \leq \ \beta(|z(t_0)|, t - t_0) + \gamma\left(|\delta|_{[t_0,t]}\right) \tag{8}$$

along all trajectories of the perturbed system (7). We prove:

Theorem 1: Assume that the system (3) satisfies Assumption 1. Then there exist functions $\overline{\beta}$ of class $\mathcal{KL}$ and $\overline{\gamma}$ of class $\mathcal{K}_\infty$ such that all trajectories of (3) in closed loop with the control

$$\begin{array}{l} u(t) = \\ K(t)\left[x(t) + \int_{t-\tau}^{t}\lambda(t, m+\tau)N(m+\tau)u(m)\mathrm{d}m\right] \end{array} \tag{9}$$

defined for all $t \geq t_0$ satisfy

$$\begin{array}{l} |x(t)| + |u|_{[t-\tau,t]} \ \leq \\ \overline{\beta}\big(|x(t_0)| + |u|_{[t_0-\tau,t_0]}, t - t_0\big) + \overline{\gamma}(|\delta|_{[t_0,t]}) \end{array} \tag{10}$$

for all initial times $t_0 \geq 0$ and all $t \geq t_0$. □

Remark 1: A key point in Assumption 1 is that it pertains to a system with no delay. The pair $(x(t), u(t))$ is defined for all initial functions $(x, u) : [t_0 - \tau, t_0] \to \mathbb{R}^n \times \mathbb{R}^p$ for which the matching condition

$$\begin{array}{l} u(t_0) = \\ K(t_0)\Big[x(t_0) + \int_{t_0-\tau}^{t_0}\lambda(t_0, m+\tau)N(m+\tau)u(m)\mathrm{d}m\Big] \end{array} \tag{11}$$

holds. We discuss matching conditions in the next section. □

Remark 2: Since

$$\begin{array}{l} \max\{|x(t)|, |u|_{[t-\tau,t]}\} \leq |x(t)| + |u|_{[t-\tau,t]} \ \text{ and} \\ |x(t_0)| + |u|_{[t_0-\tau,t_0]} \leq 2\max\{|x(t_0)|, |u|_{[t_0-\tau,t_0]}\}, \end{array} \tag{12}$$

the ISS condition (10) implies that

$$\begin{array}{l} \max\{|x(t)|, |u|_{[t-\tau,t]}\} \ \leq \\ \overline{\beta}\big(2\max\{|x(t_0)|, |u|_{[t_0-\tau,t_0]}\}, t - t_0\big) + \overline{\gamma}(|\delta|_{[t_0,t]}) \ . \end{array} \tag{13}$$

Also, to prove (10), it suffices to find functions $\beta_0 \in \mathcal{KL}$ and $\gamma_0 \in \mathcal{K}_\infty$ such that

$$\begin{array}{l} |(x(t), u(t))| \ \leq \\ \beta_0\big(|x(t_0)| + |u|_{[t_0-\tau,t_0]}, t - t_0\big) + \gamma_0(|\delta|_{[t_0,t]}) \end{array} \tag{14}$$

holds for all $t \geq t_0$ along all of the closed loop trajectories. To see why it suffices to prove (14), let us notice that if $t_0 \leq t-\tau$, then (14) gives $|u|_{[t-\tau,t]} \leq \ \beta_0\big(|x(t_0)| + |u|_{[t_0-\tau,t_0]}, t-\tau-t_0\big) + \gamma_0(|\delta|_{[t_0,t]})$, because β_0 is nonincreasing in its second argument. On the other hand, if t and t_0 are such that $t-\tau \leq t_0 \leq t$, then $|u|_{[t-\tau,t_0]} \leq |u|_{[t_0-\tau,t_0]}e^{\tau+t_0-t}$. Hence, if (14) is satisfied along all of the closed loop trajectories, then so is (10) with the choices $\overline{\beta}(s,r) = 2\beta_0(s, \max\{0, r-\tau\}) + se^{\tau-r}$ and $\overline{\gamma} = 2\gamma_0$. Therefore, throughout the sequel, we will just prove our ISS estimates with $|(x(t), u(t))|$ on the left side, because then we can easily convert the result into the actual ISS estimates (2) using the preceding argument.

B. Proof of Theorem 1

We define $\Lambda : \mathbb{R} \times C([-\tau, 0], \mathbb{R}^p) \to \mathbb{R}^n$ by

$$\Lambda(t, \phi) = \int_{-\tau}^{0}\lambda(t, m+\tau+t)N(m+\tau+t)\phi(m)\mathrm{d}m. \tag{15}$$

Then

$$\Lambda(t, u_t) = \int_{t-\tau}^{t}\lambda(t, m+\tau)N(m+\tau)u(m)\mathrm{d}m. \tag{16}$$

We will denote $\Lambda(t, u_t)$ simply by $\Lambda(t)$. A similar simplification of notation will be used for all operators encountered in the sequel. Then, for all $t \geq t_0$, we can use (4) to get

$$\begin{array}{rcl} \dot{\Lambda}(t) & = & \lambda(t, t+\tau)N(t+\tau)u(t) - N(t)u(t-\tau) \\ & & +\int_{t-\tau}^{t}\frac{\partial\lambda}{\partial t}(t, m+\tau)N(m+\tau)u(m)\mathrm{d}m \\ & = & M(t)\Lambda(t) + \lambda(t, t+\tau)N(t+\tau)u(t) \\ & & -N(t)u(t-\tau) \ . \end{array} \tag{17}$$

In terms of the new operator $\zeta(t) = x(t) + \Lambda(t)$, our feedback in (9) gives $u(t) = K(t)\zeta(t)$, so $\dot{\zeta}(t) = [M(t) + \lambda(t, t+\tau)N(t+\tau)K(t)]\zeta(t) + \delta(t)$ for all $t \geq t_0$. It follows from (8) that for all initial times $t_0 \geq 0$ and all $t \geq t_0$, the inequality

$$|\zeta(t)| \ \leq \ \beta(|\zeta(t_0)|, t - t_0) + \gamma\left(|\delta|_{[t_0,t]}\right) \tag{18}$$

holds. Hence, our bound $|K|_\infty$ on K from Assumption 1 gives

$$|u(t)| \ \leq \ |K|_\infty\beta(|\zeta(t_0)|, t - t_0) + |K|_\infty\gamma\left(|\delta|_{[t_0,t]}\right) \tag{19}$$

for all $t \geq t_0$ and all initial times $t_0 \geq 0$. Next, we distinguish between two cases:

<u>First case:</u> $t_0 \leq t \leq t_0+\tau$. In this case, (3) and the choice $\hat{p} = \max\{|M|_\infty, |N|_\infty, 1\}$ give $|x(t)| \leq |x(t_0)| + \hat{p}\int_{t_0}^{t}\big[|x(\ell)| + |u(\ell-\tau)| + |\delta(\ell)|\big]\mathrm{d}\ell$. Hence, Gronwall's inequality gives

$$\begin{array}{l} |x(t)| \leq \hat{p}\tau e^{\hat{p}\tau}|\delta|_{[t_0,t]} \\ +(1+\hat{p}\tau)e^{2\hat{p}\tau}(|x(t_0)| + |u|_{[t_0-\tau,t_0]})e^{\hat{p}(t_0-t)} \end{array} \tag{20}$$

when $t_0 \le t \le t_0 + \tau$, since $e^{\hat{p}\tau} \le e^{2\hat{p}\tau} e^{\hat{p}(t_0 - t)}$.

Second case: $t \ge t_0 + \tau$. In that case, (5), our choice of u, and (18) combine to give

$$\begin{aligned}|x(t)| \;&\le\; |\Lambda(t)| + |\zeta(t)| \;\le\; (1 + c\tau|K|_\infty)|\zeta|_{[t-\tau,t]} \\ &\le\; (1 + c\tau|K|_\infty) \\ &\quad \times\big(\beta(|\zeta(t_0)|, \max\{0, t-\tau-t_0\}) \\ &\quad + \gamma(|\delta|_{[t_0,t]})\big) \;.\end{aligned} \tag{21}$$

The ISS estimate (14) then follows by adding the estimates (19), (20), and (21) and then using the fact that $|\zeta(t_0)| \le |x(t_0)| + \tau c|u|_{[t_0-\tau,t_0]}$ for all $t_0 \ge 0$ to upper bound the first argument of β in (19) and (21). The result now follows from the argument in Remark 2.

IV. General Stabilization Result

A. Assumptions and Statement of Result

To state our general ISS results for

$$\dot{x}(t) = F(t)x(t) + G(t)u(t-\tau) + \delta(t) \;, \tag{22}$$

without assuming that the fundamental matrix is known, we assume that the functions F and G are continuous, that F is periodic of some period $\overline{T} > 0$, and that G is bounded (but not necessarily periodic). The periodicity of F is the key ingredient allowing us to get closed form expressions for the control u that do not use the fundamental matrix. In parallel to (9), the control will be the sum of a linear control, plus an integral term involving τ; see (31). We introduce the notation

$$\begin{aligned}&M_F = \tfrac{1}{\overline{T}} \int_0^{\overline{T}} F(\ell)\mathrm{d}\ell \quad \text{and} \\ &\mathcal{F}(t) = \tfrac{1}{\overline{T}} \int_{t-\overline{T}}^{t} \left(\int_m^t F(\ell)\mathrm{d}\ell\right) \mathrm{d}m - L^0,\end{aligned} \tag{23}$$

where $L^0 \in \mathbb{R}^{n\times n}$ is chosen as follows. For all i and j in $\{1, 2, \ldots, n\}$, let $\varphi^\sharp_{i,j}$ (resp., $\varphi^\flat_{i,j}$) denote the maximum (resp., minimum) of

$$\tfrac{1}{\overline{T}} \int_{t-\overline{T}}^{t} \left(\int_m^t F_{i,j}(\ell)\mathrm{d}\ell\right) \mathrm{d}m$$

over all t. These extrema exist because F is continuous and periodic. Then we pick the (i, j) entry of L^0 to be $\frac{1}{2}(\varphi^\sharp_{i,j} + \varphi^\flat_{i,j})$ for all i and j. This choice of L^0 minimizes $|\mathcal{F}_{i,j}|_\infty$ for all i and j, and it ensures that $|\mathcal{F}|_\infty \to 0$ as $\overline{T} \to 0^+$; see Remark 4 below. Note that $\dot{\mathcal{F}}(t) = F(t) - M_F$ (by the periodicity of F), and

$$|M_F| \le |F|_\infty \;. \tag{24}$$

We introduce two assumptions:

Assumption 2: There exist a bounded continuous function $K : \mathbb{R} \to \mathbb{R}^{p\times n}$ and a C^1 function $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ such that the time derivative of

$$Q(t, z) = z^\top P(t)z \tag{25}$$

along all trajectories of $\dot{z}(t) = H(t)z(t)$, where $H(t) = F(t) + e^{-M_F\tau}G(t + \tau)K(t)$ satisfies

$$\dot{Q}(t) \le -|z(t)|^2. \tag{26}$$

Also, there are positive constants p_* and p_s such that $|P(t)| \le p_*$ and

$$p_s I_n \le P(t) \le p_* I_n \tag{27}$$

hold for all $t \in \mathbb{R}$. □

Assumption 3: The inequalities

$$|\mathcal{F}|_\infty|K|_\infty p_* e^{|F|_\infty\tau}|G|_\infty \;\le\; \frac{1}{16} \;, \tag{28}$$

$$|G|_\infty|\mathcal{F}|_\infty|K|_\infty e^{(|F|_\infty+1)\tau}\sqrt{\tau} \;\le\; \frac{1}{\sqrt{2}} \;, \text{ and} \tag{29}$$

$$|\mathcal{F}|_\infty|K|_\infty p_*|G|_\infty e^\tau \max\left\{1, J_* e^{|F|_\infty\tau}\sqrt{\tau}\right\} \;\le\; 0.19 \tag{30}$$

hold, where $J_* = 2|F|_\infty + e^{|F|_\infty\tau}|G|_\infty|K|_\infty(1 + |\mathcal{F}|_\infty)$. □

The choices of (28)-(30) will become clear in our proof of:

Theorem 2: If Assumptions 2-3 hold, then (22) in closed loop with the control

$$\begin{aligned}&u(t) = \\ &K(t)\left[x(t) + \int_{t-\tau}^t e^{M_F(t-m-\tau)}G(m + \tau)u(m)\mathrm{d}m\right]\end{aligned} \tag{31}$$

defined for all $t \ge t_0$ is exponentially ISS. □

Remark 3: A key point in Assumption 2 is that it pertains to a system with no delay, and implies that $P(t)$ is positive definite for all t. See Section IV-C for a detailed discussion on how the control u can be computed so that the closed loop system has a well defined flow map. □

Remark 4: Our choice of the constant matrix L^0 in the formula (23) for $\mathcal{F}$ gives $|\mathcal{F}_{i,j}|_\infty = \frac{1}{2}(\varphi^\sharp_{i,j} - \varphi^\flat_{i,j})$ for all i and j. It follows that for all pairs (i, j), we have the following:

$$L^0_{i,j} = \operatorname{argmin}_{r\in\mathbb{R}} \left|\tfrac{1}{\overline{T}}\int_{t-\overline{T}}^t \left(\int_m^t F_{i,j}(\ell)\mathrm{d}\ell\right)\mathrm{d}m - r\right|_\infty \tag{32}$$

To see why $L^0_{i,j}$ solves the preceding minimization problem for all i and j, notice that if the (i, j) entry of some constant matrix $\mathcal{M} \in \mathbb{R}^{n\times n}$ takes some value $c_* > \frac{1}{2}(\varphi^\sharp_{i,j} + \varphi^\flat_{i,j})$, then

$$\begin{aligned}&\left|\tfrac{1}{\overline{T}} \int_{t-\overline{T}}^t \left(\int_m^t F_{i,j}(\ell)\mathrm{d}\ell\right)\mathrm{d}m - \mathcal{M}_{i,j}\right|_\infty \\ &\ge \left|\varphi^\flat_{i,j} - c_*\right| \\ &> \tfrac{1}{2}(\varphi^\sharp_{i,j} + \varphi^\flat_{i,j}) - \varphi^\flat_{i,j} \;=\; |\mathcal{F}_{i,j}|_\infty.\end{aligned} \tag{33}$$

A similar argument applies if $c_* < \frac{1}{2}(\varphi^\sharp_{i,j} + \varphi^\flat_{i,j})$, by replacing $|\varphi^\flat_{i,j} - c_*|$ by $|\varphi^\sharp_{i,j} - c_*|$ in the preceding argument. Also, $|\mathcal{F}|_\infty \le \overline{T}|F|_\infty \to 0$ as $\overline{T} \to 0^+$, so Assumption 3 is satisfied when the period $\overline{T}$ is small enough. □

Remark 5: Our proof of Theorem 2 finds positive constants a_i such that for all $t_0 \ge 0$, all choices of δ, and all admissible initial functions $u : [t_0 - \tau, t_0] \to \mathbb{R}^p$ and $x : [t_0 - \tau, t_0] \to \mathbb{R}^n$, the closed loop trajectories all satisfy

$$\begin{aligned}&|x(t)| + |u|_{[t-\tau,t]} \;\le \\ &a_1|(x, u)|_{[t_0-\tau,t_0]} e^{a_2(t_0-t)} + a_3|\delta|_{[t_0,t]}\end{aligned} \tag{34}$$

for all $t \ge t_0$. □

B. Proof of Theorem 2

We use the operator $\Theta(t) = x(t) + \kappa(t)$, where

$$\kappa(t) = \int_{t-\tau}^t e^{M_F(t-m-\tau)}G(m + \tau)u(m)\mathrm{d}m \;. \tag{35}$$

It satisfies

$$\dot{\kappa}(t) = M_F\kappa(t) + e^{-M_F\tau}G(t+\tau)u(t) - G(t)u(t-\tau) \;. \tag{36}$$

Recalling the dynamic (22) for x, we get $\dot{\Theta}(t) = F(t)x(t) + M_F\kappa(t) + e^{-M_F\tau}G(t+\tau)u(t) + \delta(t)$. Consequently, our choice $u(t) = K(t)\Theta(t)$ from (31) gives

$$\begin{aligned}\dot{\Theta}(t) &= \left[F(t) + e^{-M_F\tau}G(t + \tau)K(t)\right]\Theta(t) \\ &\quad + (M_F - F(t))\kappa(t) + \delta(t).\end{aligned}$$

From (23) and the definition of H in Assumption 2, we deduce that $\dot{\Theta}(t) = H(t)\Theta(t) - \dot{\mathcal{F}}(t)\kappa(t) + \delta(t)$.

This leads us to consider the new operator $\rho : \mathbb{R} \times C([-\tau, 0], \mathbb{R}^{n+p}) \to \mathbb{R}^n$ defined by $\rho(t) = \Theta(t) + \mathcal{F}(t)\kappa(t)$, where we continue our convention of omitting arguments of functions for brevity when they are clear from the context. Then the choice (35) gives $\dot{\rho}(t) = H(t)\Theta(t) + \mathcal{F}(t)\dot{\kappa}(t) + \delta(t)$. From (36), our choice of $u(t)$, and the definition of ρ, we get

$$\begin{array}{rcl} \dot{\rho}(t) & = & H(t)\rho(t) + \big[- H(t)\mathcal{F}(t) + \mathcal{F}(t)M_F\big]\kappa(t) \\ & & + \mathcal{F}(t)\big[e^{-M_F\tau}G(t+\tau)K(t)\Theta(t) \\ & & \quad -G(t)u(t-\tau)\big] + \delta(t) \\ & = & \big[H(t) + \mathcal{K}_0(t)\big]\rho(t) + \mathcal{L}(t)\kappa(t) \\ & & - \mathcal{F}(t)G(t)u(t-\tau) + \delta(t), \end{array} \tag{37}$$

for all $t \geq 0$, where $\mathcal{K}_0(t) = \mathcal{F}(t)e^{-M_F\tau}G(t+\tau)K(t)$ and

$$\begin{array}{rcl} \mathcal{L}(t) & = & -H(t)\mathcal{F}(t) + \mathcal{F}(t)M_F \\ & & -\mathcal{F}(t)e^{-M_F\tau}G(t+\tau)K(t)\mathcal{F}(t). \end{array}$$

From Assumption 2, we deduce that the time derivative of (25) along all trajectories of (37) satisfies

$$\begin{array}{rcl} \dot{Q}(t) & \leq & -|\rho(t)|^2 + 2\rho(t)^\top P(t)\big[\mathcal{K}_0(t)\rho(t) \\ & & + \mathcal{L}(t)\kappa(t) - \mathcal{F}(t)G(t)u(t-\tau)\big] \\ & & + 2\rho(t)^\top P(t)\delta(t) \; . \end{array} \tag{38}$$

Then we can use (24) and (28) to prove that $2|P(t)||\mathcal{K}_0(t)| \leq 2p_*|\mathcal{F}|_\infty e^{|F|_\infty\tau}|G|_\infty|K|_\infty \leq \frac{1}{8}$, so (38) gives

$$\begin{array}{rcl} \dot{Q}(t) & \leq & -\frac{7}{8}|\rho(t)|^2 + \{|\rho(t)|\}\{2|P(t)\big[\mathcal{L}(t)\kappa(t) \\ & & \quad - \mathcal{F}(t)G(t)u(t-\tau)\big]|\} \\ & & + 2\rho(t)^\top P(t)\delta(t). \end{array} \tag{39}$$

Hölder's inequality $ab \leq \frac{1}{4}a^2 + b^2$ where a and b are the terms in braces in (39) gives the following:

$$\begin{array}{rcl} \dot{Q}(t) & \leq & -\frac{5}{8}|\rho(t)|^2 + 2\rho(t)^\top P(t)\delta(t) \\ & & + 4\big|P(t)\big[\mathcal{L}(t)\kappa(t) - \mathcal{F}(t)G(t)u(t-\tau)\big]\big|^2. \end{array} \tag{40}$$

Using the bound p_* on P and the Cauchy inequality gives

$$\begin{array}{rcl} \dot{Q}(t) & \leq & -\frac{5}{8}|\rho(t)|^2 + 8p_*^2\big[|\mathcal{L}(t)\kappa(t)|^2 \\ & & + |\mathcal{F}(t)G(t)|^2|u(t-\tau)|^2\big] \\ & & + 2\rho(t)^\top P(t)\delta(t) \; . \end{array} \tag{41}$$

