Journal of Business & Economic Statistics

ISSN: 0735-0015 (Print) 1537-2707 (Online) Journal homepage: http://amstat.tandfonline.com/loi/ubes20

The Estimation and Testing of the Cointegration Order Based on the Frequency Domain

Igor Viveiros Melo Souza, Valderio Anselmo Reisen, Glaura da Conceição Franco & Pascal Bondon

To cite this article: Igor Viveiros Melo Souza, Valderio Anselmo Reisen, Glaura da Conceição Franco & Pascal Bondon (2016): The Estimation and Testing of the Cointegration Order Based on the Frequency Domain, Journal of Business & Economic Statistics, DOI: 10.1080/07350015.2016.1251442

To link to this article: http://dx.doi.org/10.1080/07350015.2016.1251442

Accepted author version posted online: 31 Oct 2016.
Published online: 31 Oct 2016.

The Estimation and Testing of the Cointegration Order Based on the Frequency Domain

Igor Viveiros Melo Souza
DEECO, UFOP and Departamento de Estatística, UFMG, Rua do Catete, no. 166 Mariana, Minas Gerais 30420-000, Brazil (*igorviveiros@gmail.com*)

Valderio Anselmo Reisen
Departamento de Estatística, PPGECON, PPGEA, UFES, Av. Fernando Ferrari s/n Goiabeiras, Vitoria, Espírito Santo 29060-900, Brazil (*valderioanselmoreisen@gmail.com*)

Glaura da Conceição Franco
Departamento de Estatística, UFMG, Av. Antonio Carlos, no. 6627 Predio do ICEx 4067 Belo Horizonte, Minas Gerais 31270-901, Brazil (*glaura@est.ufmg.br*)

Pascal Bondon
Laboratoire des Signaux et Systèmes, CNRS - CentraleSupélec Université Paris-Sud, France (*bondon@lss.supelec.fr*)

This article proposes a method to estimate the degree of cointegration in bivariate series and suggests a test statistic for testing noncointegration based on the determinant of the spectral density matrix for the frequencies close to zero. In the study, series are assumed to be $I(d)$, $0 < d \leq 1$, with parameter d supposed to be known. In this context, the order of integration of the error series is $I(d-b)$, $b \in [0, d]$. Besides, the determinant of the spectral density matrix for the dth difference series is a power function of b. The proposed estimator for b is obtained here performing a regression of logged determinant on a set of logged Fourier frequencies. Under the null hypothesis of noncointegration, the expressions for the bias and variance of the estimator were derived and its consistency property was also obtained. The asymptotic normality of the estimator, under Gaussian and non-Gaussian innovations, was also established. A Monte Carlo study was performed and showed that the suggested test possesses correct size and good power for moderate sample sizes, when compared with other proposals in the literature. An advantage of the method proposed here, over the standard methods, is that it allows to know the order of integration of the error series without estimating a regression equation. An application was conducted to exemplify the method in a real context.

KEY WORDS: Consistency; Determinant of spectral density matrix; Estimator; Fractional cointegration; Test of noncointegration.

1. INTRODUCTION

To study the relationship among economic variables, the concept of cointegration, introduced by Granger (1981), has been widely employed, mainly due to the spurious regression problem. After this seminal work, several studies about this topic have been developed. In the classic context, the most used tests for cointegration are the Engle and Granger (1987) test (EG), the Phillips and Ouliaris (1988) test, and the Johansen (1991) procedure. Besides, tests to verify the presence of a unit root are necessary to use appropriate procedures for modeling the data.

Despite its widespread use, the classical set up of cointegration is lately being considered quite restrictive for many real problems. As an alternative, fractional cointegration has emerged as a more adequate methodology and examples to real problems can be seen in Cheung and Lai (1993), Baillie and Bollerslev (1994), Dittmann (2001), McHale and Peel (2010), and Cuestas, Gil-Alana, and Staehr (2014). Different approaches have been implemented in the estimation and construction of hypothesis tests concerning fractional processes. See, for example, Robinson (1994), Robinson and Marinucci (2001), Marinucci and Robinson (2001), Robinson and Yajima (2002), and Velasco (2003).

The first step in cointegration analysis is to verify the order of integration of the series $X_{i,t}$, $t = 1, 2, \ldots, i = 1, \ldots, h$, of the vector $\mathbf{X_t} = (X_{1,t}, \ldots, X_{h,t})'$. A series $X_{i,t}$ is said to be integrated of order d, $d \in \Re$, denoted by $X_{i,t} \sim I(d)$, if d is the minimum number of differences required to obtain a process that admits an autoregressive moving average representation (ARMA). In this context, parameter d measures the memory of the series. The series $X_{i,t}$ can be written as $X_{i,t} = (1-B)^{-d} e_{i,t}$ where $e_{i,t} = \theta_q(B)\phi_p^{-1}(B)u_{i,t}$ with $u_{i,t}$ being a zero-mean white-noise process with constant variance $\sigma^2_{u_i}$ and $\theta_q(B)$ and $\phi_p(B)$

Journal of Business & Economic Statistics
XXXX 2017, Vol. 0, No. 0
DOI: 10.1080/07350015.2016.1251442

are polynomials in B with order q and p, respectively, with all roots outside of the unit circle. B is the backshift operator, that is, $B^{\tau}X_{i,t} = X_{i,t-\tau}\ \forall\ \tau \in \mathbb{N}$. In this case, the series $X_{i,t}$ is said to be an Autoregressive Fractionally Integrated Moving Average process, denoted by ARFIMA (p, d, q) (see Hosking 1981).

A general definition of fractional cointegration was given by Robinson and Marinucci (1998), allowing a different order of integration for $X_{i,t}$, that is, $X_{i,t} \sim I(d_i)$, $d_i > 0$, $\forall\ i$. Therefore, an $h \times 1$ vector $\mathbf{X_t}$, $t = 1, 2, \ldots,$ is called fractionally cointegrated, denoted by $\mathbf{X}_t \sim \mathrm{FCI}(d_1, \ldots, d_h, d_\varepsilon)$, if there exists a unique $h \times 1$ vector $\boldsymbol{\beta} \neq 0$ such that $\varepsilon_t = \boldsymbol{\beta}^{'}\mathbf{X}_t \sim I(d_\varepsilon)$, where $0 \leq d_\varepsilon \leq \min_{1\leq i\leq h} d_i$. This definition is valid if and only if $d_i = d_j$ for some $i \neq j$, $i, j = 1, \ldots, h$. The vector $\boldsymbol{\beta}$ is called cointegration vector. In the case that $d_1 = \cdots = d_h = d$, it is usual to write $\mathbf{X}_t \sim \mathrm{CI}(d, b)$ where $b = d - d_\varepsilon$. When $b = 0$ the vector $\mathbf{X}_t$ is noncointegrated. In this sense, parameter b measures the reduction in the order of integration of the error series ε_t. Various estimators of d_ε can be used in hypothesis tests for fractionally cointegrated processes (see, e.g., Dittmann 2000).

The main objectives of this article are twofold. The first one is to propose a new method for estimating the reduction in the order of integration, b, of the error series ε_t in a bivariate vector $\mathbf{X_t} = (X_{1,t}, X_{2,t})'$. The second goal is to suggest a test of noncointegration based on the determinant of the spectral density matrix of the vector $(\Delta^d X_{1,t}, \Delta^d X_{2,t})'$, where Δ^d is the dth difference operator, that is, $\Delta^d = (1 - B)^d$.

Asymptotic results for the estimation method are established and an empirical Monte Carlo study is conducted to evaluate its performance for small sample sizes across different regression coefficients in the cointegration system. In addition, for comparison purposes, the noncointegrated test derived from the estimator of b is compared with the standard ones.

The article is structured as follows. In Section 2, some properties of the determinant of the spectral density matrix for cointegrated and noncointegrated bi-variate vector $\mathbf{X_t} = (X_{1,t}, X_{2,t})'$ at the first difference are analyzed. Section 3 discusses the proposed estimator of the parameter b; the log-determinant regression estimator. Section 4 is motivated by a Monte Carlo study that is conducted to analyze the empirical performance of proposed estimation method in terms of the bias, size, and power. This section also presents an empirical comparison among the methods proposed here with the well-known cointegration tests in the literature: the residual-based tests and the Log Coherence Regression given in Dittmann (2000) and Velasco (2003), respectively. Section 5 shows an application of the proposed methodology to a real time series dataset and Section 6 concludes the work.