Using (24), we can easily prove that

$$\begin{array}{l} |\mathcal{L}(t)| \leq |\mathcal{F}|_\infty\big(2|F|_\infty + e^{|F|_\infty\tau}|G|_\infty|K|_\infty(1 + |\mathcal{F}|_\infty)\big) \\ \text{and } \; |\kappa(t)| \leq e^{|F|_\infty\tau}|G|_\infty \int_{t-\tau}^t |u(m)|\mathrm{d}m. \end{array}$$

Hence, our choice $J_* = 2|F|_\infty + e^{|F|_\infty\tau}|G|_\infty|K|_\infty(1+|\mathcal{F}|_\infty)$ from Assumption 3 gives $|\mathcal{L}(t)| \leq |\mathcal{F}|_\infty J_*$. Since Jensen's inequality gives

$$\left(\int_{t-\tau}^t |u(m)|\mathrm{d}m\right)^2 \; \leq \; \tau \int_{t-\tau}^t |u(m)^2|\mathrm{d}m \; , \tag{42}$$

we deduce that

$$\begin{array}{rcl} \dot{Q}(t) & \leq & -\frac{5}{8}|\rho(t)|^2 + 8|\mathcal{F}|_\infty^2 p_*^2|G|_\infty^2 \\ & & \times \Big[J_*^2 e^{2|F|_\infty\tau}\tau \int_{t-\tau}^t |u(m)|^2\mathrm{d}m \\ & & + |u(t-\tau)|^2\Big] + 2\rho(t)^\top P(t)\delta(t). \end{array} \tag{43}$$

Next, observe that our choice $\rho(t) = \Theta(t) + \mathcal{F}(t)\kappa(t)$ gives $u(t) = K(t)\Theta(t) = K(t)[\rho(t) - \mathcal{F}(t)\kappa(t)]$, so if we set

$$\zeta(t) = \int_{t-\tau}^t e^{2(m-t+\tau)}|u(m)|^2\mathrm{d}m \; , \tag{44}$$

then we can again use Jensen's inequality from (42) to get

$$\begin{array}{rcl} \dot{\zeta}(t) & = & -2\int_{t-\tau}^t e^{2(m-t+\tau)}|u(m)|^2\mathrm{d}m \\ & & + e^{2\tau}|K(t)(\rho(t) - \mathcal{F}(t)\kappa(t))|^2 \\ & & - |u(t-\tau)|^2 \\ & \leq & -2\int_{t-\tau}^t |u(m)|^2\mathrm{d}m + 2e^{2\tau}|K|_\infty^2|\rho(t)|^2 \\ & & + 2e^{2\tau}|K|_\infty^2|\mathcal{F}|_\infty^2|\kappa(t)|^2 - |u(t-\tau)|^2 \\ & \leq & 2\big[e^{2\tau}|\mathcal{F}|_\infty^2|K|_\infty^2 e^{2|F|_\infty\tau}|G|_\infty^2\tau \; - 1\big] \\ & & \quad \times \int_{t-\tau}^t |u(m)|^2\mathrm{d}m \\ & & + 2e^{2\tau}|K|_\infty^2|\rho(t)|^2 - |u(t-\tau)|^2. \end{array} \tag{45}$$

We deduce from (29) that

$$\begin{array}{rcl} \dot{\zeta}(t) & \leq & -\int_{t-\tau}^t |u(m)|^2\mathrm{d}m \\ & & + 2e^{2\tau}|K|_\infty^2|\rho(t)|^2 - |u(t-\tau)|^2 \; . \end{array} \tag{46}$$

Then we set

$$V(t, (\rho, \zeta)(t)) = Q(t, \rho(t)) + 8p_*^2|\mathcal{F}|_\infty^2|G|_\infty^2 s\zeta(t) \; , \tag{47}$$

where the constant $s > 0$ will be selected later. It follows that

$$\begin{array}{rcl} \dot{V}(t) & \leq & \big[-\frac{5}{8} + 16sp_*^2|\mathcal{F}|_\infty^2|G|_\infty^2 e^{2\tau}|K|_\infty^2\big]\,|\rho(t)|^2 \\ & & + 8p_*^2|G|_\infty^2|\mathcal{F}|_\infty^2\,\big[J_*^2 e^{2|F|_\infty\tau}\tau - s\big] \\ & & \quad \times \int_{t-\tau}^t |u(m)|^2\mathrm{d}m \\ & & + 8p_*^2|\mathcal{F}|_\infty^2|G|_\infty^2\,(-s+1)\,|u(t-\tau)|^2 \\ & & + 2\rho(t)^\top P(t)\delta(t) \; . \end{array} \tag{48}$$

Let $s = \max\big\{1, \frac{21}{20}J_*^2 e^{2|F|_\infty\tau}\tau\big\}$. Then

$$\begin{array}{rcl} \dot{V}(t) & \leq & \big[-\frac{5}{8} + \frac{84}{5}\max\big\{1, J_*^2 e^{2|F|_\infty\tau}\tau\big\} \\ & & \quad \times p_*^2|\mathcal{F}|_\infty^2|G|_\infty^2 e^{2\tau}|K|_\infty^2\big]|\rho(t)|^2 \\ & & - \frac{2}{5}p_*^2|G|_\infty^2|\mathcal{F}|_\infty^2 J_*^2 e^{2|F|_\infty\tau}\tau \\ & & \quad \times \int_{t-\tau}^t |u(m)|^2\mathrm{d}m \\ & & + 2\rho(t)^\top P(t)\delta(t) \; . \end{array} \tag{49}$$

Hence, (30) gives

$$\begin{array}{l} \dot{V}(t) \; \leq \; -\frac{1}{100}|\rho(t)|^2 \\ -\frac{2}{5}p_*^2|G|_\infty^2|\mathcal{F}|_\infty^2 J_*^2 e^{2|F|_\infty\tau}\tau \int_{t-\tau}^t |u(m)|^2\mathrm{d}m \\ + 2\rho(t)^\top P(t)\delta(t) \; . \end{array} \tag{50}$$

Using our bounds for P from Assumption 2 and the triangle inequality, we find constants $b_1 > 0$ and $b_2 > 0$ such that

$$\dot{V}(t) \; \leq \; -b_1 V(t, (\rho, \zeta)(t)) + b_2|\delta(t)|^2 \tag{51}$$

along all of the closed loop trajectories. This gives the exponential ISS estimate (34). See Appendix A.1 for more details.

C. Computing the Closed Loop Trajectories for (22)

Computing the closed loop control values u from (31) is nontrivial because u is defined in terms of an integral equation involving the state, and because one cannot change the initial control values after the initial time $t_0 \geq 0$. See [32] for methods for computing other controllers based on reduction. Here we show how the controller (31) and the closed loop trajectories of (22) can be computed recursively on $[t_0 - \tau, +\infty)$ for any initial time $t_0 \geq 0$, any constant delay $\tau > 0$, any initial function $x : [t_0 - \tau, t_0] \to \mathbb{R}^n$ for the state, and any continuous initial function $u : [t_0 - \tau, t_0] \to \mathbb{R}^p$ for our controller u for which the matching condition

$$u(t_0) = K(t_0)\,[x(t_0) + \int_{t_0-\tau}^{t_0} e^{M_F(t_0-m-\tau)}G(m+\tau)u(m)\mathrm{d}m\Big] \tag{52}$$

is satisfied. While nonessential from the dynamical system standpoint, an interesting mathematical feature of this result is that it gives a continuous control function u and that it stipulates an admissibility condition on the initial input. See also [33] for an analogous construction involving differentiating a distributed delay term. However, [33] deals with linear time-invariant systems where the problem of finding the fundamental matrix does not arise. To handle our time-varying system, we first solve for $x(t)$ on $[t_0, t_0+\tau]$ using (22) and the initial function for $u : [t_0-\tau, t_0] \to \mathbb{R}^p$, and then we solve the initial value problem

$$\begin{array}{rcl} \dot{\mathcal{R}}(t) & = & M_F\mathcal{R}(t) - G(t)u(t-\tau) \\ & & + e^{-M_F\tau}G(t+\tau)K(t)[x(t)+\mathcal{R}(t)] \\ \mathcal{R}(t_0) & = & \int_{t_0-\tau}^{t_0} e^{M_F(t_0-m-\tau)}G(m+\tau)u(m)\mathrm{d}m \end{array} \tag{53}$$

on $[t_0, t_0+\tau]$. Then we define u on $[t_0, t_0+\tau]$ by the formula $u(t) = K(t)[x(t)+\mathcal{R}(t)]$. Then u is continuous on $[t_0-\tau, t_0+\tau]$, using (52) and the continuity of the initial function for u. By the uniqueness of the solution of the initial value problem (53), our formula for $\kappa(t_0)$ obtained from (35), and our formula (36) for $\dot{\kappa}$, it follows that $\mathcal{R}(t)$ agrees with the function $\kappa(t)$ we defined in (35) on the interval $[t_0, t_0+\tau]$, so $u(t) = K(t)[x(t)+\mathcal{R}(t)]$ agrees with (31) on $[t_0, t_0+\tau]$. We can repeat this process on the intervals $[t_0+\ell\tau, t_0+(\ell+1)\tau]$ for all $\ell \in \mathbb{N}$, and this gives the continuous controller and the closed loop trajectories.

Given any initial state $x(t_0)$, we can argue as in [34] to find initial functions for u that satisfy (52). To see how, first pick any constant $h > 0$ such that $e^{\tau|F|_\infty}|K|_\infty|G|_\infty < h$. Then

$$\mathcal{G} = I_p - \int_{t_0-\tau}^{t_0} K(t_0)e^{M_F(t_0-m-\tau)}G(m+\tau)e^{-h(t_0-m)}\mathrm{d}m \tag{54}$$

is invertible, because the norm of the integral in (54) is strictly less than 1, and we can bound $\mathcal{G}^{-1}$ by a function of h and τ; see Lemma A.1 below. Then $u : [t_0-\tau, t_0] \to \mathbb{R}^p$ defined by $u(t) = \mathcal{G}^{-1}e^{-h(t_0-t)}K(t_0)x(t_0)$ satisfies (52), and we can find a constant $\bar{c}$ that is independent of t_0 such that $|(x,u)|_{[t_0-\tau,t_0]} \le \bar{c}|x|_{[t_0-\tau,t_0]}$. This allows us to convert our ISS estimates into standard ISS estimates, where the control u does not appear.

Remark 6: By applying Gronwall's inequality to (53) on $[t_0, t_0+\tau]$, we can find constants $\tilde{c}_i > 0$ such that $|\mathcal{R}(t)| \le \tilde{c}_1(|x|_{[t_0,t_0+\tau]} + |u|_{[t_0-\tau,t_0]})$ holds for all $t \in [t_0, t_0+\tau]$, and therefore also $|\mathcal{R}(t)| \le \tilde{c}_2(|(x,u)|_{[t_0-\tau,t_0]} + |\delta|_{[t_0,t]})$ for all $t \in [t_0, t_0+\tau]$, by using the linear growth of the original dynamics for x. Here and in the sequel, the constants $\tilde{c}_i$ are independent of t_0 and the choice of the trajectory of (53). Then the formula $u(t) = K(t)[x(t)+\mathcal{R}(t)]$ and the boundedness of K provide a constant $\tilde{c}_3 > 0$ such that

$$|(x,u)|_{[t_0,t_0+\tau]} \;\le\; \tilde{c}_3\big(|(x,u)|_{[t_0-\tau,t_0]} + |\delta|_{[t_0,t]}\big). \tag{55}$$

We will use analogous arguments in our treatment of slowly time-varying systems. □

V. Special Case of Rapidly Time-Varying Periodic Systems

A. Assumptions and Theorem

In this section, we exhibit a family of systems of the form

$$\dot{x}(t) = A(\omega t)x(t) + Bu(t-\tau) + \delta(t) \tag{56}$$

that satisfy Assumptions 2-3, where x is valued in $\mathbb{R}^n$, the control u is valued in $\mathbb{R}^p$ and is to be specified, $\tau \ge 0$ is a known constant delay, $\omega \ge 0$ is a known constant (called the rapidness parameter), B is any constant $n \times p$ matrix, and A is a C^0 function having some period $T > 0$. Our result for the rapidly time-varying system (56) will follow from Theorem 2, by taking $F(t) = A(\omega t)$. In the context of Theorem 2, F is assumed to have some period $\overline{T}$, while A in this section has period T. On the other hand, the vector field $F(t) = A(\omega t)$ in (56) has period $\overline{T} = T/\omega$, which motivates our use of both $\overline{T}$ in the preceding section and T in this section. We will design a dynamic feedback control u for (56) and find a lower bound on the admissible rapidness parameters ω that ensure ISS for (56). To this end, we set $M_A = \int_0^T A(\ell)\mathrm{d}\ell/T$, and we use the constant $d_* = |A - M_A|_\infty$. We assume:

Assumption 4: The pair (M_A, B) is stabilizable. □

Assumption 4 provides a constant matrix $K \in \mathbb{R}^{p\times n}$ such that the matrix $H_0 = M_A + e^{-M_A\tau}BK$ is Hurwitz, and a constant symmetric positive definite matrix S such that

$$SH_0 + H_0^\top S \le -2I_n\ . \tag{57}$$

We also use the following function (but see Remark 8 for generalizations that are analogous to the norm minimizing matrix L^0 we used in Section IV-A):

$$\mathcal{A}_\omega(t) = \frac{\omega}{T}\int_{t-\frac{T}{\omega}}^t \left(\int_m^t [A(\omega\ell) - M_A]\mathrm{d}\ell\right)\mathrm{d}m\ . \tag{58}$$

Notice for later use that $\dot{\mathcal{A}}_\omega(t) = A(\omega t) - M_A$,

$$|\dot{\mathcal{A}}_\omega|_\infty \le d_*\ ,\ \text{and}\ |\mathcal{A}_\omega|_\infty \le \frac{T}{2\omega}d_*\ . \tag{59}$$

We also assume:

Assumption 5: The inequalities

$$\frac{Td_*}{\omega} \le 1\ , \tag{60}$$

$$2|S|\big[2|A|_\infty + d_* + 2e^{|A|_\infty\tau}|B||K|\big]\frac{\frac{T}{2\omega}d_*}{1-\frac{T}{2\omega}d_*} \le 1\ , \tag{61}$$

$$\frac{Td_*}{2\omega}|K|\overline{p}e^{|A|_\infty\tau}|B| \;\le\; \frac{1}{16}\ , \tag{62}$$

$$\frac{Td_*}{2\omega}|B||K|e^{(|A|_\infty+1)\tau}\sqrt{\tau} \;\le\; \frac{1}{\sqrt{2}}\ ,\ \text{and} \tag{63}$$

$$\frac{Td_*}{2\omega}|K|\max\big\{1, \big[2|A|_\infty + e^{|A|_\infty\tau}|B||K|\left(1+\tfrac{Td_*}{2\omega}\right)\big] \times e^{|A|_\infty\tau}\sqrt{\tau}\big\}\,\overline{p}|B|e^\tau \;\le\; 0.19 \tag{64}$$

hold, where $\overline{p} = |S|(1 - Td_*/(2\omega))^{-2}(1 + Td_*/(2\omega))^2$. □

Notice that Assumption 5 holds when ω is large enough, or if T is small enough. The choices of the bounds in Assumption 5 will become clear when we prove:

Theorem 3: Let Assumptions 4-5 hold. Then (56) in closed loop with the control

$$u(t) = K\left[x(t) + \int_{t-\tau}^t e^{M_A(t-m-\tau)}Bu(m)\mathrm{d}m\right] \tag{65}$$

defined for all $t \ge t_0$ is exponentially ISS. □

Remark 7: Our proof of Theorem 3 will provide constants $c_i > 0$ for $i = 1, 2, 3$ such that for all $t_0 \geq 0$ and $t \geq t_0$, all of the corresponding closed loop trajectories satisfy

$$\begin{array}{l} |x(t)| + |u|_{[t-\tau,t]} \leq \\ c_1|(x,u)|_{[t_0-\tau,t_0]}e^{-c_2(t-t_0)} + c_3|\delta|_{[t_0,t]}. \end{array} \tag{66}$$

The proof that u and state can be computed on $[t_0, +\infty)$ under matching conditions is as in Section IV-C. □

Remark 8: We can extend Theorem 3 to systems of the type $\dot{x}(t) = A(\omega t)x(t) + F(\theta t)u(t-\tau)$, when ω and θ are large enough constants. If A and B are constant, then (65) is the reduction control from [1]. Reasoning as we did in Section IV-A, we can replace $\mathcal{A}_\omega(t)$ by

$$\frac{\omega}{T}\int_{t-\frac{T}{\omega}}^{t}\left(\int_m^t A(\omega\ell)\mathrm{d}\ell\right)\mathrm{d}m - L^0, \tag{67}$$

where L^0 is an optimizing $n \times n$ matrix; see Remark 4. For simplicity, we used the simpler form of $\mathcal{A}_\omega$ from (58) involving the average M_A of A. □

B. Proof of Theorem 3

The proof entails showing that if Assumptions 4-5 hold, then Assumptions 2-3 are satisfied with the choices $F(t) = A(\omega t)$ and $G(t) = B$ and the period $\overline{T} = T/\omega$ for F. The result then follows from Theorem 2 and Remark 5, because (65) agrees with the control (31) from Theorem 2. In the notation of Section IV-A, we choose $K(t) = K$ and $H(t) = A(\omega t) + e^{-M_A\tau}BK = H_0 + \dot{\mathcal{A}}_\omega(t)$. Consider the system $\dot{z}(t) = H(t)z(t)$, and set $J_\omega(t) = I_n - \mathcal{A}_\omega(t)$. Then the function $y(t) = J_\omega(t)z(t)$ satisfies

$$\begin{array}{rcl} \dot{y}(t) &=& J_\omega(t)\big[H_0 + \dot{\mathcal{A}}_\omega(t)\big]z(t) - \dot{\mathcal{A}}_\omega(t)z(t) \\ &=& H_0z(t) - \mathcal{A}_\omega(t)\big[H_0 + \dot{\mathcal{A}}_\omega(t)\big]z(t)\ . \end{array}$$

From (59)-(60), we have $|\mathcal{A}_\omega(t)| \leq 1/2$ for all t. Hence, Lemma A.1 implies that $J_\omega(t)$ is invertible for all t, and then

$$\begin{array}{l} \dot{y}(t) = \\ H_0J_\omega^{-1}(t)y(t) - \mathcal{A}_\omega(t)\big[H_0 + \dot{\mathcal{A}}_\omega(t)\big]J_\omega^{-1}(t)y(t)\ . \end{array} \tag{68}$$

Let $W(y) = y^\top Sy$. Adding and subtracting $H_0y(t)$ on the right side of (68) and using (57) gives

$$\begin{array}{rcl} \dot{W} &\leq& -2|y(t)|^2 + 2y^\top(t)SH_0\big[J_\omega^{-1}(t) - I_n\big]y(t) \\ && -2y^\top(t)S\mathcal{A}_\omega(t)\big(H_0 + \dot{\mathcal{A}}_\omega(t)\big)J_\omega^{-1}(t)y(t)\ . \end{array} \tag{69}$$

By Lemma A.1, we have $|J_\omega^{-1}(t) - I_n| \leq |\mathcal{A}_\omega(t)|/(1 - |\mathcal{A}_\omega(t)|)$ and $\left|J_\omega^{-1}(t)\right| \leq 1/(1 - |\mathcal{A}_\omega(t)|)$ for all $t \in \mathbb{R}$. We deduce the following:

$$\begin{array}{l} \dot{W} \leq \\ -2|y(t)|^2 + 2|S|\left[2|H_0| + \left|\dot{\mathcal{A}}_\omega(t)\right|\right]\frac{|\mathcal{A}_\omega(t)|}{1-|\mathcal{A}_\omega(t)|}|y(t)|^2. \end{array} \tag{70}$$

Since $|H_0| \leq |A|_\infty + e^{|A|_\infty\tau}|B||K|$ and $|z(t)| = |J_\omega^{-1}(t)y(t)| \leq |y(t)|/(1-|\mathcal{A}_\omega(t)|)$, we can use (59) to obtain

$$\begin{array}{rcl} \dot{W} &\leq& -2|y(t)|^2 + 2|S|\big[2|H_0| + d_*\big]\frac{\frac{T}{2\omega}d_*}{1-\frac{T}{2\omega}d_*}|y(t)|^2 \\ &\leq& -|y(t)|^2 \ \leq\ -\left(1 - \frac{Td_*}{2\omega}\right)^2|z(t)|^2\ , \end{array} \tag{71}$$

where the last two inequalities followed from (60)-(61).