2. THE DETERMINANT OF THE SPECTRAL DENSITY MATRIX FOR A BIVARIATE SERIES

This section presents the properties of the determinant of the spectral density matrix for the vector $(\Delta^d X_{1,t}, \Delta^d X_{2,t})'$ where both components are $I(0)$ and satisfies the linear relationship $X_{1,t} = \beta X_{2,t} + \varepsilon_t$ for a unique β such that $\beta \neq 0$. The error term ε_t is assumed to be $I(d - b)$, with $0 \leq b \leq d$ and $0 < d \leq 1$.

Let the observable bivariate time series $(X_{1,t}, X_{2,t})'$ be formed by the following system:

$$\begin{aligned} X_{1,t} &= \beta_1 T_t + w_{1,t} \\ X_{2,t} &= \beta_2 T_t + w_{2,t} \end{aligned} \tag{2.1}$$

for $t = 1, 2, \ldots, \beta_1, \beta_2, \in \Re$. The series T_t is a common unobservable stochastic trend such that

$$T_t = (1 - B)^{-d}\eta_t \tag{2.2}$$

and the innovations η_t is a zero-mean stationary ARMA process with finite fourth moment, that is, $\mathrm{E}(\eta_t^4) < \infty$. The pair of innovations $(w_{1,t}, w_{2,t})'$ follows the processes:

$$w_{1,t} = (1 - B)^{-(d-b_1)} e_{1,t} \tag{2.3a}$$

$$w_{2,t} = (1 - B)^{-(d-b_2)} e_{2,t}, \tag{2.3b}$$

where $b_1, b_2 \in [0, 1]$. The vector $(e_{1,t}, e_{2,t})'$ is uncorrelated with η_t and follows a zero-mean vector ARMA process with finite fourth moment and covariance matrix $\boldsymbol{\Omega}_\tau$, whose diagonal elements are such that $\sum_{\tau=0}^{\infty} \sqrt{\tau}|\gamma_{e_1}(\tau)| < \infty$, $\sum_{\tau=0}^{\infty} \sqrt{\tau}|\gamma_{e_2}(\tau)| < \infty$. The system described in Equation (2.1) can be rewritten as follows:

$$X_{1,t} = \beta X_{2,t} + \varepsilon_t \tag{2.4a}$$

$$\varepsilon_t = w_{1,t} - \beta w_{2,t}, \tag{2.4b}$$

where ε_t is a nonobservable error term such that $\varepsilon_t \sim I(d - b)$ with $b = \min(b_1, b_2)$ and $\beta = \beta_1/\beta_2$. The vector $(X_{1,t}, X_{2,t})'$ will be noncointegrated if and only if $b = 0$. Taking the dth-order difference, Equation (2.1) can be rewritten as

$$\begin{bmatrix} \Delta^d X_{1,t} \\ \Delta^d X_{2,t} \end{bmatrix} = \begin{bmatrix} \beta_1 & (1-B)^{b_1} & 0 \\ \beta_2 & 0 & (1-B)^{b_2} \end{bmatrix} \begin{bmatrix} \eta_t \\ e_{1,t} \\ e_{2,t} \end{bmatrix}. \tag{2.5}$$

The spectral density matrix of the vector $(\Delta^d X_{1,t}, \Delta^d X_{2,t})'$, say $\mathbf{F}_{\Delta^d}(\lambda)$, $\lambda \in [0, \pi)$, can be written as (see Priestley 1981, pp. 658–659)

$$\mathbf{F}_{\Delta^d}(\lambda) = \begin{bmatrix} f_{\Delta^d X_1}(\lambda) & f_{\Delta^d X_1 \Delta^d X_2}(\lambda) \\ f_{\Delta^d X_2 \Delta^d X_1}(\lambda) & f_{\Delta^d X_2}(\lambda) \end{bmatrix}, \tag{2.6}$$