Hence, in terms of the function Q from Theorem 2, it follows that $\dot{Q} \leq -|z|^2$ holds along all trajectories of $\dot{z} = H(t)z$, when we set

$$Q(t,z) = \left(1 - \frac{Td_*}{2\omega}\right)^{-2}W\big(J_\omega(t)z\big) \ = z^\top P(t)z \tag{72}$$

and

$$P(t) = \left(1 - \frac{Td_*}{2\omega}\right)^{-2}[J_\omega(t)]^\top SJ_\omega(t). \tag{73}$$

Also, the second inequality in (59) gives

$$\begin{array}{rcl} |P(t)| &\leq& \left(1 - \frac{Td_*}{2\omega}\right)^{-2}|S||J_\omega(t)|^2 \\ &\leq& |S|\left(1 - \frac{Td_*}{2\omega}\right)^{-2}\left(1 + \frac{Td_*}{2\omega}\right)^2\ . \end{array} \tag{74}$$

Hence, using the choice (73), we can satisfy the upper bound requirement on P from Assumption 2 using the constant $p_* = |S|(1 - Td_*/(2\omega))^{-2}(1 + Td_*/(2\omega))^2$.

Let $s_m > 0$ be the smallest eigenvalue of S. Then, for all $z \in \mathbb{R}^n$ and $t \geq 0$, we get $s_m\left|[I_n - \mathcal{A}_\omega(t)]z\right|^2 \leq Q(t,z)$. We deduce that $s_m(|z| - |\mathcal{A}_\omega(t)||z|)^2 \leq Q(t,z)$. It follows from (59)-(60) that $s_m\left(|z| - \frac{1}{2}|z|\right)^2 \leq Q(t,z)$. Hence, we can satisfy the lower bound condition on P from Assumption 2 using $p_s = 0.25s_m$. Therefore, Assumption 2 is satisfied. Since the period of $A(\omega t)$ is $\overline{T} = \frac{T}{\omega}$, and since (62)-(64) in Assumption 5 imply that Assumption 3 holds with $p_* = \overline{p}$, the result now follows from Theorem 2.

VI. Special Case of Slowly Time-Varying Systems

A. Statement of Result and Discussion

As noted in [24], [26], stabilization and ISS methods for undelayed rapidly time-varying systems do not apply to slowly time-varying systems. The same is true in the input delayed case, so the method of the preceding section does not apply to slowly time-varying vector fields. See (71) above, where the final right side becomes positive when ω is small enough. On the other hand, slowly time-varying systems arise in important engineering applications [24], [26]. This motivates the next theorem. Slowly time-varying systems with delays were also studied in [35, Section 12.8], but our results are novel because of our ISS result coupled with our ability to find upper bounds on the admissible values of the slowness parameter.

Consider the system

$$\dot{x}(t) = A(\epsilon t)x(t) + F(\epsilon t)u(t-\tau) + \delta(t), \tag{75}$$

where x is valued in $\mathbb{R}^n$, u is valued in $\mathbb{R}^p$ and will be specified, $\tau > 0$ and $\epsilon > 0$ are known constants, A is C^1, and F is globally Lipschitz with a Lipschitz constant $\kappa_F > 0$. As before, δ is an unknown measurable essentially bounded perturbation representing modeling or control uncertainty, or approximation errors from linearization. We assume:

Assumption 6: There exist a continuous function $L : \mathbb{R} \rightarrow \mathbb{R}^{p\times n}$ and a C^1 function $Q : \mathbb{R} \rightarrow \mathbb{R}^{n\times n}$ such that $Q(p)$ is symmetric and positive definite for all $p \in \mathbb{R}$ and such that $H(p) = A(p) + F(p + \tau\epsilon)L(p)$ satisfies the inequality

$$Q(p)H(p) + H(p)^\top Q(p) \leq -2I_n \tag{76}$$

for all $p \in \mathbb{R}$. Also, the functions F and L are bounded. □

Our upper bound on ϵ is in terms of the constants from:

Assumption 7: There are constants $\bar{c}_j > 0$ such that $|A(p)| \leq \bar{c}_1$, $|F(p)L(q)| \leq \bar{c}_2$, $|Q(p)| \leq \bar{c}_3$, $|A'(p)| \leq \bar{c}_4$, and $|Q'(p)| \leq \bar{c}_5$ for all p and q in $\mathbb{R}$. Moreover, there exists a constant $q_c > 0$ such that $q_cI_n \leq Q(p)$ for all $p \in \mathbb{R}$. □

We emphasize that the derivatives in Assumption 7 are with respect to the arguments of the corresponding functions, which will be different from time t in general; see (87). Setting

$$\epsilon_* = \min\left\{\tfrac{1}{2[\bar{c}_5+2\tau\bar{c}_3\bar{c}_4e^{2\bar{c}_1\tau}]}, \tfrac{1}{2\tau^2\bar{c}_2\bar{c}_3\bar{c}_4e^{2\bar{c}_1\tau}}\right\}, \qquad (77)$$

we prove:

Theorem 4: Let $\tau > 0$ be any constant. If Assumptions 6-7 hold, then for each constant $\epsilon \in (0, \epsilon_*)$, the system (75) with the slowness parameter ϵ in closed loop with the control

$$\begin{aligned} u(t) = L(\epsilon t)e^{A(\epsilon t)\tau}\big[x(t) \\ + \textstyle\int_{t-\tau}^{t} e^{A(\epsilon m)(t-m-\tau)}F(\epsilon(m+\tau))u(m)\mathrm{d}m\big] \end{aligned} \qquad (78)$$

defined for all $t \geq t_0$ is exponentially ISS. □

Remark 9: As in the rapidly time-varying case, our proof of Theorem 4 provides constants c_i for $i = 1, 2, 3$ such that all of the corresponding closed loop trajectories satisfy

$$\begin{aligned} |x(t)| + |u|_{[t-\tau,t]} \leq \\ c_1|(x,u)|_{[t_0-\tau,t_0]}e^{-c_2(t-t_0)} + c_3|\delta|_{[t_0,t]}, \end{aligned} \qquad (79)$$

for all initial times $t_0 \geq 0$ and all $t \geq t_0$. □

Remark 10: Assumption 6 implies that for all $p \in \mathbb{R}$, the pair $(A(p), F(p+\tau\epsilon))$ is stabilizable. This is an analog of our Hurwitzness condition in the rapidly time-varying vector fields case. Assumptions 6-7 do not imply that A and F are periodic. Two novel features of our proof are (a) its use of a new Lyapunov-Krasovskii functional based on transforming a time-varying quadratic Lyapunov function for a suitable undelayed system and (b) our ability to find explicit upper bounds on the allowable values of ϵ, which were not previously available for input-delayed slowly time-varying systems. □

B. Proof of Theorem 4

Pick any initial time $t_0 \geq 0$ and any initial functions for x and u. We use the operator $z : \mathbb{R} \times C([-\tau, 0], \mathbb{R}^{n+p}) \to \mathbb{R}^n$ defined for all $t \geq t_0$ by $z(t) = x(t) + \Gamma(t)$, where

$$\Gamma(t) = \textstyle\int_{t-\tau}^{t} e^{A(\epsilon m)(t-m-\tau)}F(\epsilon(m+\tau))u(m)\mathrm{d}m\ . \qquad (80)$$

By differentiating, we get the closed loop coupled system

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & A(\epsilon t)x(t) + \delta(t) \\ & & + F(\epsilon t)L(\epsilon[t-\tau])e^{A(\epsilon[t-\tau])\tau}z(t-\tau) \\ \dot{z}(t) & = & e^{-A(\epsilon t)\tau}H(\epsilon t)e^{A(\epsilon t)\tau}z(t) + \overline{\rho}(\epsilon, t, z_t) \\ & & + \delta(t) \end{array}\right. \qquad (81)$$

defined for all $t \geq t_0 + \tau$, where

$$\begin{aligned} &\overline{\rho}(\epsilon, t, z_t) = \\ &\textstyle\int_{t-\tau}^{t}[A(\epsilon m) - A(\epsilon t)]e^{A(\epsilon m)(t-m-\tau)} \\ &\quad\times F(\epsilon(m+\tau))L(\epsilon m)e^{A(\epsilon m)\tau}z(m)\mathrm{d}m\ . \end{aligned} \qquad (82)$$

We next establish stability properties for the z dynamics

$$\dot{z}(t) = e^{-A(\epsilon t)\tau}H(\epsilon t)e^{A(\epsilon t)\tau}z(t) + \overline{\rho}(\epsilon, t, z_t) + \delta(t) \qquad (83)$$

for all $t \geq t_0 + \tau$. Take $V(t, z) = z^\top e^{A(\epsilon t)^\top\tau}Q(\epsilon t)e^{A(\epsilon t)\tau}z$. For all $t \geq t_0 + \tau$, its time derivative along (83) satisfies

$$\begin{aligned} &\tfrac{d}{dt}[V(t, z(t))] = \\ &2z(t)^\top e^{A(\epsilon t)^\top\tau}Q(\epsilon t)H(\epsilon t)e^{A(\epsilon t)\tau}z(t) \\ &+ 2z(t)^\top e^{A(\epsilon t)^\top\tau}Q(\epsilon t)e^{A(\epsilon t)\tau}\overline{\rho}(\epsilon, t, z_t) \\ &+ z(t)^\top e^{A(\epsilon t)^\top\tau}\beta(\epsilon t)e^{A(\epsilon t)\tau}z(t) \\ &+ 2z(t)^\top e^{A(\epsilon t)^\top\tau}Q(\epsilon t)e^{A(\epsilon t)\tau}\delta(t)\ , \end{aligned} \qquad (84)$$

with the following definition:

$$\beta(\epsilon t) = e^{-A(\epsilon t)^\top\tau}\tfrac{d}{dt}\left(e^{A(\epsilon t)^\top\tau}Q(\epsilon t)e^{A(\epsilon t)\tau}\right)e^{-A(\epsilon t)\tau} \qquad (85)$$

From Assumptions 6-7, we deduce that for all $t \geq t_0 + \tau$,

$$\begin{aligned} &\tfrac{d}{dt}[V(t, z(t))] \leq \\ &(-2 + |\beta(\epsilon t)|)|e^{A(\epsilon t)\tau}z(t)|^2 \\ &+ \left\{|e^{A(\epsilon t)\tau}z(t)|\right\}\left\{2\bar{c}_3e^{\bar{c}_1\tau}\left[|\overline{\rho}(\epsilon, t, z_t)| + \delta(t)\right]\right\} \\ &\leq \left(-\tfrac{3}{2} + |\beta(\epsilon t)|\right)|e^{A(\epsilon t)\tau}z(t)|^2 \\ &\quad + 4\bar{c}_3^2e^{2\bar{c}_1\tau}\left[|\overline{\rho}(\epsilon, t, z_t)|^2 + |\delta(t)|^2\right]\ , \end{aligned} \qquad (86)$$

where the last inequality followed from applying the triangle inequality to the terms in braces and the relation $(a+b)^2 \leq 2a^2 + 2b^2$, which holds for all real numbers a and b.

Next observe that

$$\begin{aligned} &\beta(\epsilon t) = \epsilon Q'(\epsilon t) + e^{-A(\epsilon t)^\top\tau}\tfrac{d}{dt}\left(e^{A(\epsilon t)^\top\tau}\right)Q(\epsilon t) \\ &+ Q(\epsilon t)\tfrac{d}{dt}\left(e^{A(\epsilon t)\tau}\right)e^{-A(\epsilon t)\tau}\ . \end{aligned} \qquad (87)$$

Furthermore, the function $\alpha_k(\epsilon t) = \frac{1}{k!}\frac{d}{dt}(A(\epsilon t)^k)$ satisfies $|\alpha_k(\epsilon t)| \leq \epsilon\bar{c}_1^{k-1}\bar{c}_4/(k-1)!$. It follows from differentiating the Maclaurin series for $e^{A(\epsilon t)\tau}$ that

$$\left|\frac{d(e^{A(\epsilon t)\tau})}{dt}\right| \leq \sum_{k=1}^{+\infty}\tau^k\epsilon\frac{\bar{c}_1^{k-1}\bar{c}_4}{(k-1)!} = \epsilon\tau\bar{c}_4e^{\bar{c}_1\tau}.$$

The Maclaurin series is needed to quantify the effects of the time variation in the function β from (85). This gives $|\beta(\epsilon t)| \leq \epsilon(\bar{c}_5 + 2\tau\bar{c}_3\bar{c}_4e^{2\bar{c}_1\tau})$. Next note that

$$\begin{aligned} &|\overline{\rho}(\epsilon, t, z_t)| \leq \\ &\textstyle\int_{t-\tau}^{t}|A(\epsilon m) - A(\epsilon t)|e^{|A(\epsilon m)|\tau} \\ &\quad\times|F(\epsilon(m+\tau))L(\epsilon m)|e^{A(\epsilon m)\tau}z(m)|\mathrm{d}m \\ &\leq \epsilon\tau\bar{c}_4e^{\bar{c}_1\tau}\bar{c}_2\textstyle\int_{t-\tau}^{t}|e^{A(\epsilon m)\tau}z(m)|\mathrm{d}m. \end{aligned} \qquad (88)$$

Combining (86) with (88), noting that our upper bound (77) on ϵ gives $|\beta(\epsilon t)| \leq \frac{1}{2}$, and using Jensen's Inequality gives the following for all $t \geq t_0 + \tau$:

$$\begin{array}{rcl} \tfrac{d}{dt}[V(t, z(t))] & \leq & -\left|e^{A(\epsilon t)\tau}z(t)\right|^2 \\ & & + 4\bar{c}_3^2e^{2\bar{c}_1\tau}|\overline{\rho}(\epsilon, t, z_t)|^2 \\ & & + 4\bar{c}_3^2e^{2\bar{c}_1\tau}|\delta(t)|^2 \\ & \leq & -\left|e^{A(\epsilon t)\tau}z(t)\right|^2 \\ & & + 4\tau^3(\bar{c}_2\bar{c}_3\bar{c}_4)^2e^{4\bar{c}_1\tau}\epsilon^2 \\ & & \times\int_{t-\tau}^{t}|e^{A(\epsilon m)\tau}z(m)|^2\mathrm{d}m \\ & & + 4\bar{c}_3^2e^{2\bar{c}_1\tau}|\delta(t)|^2\ . \end{array} \qquad (89)$$

We used Jensen's inequality to get

$$\begin{aligned} &\left(\textstyle\int_{t-\tau}^{t}|e^{A(\epsilon m)\tau}z(m)|\mathrm{d}m\right)^2 \leq \\ &\tau\textstyle\int_{t-\tau}^{t}|e^{A(\epsilon m)\tau}z(m)|^2\mathrm{d}m. \end{aligned} \qquad (90)$$

The second term in the min defining our upper bound on ϵ from (77) provides a constant $\mathcal{L} \in \left(4\tau^3(\bar{c}_2\bar{c}_3\bar{c}_4)^2e^{4\bar{c}_1\tau}\epsilon^2, \frac{1}{\tau}\right)$. Hence, the formula

$$\begin{aligned} &\tfrac{d}{dt}\textstyle\int_{t-\tau}^{t}\int_{\ell}^{t}|e^{A(\epsilon m)\tau}z(m)|^2\mathrm{d}m\mathrm{d}\ell = \\ &\tau|e^{A(\epsilon t)\tau}z(t)|^2 - \textstyle\int_{t-\tau}^{t}|e^{A(\epsilon m)\tau}z(m)|^2\mathrm{d}m \end{aligned} \qquad (91)$$

and the decay estimate (89) on V imply that the functional

$$\begin{aligned} &U(t, z_t) = \\ &V(t, z(t)) + \mathcal{L}\textstyle\int_{t-\tau}^{t}\int_{\ell}^{t}|e^{A(\epsilon m)\tau}z(m)|^2\mathrm{d}m\mathrm{d}\ell. \end{aligned} \qquad (92)$$

satisfies the following:

$$\begin{aligned} &\tfrac{d}{dt}[U(t, z_t)] \leq \\ &-(1-\mathcal{L}\tau)\left|e^{A(\epsilon t)\tau}z(t)\right|^2 + 4\bar{c}_3^2e^{2\bar{c}_1\tau}|\delta(t)|^2 \\ &+ \left(4\tau^3(\bar{c}_2\bar{c}_3\bar{c}_4)^2e^{4\bar{c}_1\tau}\epsilon^2 - \mathcal{L}\right)\textstyle\int_{t-\tau}^{t}|e^{A(\epsilon m)\tau}z(m)|^2\mathrm{d}m \end{aligned} \qquad (93)$$

We deduce that there are positive constants C_* and C_{**} such that for all $t \geq t_0 + \tau$, we have

$$\frac{d}{dt}[U(t, z_t)] \;\leq\; -C_* U(t, z_t) + C_{**}|\delta(t)|^2 \;\text{ and} \tag{94}$$

$$C_* \left|z(t)\right|^2 \;\leq\; U(t, z_t) \;\leq\; C_{**}|z|^2_{[t-\tau,t]} \tag{95}$$

along all trajectories of (83). Using (94)-(95) and our formulas for $z(t) = x(t) + \Gamma(t)$, Γ, and $u(t) = L(\epsilon t)e^{A(\epsilon t)\tau}z(t)$ gives the desired exponential ISS estimate; see Appendix A.3.

VII. Examples

A key contribution of our results is that they lead to the ISS decay estimates (10), (34), (66), and (79) with $|x(t)|+|u|_{[t-\tau,t]}$ on the left side. Proving these estimates requires checking several assumptions. In this section, we give examples that show how the assumptions of our theorems can be checked.

A. Scalar Example

Consider the one dimensional system

$$\begin{array}{l}\dot{x}(t) = \\ \big[\sin(t/9) + \sin(11\pi t)\big]x(t) + u(t-\tau) + \delta(t)\end{array} \tag{96}$$

with the disturbance δ. Although the vector fields are not periodic and the system is neither rapidly or slowly time-varying, the system is covered by Theorem 1. To see why, notice that with the notation of Section III, we have $M(t) = \sin(t/9) + \sin(11\pi t)$, $N = 1$,

$$\begin{array}{l}\lambda(t, t_0) = \\ e^{-9\cos(t/9) - \frac{1}{11\pi}\cos(11\pi t) + 9\cos(t_0/9) + \frac{1}{11\pi}\cos(11\pi t_0)}\,,\end{array} \tag{97}$$

and

$$\begin{array}{l}M(t) + \lambda(t, t+\tau)N(t+\tau)K(t) = \sin(t/9) + \sin(11\pi t) \\ +e^{-9\cos(t/9) - \frac{1}{11\pi}\cos(11\pi t) + 9\cos(\frac{t+\tau}{9}) + \frac{1}{11\pi}\cos(11\pi(t+\tau))}K(t)\,.\end{array}$$

Choosing

$$\begin{array}{l}K(t) = \\ e^{9\cos(t/9) + \frac{1}{11\pi}\cos(11\pi t) - 9\cos(\frac{t+\tau}{9}) - \frac{1}{11\pi}\cos(11\pi(t+\tau))} \\ \times\big[-1 - \sin(t/9) - \sin(11\pi t)\big]\end{array} \tag{98}$$

gives

$$M(t) + \lambda(t, t+\tau)N(t+\tau)K(t) = -1, \tag{99}$$

so Assumption 1 is satisfied. If we change (96) by instead having $M(t) = \sin(t/r) + \sin(t/(vr))$ for any positive integers v and r, then the example is covered by Theorem 2, with $\bar{T} = 2\pi vr$, $F(t) = M(t)$, and $K = -F(t) - 1$.

B. Rapidly Time-Varying Pendulum

We next illustrate our Theorem 3 on rapidly time-varying systems, using the model

$$\left\{\begin{array}{rcl}\dot{r}_1(t) &=& r_2(t)\\ \dot{r}_2(t) &=& -\frac{g}{l}\sin(r_1(t)) + \frac{1}{ml^2}v(t-\tau)\end{array}\right. \tag{100}$$

of the simple pendulum, where $g = 9.8$ m/s is the gravity constant, l is the pendulum length in meters, m is the mass, v is the input, and the constant delay is $\tau > 0$. Assume that we wish to follow any C^1 reference trajectory $(r_{1,s}(t), r_{2,s}(t))$ such that $\dot{r}_{1,s}(t) = r_{2,s}(t)$. The error variables $\tilde{r}_i = r_i - r_{i,s}(t)$ for $i = 1, 2$ and the change of feedback $u(t-\tau) = \frac{1}{ml^2}v(t-\tau) - \dot{r}_{2,s}(t) - \frac{g}{l}\sin(r_{1,s}(t))$ produce the system

$$\left\{\begin{array}{rcl}\dot{\tilde{r}}_1(t) &=& \tilde{r}_2(t)\\ \dot{\tilde{r}}_2(t) &=& \frac{g}{l}[\sin(r_{1,s}(t)) - \sin(\tilde{r}_1(t) + r_{1,s}(t))] \\ && +\, u(t-\tau).\end{array}\right. \tag{101}$$

We now specialize to the case where $r_{1,s}(t) = \omega t$, where $\omega > 0$ is a constant. The linear approximation of (101) at 0 is

$$\left\{\begin{array}{rcl}\dot{x}_1(t) &=& x_2(t)\\ \dot{x}_2(t) &=& k\cos(\omega t)x_1(t) + u(t-\tau),\end{array}\right. \tag{102}$$

where $k = -\frac{g}{l}$. One can easily check that Assumption 4 is satisfied with the choices $T = 2\pi$,

$$M_A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \;\text{ and }\; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, Theorem 3 applies for large enough ω.

We next build the controller (65). Notice that

$$e^{M_A s} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}, \quad \text{so} \quad e^{M_A s}B = \begin{bmatrix} s \\ 1 \end{bmatrix}$$

for all $s \in \mathbb{R}$. Hence, choosing $K = (-0.6, -0.4)$, we can satisfy (57) for $\tau = 1$ using

$$S = \begin{bmatrix} \frac{22}{3} & -1 \\ -1 & 4 \end{bmatrix}, \tag{103}$$

which has the eigenvalues 7.61032 and 3.72302. The stabilizing controller from (65) is then

$$\begin{array}{l}u(t) = -0.6x_1(t) - 0.4x_2(t) \\ -\int_{t-1}^{t}\big(0.6(t-m-1) + 0.4\big)u(m)\mathrm{d}m\end{array} \tag{104}$$

which exponentially stabilizes the system (102).

We simulated (102) in closed loop with (104), with $\tau = 1$ and the parameter choices $k = -0.0544$ and $\omega = 180$, which satisfy our assumptions with the preceding choices of K and S. We chose the initial control $u_0 = 0$ and the initial state $x(0) = (0.4, -0.6)$ (which satisfy the matching condition (52)). We report our results in Figure 1. It shows rapid convergence of the tracking error to zero and therefore helps validate our results. Then we repeated the simulations exactly as before except with the disturbance $\delta(t) = 0.1(\sin(t), \cos(t))$ added on the right side of (102) producing the perturbed system

$$\left\{\begin{array}{rcl}\dot{x}_1(t) &=& x_2(t) + 0.1\sin(t)\\ \dot{x}_2(t) &=& k\cos(\omega t)x_1(t) + u(t-\tau) + 0.1\cos(t),\end{array}\right. \tag{105}$$

and we report the results in Figure 2.

C. Slowly Time-Varying Pendulum

We next illustrate our Theorem 4 on slowly time-varying systems, using the pendulum dynamics (102) with the slowness parameter ϵ instead of the rapidness parameter ω, i.e.,

$$\left\{\begin{array}{rcl}\dot{x}_1(t) &=& x_2(t)\\ \dot{x}_2(t) &=& k\cos(\epsilon t)x_1(t) + u(t-\tau).\end{array}\right. \tag{106}$$

Then Theorem 4 applies with the choices

$$\begin{array}{l}A(\epsilon t) = \begin{bmatrix} 0 & 1 \\ k\cos(\epsilon t) & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ L = \begin{bmatrix} -1 \\ -1 \end{bmatrix}^\top, \;\text{ and }\; Q = \begin{bmatrix} 3.3 & 1.1 \\ -1.1 & 2.2 \end{bmatrix}.\end{array} \tag{107}$$

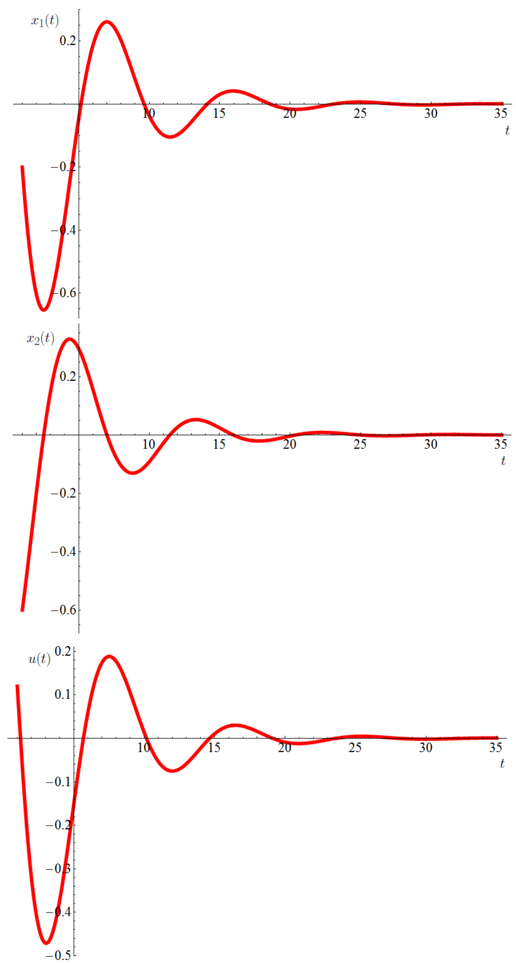

Fig. 1. Simulation of States and Closed Loop Tracking Control for Rapidly Time-Varying Pendulum Dynamics in the $\delta = 0$ Case where there are No Disturbances. Units are Meters and Seconds.

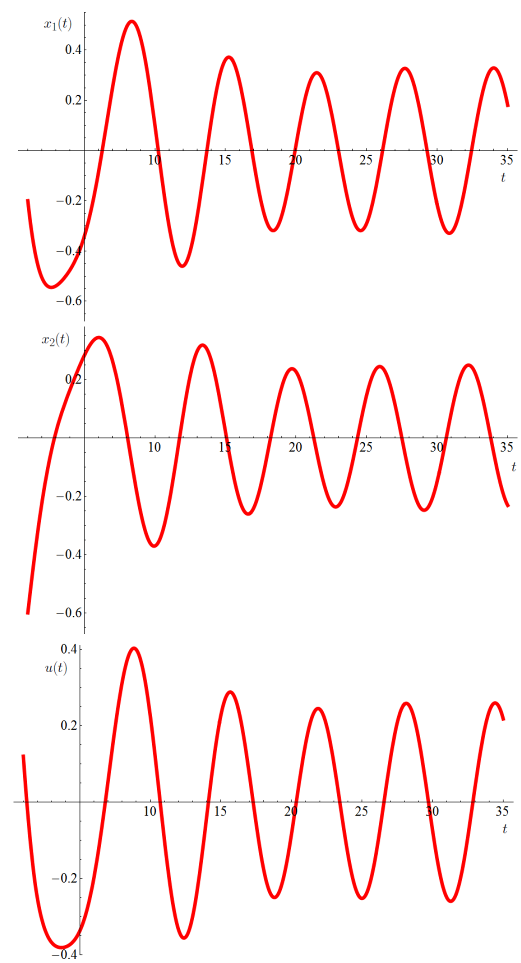

Fig. 2. Simulation of States and Closed Loop Tracking Control for Rapidly Time-Varying Pendulum Dynamics with Disturbance $\delta(t) = 0.1(\sin(t), \cos(t))$. Units are Meters and Seconds.

In fact, the preceding choices give $|A'|_\infty = |k|$, $|Q| = 3.979$, and $|L| = \sqrt{2}$, so our upper bound ϵ_* from (77) on the slowness parameter is

$$\epsilon_* = \frac{1}{4|k|e^{2\tau}(3.979)\tau} \min\left\{1, \frac{\sqrt{2}}{\tau}\right\}. \tag{108}$$

Taking $\tau = 1$ and $k = -0.0544$ as before gives the upper bound $\epsilon_* = 0.1562$. If $\tau = 0.5$ and $k = -0.0544$, then we instead have $\epsilon_* = 0.8496$. In general, $\epsilon_* \to +\infty$ as $|k| \to 0$ for each τ, so stability is maintained under arbitrarily long or short time variation in the pendulum coefficient as the pendulum becomes longer. This shows that we do not require the condition $\epsilon\tau < 1$.

VIII. Conclusions

The stabilization of time-varying linear systems under arbitrarily long input delays is a challenging problem that is beyond the scope of the standard prediction and reduction methods for time-invariant systems. We developed a new reduction approach to prove ISS with respect to additive uncertainties. This gave ISS for rapidly and slowly time-varying systems. Since our controls are dynamic, our ISS estimates quantify the decay of the state and the control, with overshoots depending on the uncertainty. This is significant, because rapidly and slowly time-varying systems are ubiquitous in engineering [6], [25] but to the best of the authors' knowledge were not covered by the existing results when long delays and uncertainties are present. We found explicit bounds on the admissible rapidness and slowness parameters. Much work remains to be done. Some desirable extensions would be to nonlinear systems, systems with state dependent delays [36], [37], and PDE state dynamics where there are also input constraints such as saturation constraints.

Appendix A.1. End of Proof of Theorem 2

We can integrate our decay estimate (51) to get

$$V\big(t, (\rho,\zeta)(t)\big) \le V\big(t_0, (\rho,\zeta)(t_0)\big)e^{b_1(t_0-t)} + b_3|\delta|^2_{[t_0,t]} \tag{A.1}$$

for all $t_0 \ge 0$ and all $t \ge t_0$. Here and in the sequel, all constants $b_i > 0$ are independent of t_0 and the trajectory. Combined with (27) and (47), we deduce from (A.1) that if $t \ge t_0 \ge 0$, then the following hold:

$$p_s|\rho(t)|^2 \le V\big(t_0, (\rho,\zeta)(t_0)\big)e^{b_1(t_0-t)} + b_3|\delta|^2_{[t_0,t]} \tag{A.2}$$

$$\zeta(t) \le b_4 V\big(t_0, (\rho,\zeta)(t_0)\big)e^{b_1(t_0-t)} + b_5|\delta|^2_{[t_0,t]}\ , \tag{A.3}$$

where b_4 and b_5 are positive constants.

Next recall our formula $\rho(t) = \Theta(t) + \mathcal{F}(t)\kappa(t) = x(t) + (1+\mathcal{F}(t))\kappa(t)$ and the formula for ζ in (44). Then

$|\rho(t)|^2 \leq 2|x(t)|^2 + 2(1+|\mathcal{F}|_\infty)^2|\kappa(t)|^2$, so $V(t,(\rho,\zeta)(t)) \leq 2p_*|x(t)|^2 + b_6\int_{t-\tau}^t |u(m)|^2 \mathrm{d}m$, where b_6 is a positive constant. Combined with (A.2), this implies that for all $t \geq t_0$,

$$\begin{array}{rcl} p_s|\rho(t)|^2 & \leq & \mathcal{J}(x_{t_0},u_{t_0})e^{b_1(t_0-t)} + b_3|\delta|^2_{[t_0,t]} \text{ and} \\ \zeta(t) & \leq & b_4\mathcal{J}(x_{t_0},u_{t_0})e^{b_1(t_0-t)} + b_5|\delta|^2_{[t_0,t]}, \end{array} \tag{A.4}$$

where $\mathcal{J}(x_{t_0},u_{t_0}) = 2p_*|x(t_0)|^2 + b_6\tau|u|^2_{[t_0-\tau,t_0]}$. Also, $|x(t)| = |\rho(t) - \mathcal{F}(t)\kappa(t) - \kappa(t)| \leq |\rho(t)| + b_7\int_{t-\tau}^t |u(m)|\mathrm{d}m$, where b_7 is a positive constant. Combining this last inequality with (A.4), we obtain

$$\begin{array}{l} |x(t)| \leq \sqrt{\frac{1}{p_s}\mathcal{J}(x_{t_0},u_{t_0})e^{b_1(t_0-t)} + \frac{b_3}{p_s}|\delta|^2_{[t_0,t]}} \\ + b_7\int_{t-\tau}^t |u(m)|\mathrm{d}m\ . \end{array} \tag{A.5}$$

Now observe that the Cauchy-Schwarz inequality gives

$$\begin{array}{l} \int_{t-\tau}^t |u(m)|\mathrm{d}m \leq \int_{t-\tau}^t e^{m-t+\tau}|u(m)|\mathrm{d}m \\ \leq \sqrt{\tau}\sqrt{\zeta(t)}, \end{array} \tag{A.6}$$

which we combine with the second inequality in (A.4) and (A.5) to get

$$\begin{array}{l} \max\left\{|x(t)|, \sqrt{\zeta(t)}\right\} \leq \sqrt{\frac{b_3}{p_s}}|\delta|_{[t_0,t]} + \\ \sqrt{\max\{b_4, 1/p_s\}\mathcal{J}(x_{t_0},u_{t_0})e^{b_1(t_0-t)}} \\ + b_7\sqrt{\tau}\sqrt{b_4\mathcal{J}(x_{t_0},u_{t_0})e^{b_1(t_0-t)}} \\ + \max\{1, b_7\sqrt{\tau}\}\sqrt{b_5}|\delta|_{[t_0,t]}. \end{array} \tag{A.7}$$

Also, our formula $u(t) = K(t)[x(t)+\kappa(t)]$ and (A.6) give

$$\begin{array}{l} |(x(t),u(t))| \leq (1+|K|_\infty) \\ \times\left(|x(t)| + e^{|F|_\infty\tau}|G|_\infty\int_{t-\tau}^t |u(m)|\mathrm{d}m\right) \\ \leq 2(1+|K|_\infty)\max\{1, e^{|F|_\infty\tau}|G|_\infty\sqrt{\tau}\} \\ \quad\times\max\left\{|x(t)|, \sqrt{\zeta(t)}\right\}\ . \end{array} \tag{A.8}$$

The desired exponential ISS estimate then follows by using the right side of (A.7) to upper bound the right side (A.8), and then reasoning as in the last part of Remark 2 to get $|x(t)| + |u|_{[t-\tau,t]}$ on the left side of the ISS estimate.

Appendix A.2. Lemma

We used the following in Section IV-C, and in the proof of Theorem 3 with the choice $\mathcal{F}_0 = I_n - \mathcal{A}_\omega(t)$:

Lemma A.1: Let $\mathcal{F}_0 = I_n - \mathcal{G}$ where the matrix $\mathcal{G} \in \mathbb{R}^{n\times n}$ is such that $|\mathcal{G}| < 1$. Then $\mathcal{F}_0$ is invertible and the inequality $|\mathcal{F}_0^{-1} - I_n| \leq |\mathcal{G}|/(1-|\mathcal{G}|)$ is satisfied. □

Proof. If $v \in \mathbb{R}^n$ is such that $\mathcal{F}_0 v = 0$, then $v = \mathcal{G}v$. Then $|v| \leq |\mathcal{G}||v|$. Since $|\mathcal{G}| < 1$, we deduce that $v = 0$, so $\mathcal{F}_0$ is invertible. Next note that $\mathcal{F}_0^{-1} - I_n = [\mathcal{F}_0^{-1} - I_n]\mathcal{G} + \mathcal{G}$, so $|\mathcal{F}_0^{-1} - I_n| \leq |\mathcal{F}_0^{-1} - I_n||\mathcal{G}| + |\mathcal{G}|$. Hence, $|\mathcal{F}_0^{-1} - I_n|(1-|\mathcal{G}|) \leq |\mathcal{G}|$, which gives the result. □

Appendix A.3. End of Proof of Theorem 4

We show how to convert the decay estimate (94) and upper and lower bounds (95) for U along all trajectories $z(t)$ into the desired exponential ISS result. First note that (94) gives $U(t,z_t) \leq e^{C_*(t_0+\tau-t)}U(t_0+\tau, z_{t_0+\tau}) + C_{**}|\delta|^2_{[t_0,t]}/C_*$ for all $t \geq t_0+\tau$, which we combine with (95) to get:

$$C_*|z(t)|^2 \leq C_{**}e^{C_*(t_0+\tau-t)}|z|^2_{[t_0,t_0+\tau]} + \frac{C_{**}}{C_*}|\delta|^2_{[t_0,t]} \tag{A.9}$$

This gives the estimate

$$\begin{array}{rcl} |z(t)| & \leq & \sqrt{\frac{C_{**}}{C_*}}|z|_{[t_0,t_0+\tau]}e^{0.5C_*(t_0+\tau-t)} \\ & & + \frac{\sqrt{C_{**}}}{C_*}|\delta|_{[t_0,t]} \end{array} \tag{A.10}$$

for all $t \geq t_0+\tau$. Our choice of Γ from (80) gives

$$|z|_{[t_0,t_0+\tau]} \leq |x|_{[t_0,t_0+\tau]} + \tau e^{\tau\bar{c}_1}|F|_\infty|u|_{[t_0-\tau,t_0+\tau]} \tag{A.11}$$

and $u(t) = L(\epsilon t)e^{A(\epsilon t)\tau}z(t)$ for all $t \geq t_0$. Hence, for all $t \geq t_0+2\tau$, the estimates (A.10)-(A.11) give

$$\begin{array}{rcl} |x(t)| & \leq & |z(t) - \Gamma(t)| \\ & \leq & |z(t)| + e^{2\bar{c}_1\tau}\bar{c}_2\int_{t-\tau}^t |z(m)|\mathrm{d}m \\ & \leq & (1+e^{2\bar{c}_1\tau}\bar{c}_2\tau)|z|_{[t-\tau,t]} \\ & \leq & \sqrt{\frac{C_{**}}{C_*}}(1+e^{2\bar{c}_1\tau}\bar{c}_2\tau)(1+\tau e^{\tau\bar{c}_1}|F|_\infty) \\ & & \times|(x,u)|_{[t_0-\tau,t_0+\tau]}e^{0.5C_*(t_0+\tau-t)} \\ & & + (1+e^{2\bar{c}_1\tau}\bar{c}_2\tau)\frac{\sqrt{C_{**}}}{C_*}|\delta|_{[t_0,t]}\ . \end{array} \tag{A.12}$$

Reasoning as in Remark 6 gives a constant $d_0 > 0$ such that

$$|(x,u)|_{[t_0-\tau,t_0+\tau]} \leq d_0\left(|(x,u)|_{[t_0-\tau,t_0]} + |\delta|_{[t_0,t]}\right). \tag{A.13}$$

On the other hand, if $t_0 \leq t \leq t_0+2\tau$, then since the formula for the open loop dynamics gives $|\dot{x}(t)| \leq \max\{\bar{c}_1, |F|_\infty\}\left(|x(t)| + |u(t-\tau)|\right) + |\delta(t)|$, we can apply Gronwall's inequality on $[t_0,t]$ to get

$$\begin{array}{l} |x(t)| \leq \\ e^{2\max\{\bar{c}_1,|F|_\infty\}\tau}\big((1+2\tau\max\{\bar{c}_1,|F|_\infty\}) \\ \quad\times|(x,u)|_{[t_0-\tau,t_0+\tau]} + 2\tau|\delta|_{[t_0,t]}\big) \end{array} \tag{A.14}$$

and then use (A.13) to get

$$\begin{array}{l} |x(t)| \leq \\ \big\{(1+2\tau\max\{\bar{c}_1,|F|_\infty\}) \\ \times\left(d_0|(x,u)|_{[t_0-\tau,t_0]} + |\delta_{[t_0,t]}\right)e^{2\max\{\bar{c}_1,|F|_\infty\}\tau} \\ + 2\tau e^{2\tau\max\{\bar{c}_1,|F|_\infty\}}|\delta|_{[t_0,t]}\big\}e^{0.5C_*(t_0-t)}e^{C_*\tau}. \end{array} \tag{A.15}$$

Also, our estimates (A.10), (A.11), and (A.13) provide a constant $d_1 > 0$ such that

$$\begin{array}{l} |z(t)| \leq \\ d_1\left(|(x,u)|_{[t_0-\tau,t_0]}e^{0.5C_*(t_0-t-\tau)} + |\delta|_{[t_0,t]}\right), \end{array} \tag{A.16}$$

for all $t \geq t_0$, and our formula for the control gives

$$\begin{array}{l} |u(t)| \leq \\ |L|_\infty e^{\bar{c}_1\tau}\left[|x(t)| + e^{2\bar{c}_1\tau}\bar{c}_2\int_{t-\tau}^t |z(m)|\mathrm{d}m\right] \end{array} \tag{A.17}$$

for all $t \geq t_0$. The exponential ISS estimate now follows by substituting (A.13) into (A.12), substituting (A.16) into (A.17), and then adding (A.12), (A.15), and (A.17), and then finally reasoning as in the last part of Remark 2, which again gives $|x(t)| + |u|_{[t-\tau,t]}$ on the left side of the ISS estimate.

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