where $f_{\Delta^d X_1}(\lambda)$ and $f_{\Delta^d X_2}(\lambda)$ are the spectral densities of $\Delta^d X_{1,t}$ and $\Delta^d X_{2,t}$, respectively, and $f_{\Delta^d X_1 \Delta^d X_2}(\lambda)$ and $f_{\Delta^d X_2 \Delta^d X_1}(\lambda)$ are the cross-spectrum between $\Delta^d X_{1,t}$ and $\Delta^d X_{2,t}$, respectively. The matrix $\mathbf{F}_{\Delta^d}(\lambda)$ is Hermitian, which means that $f_{\Delta^d X_1 \Delta^d X_2}(\lambda) = \overline{f_{\Delta^d X_2 \Delta^d X_1}(\lambda)}$, where the over line means the complex conjugate. Using the standard spectral properties of multivariate time series, $\mathbf{F}_{\Delta^d}(\lambda)$ can be rewritten as (see Priestley 1981)

$$\mathbf{F}_{\Delta^d}(\lambda) = \begin{bmatrix} \beta_1^2 f_\eta(\lambda) + |1 - e^{-\iota\lambda}|^{2b_1} f_{e_1}(\lambda) & \beta_1\beta_2 f_\eta(\lambda) + |1 - e^{-\iota\lambda}|^{2(b_1+b_2)} f_{e_1e_2}(\lambda) \\ \beta_2\beta_1 f_\eta(\lambda) + |1 - e^{-\iota\lambda}|^{2(b_1+b_2)} f_{e_2e_1}(\lambda) & \beta_2^2 f_\eta(\lambda) + |1 - e^{-\iota\lambda}|^{2b_2} f_{e_2}(\lambda) \end{bmatrix}. \tag{2.7}$$

The determinant of matrix $\mathbf{F}_{\Delta^d}(\lambda)$ is given by

$$D(\lambda) = |1 - e^{-\iota\lambda}|^{2b_1} \beta_2^2 f_{e_1}(\lambda) f_\eta(\lambda) + |1 - e^{-\iota\lambda}|^{2b_2} \beta_1^2 f_{e_2}(\lambda) f_\eta(\lambda)$$

$$+|1-e^{-\iota\lambda}|^{2(b_1+b_2)}\left[f_{e_1}(\lambda)f_{e_2}(\lambda)-2\beta_1\beta_2 f_\eta(\lambda)\operatorname{Re}\left(f_{e_1e_2}(\lambda)\right)\right] - |1-e^{-\iota\lambda}|^{4(b_1+b_2)}|f_{e_1e_2}(\lambda)|^2. \tag{2.8}$$

In the above, $\operatorname{Re}(f_{e_1e_2}(\lambda))$ means the real part of $f_{e_1e_2}(\lambda)$. Assuming, without loss of generality, that $b_1 \leq b_2$, which makes $b = b_1$, and using the facts that $|1-e^{-\iota\lambda}|^{2b^*} = (2-2\cos\lambda)^{b^*}$, for any $b^* \in \Re$, and $\lim_{\lambda\to 0^+}\frac{(2-2\cos\lambda)^{b^*}}{\lambda^{2b^*}} = 1$, which implies that $|1-e^{-\iota\lambda}|^{2b^*} = O(\lambda^{2b^*})$ as $\lambda \longrightarrow 0^+$, the determinant $D(\lambda)$ can be rewritten as

$$D(\lambda) = |1-e^{-\iota\lambda}|^{2b}G(\lambda)+O(\lambda^{2b_2})+O(\lambda^{2(b+b_2)})+O(\lambda^{4(b+b_2)}), \tag{2.9}$$

where $G(\lambda)$ is a bounded function due to the stationarity of the processes $e_{1,t}$, $e_{2,t}$ and η_t such that $\lim_{\lambda\to 0^+} G(\lambda) = C$, where $0 < C < \infty$. From this, the determinant $D(\lambda)$ can be approximated by

$$D(\lambda) \sim |1-e^{-\iota\lambda}|^{2b}C = O(\lambda^{2b}) \text{ as } \lambda \longrightarrow 0^+, \tag{2.10}$$

where the symbol "$\sim$" means that the ratio of left- and right-hand sides tends to one as $\lambda \to 0^+$. From Equation (2.10), $D(\lambda)$ depends on the reduction of the order of integration b imposed by cointegration. Similar results are also described by Nielsen (2004). Therefore, if $(X_{1,t}, X_{2,t})'$ is cointegrated, that is, $0 < b \leq d$, then $D(\lambda) \to 0$ as $\lambda \longrightarrow 0^+$. It means that $\mathbf{F}_{\Delta^d}(\lambda)$ is a matrix with incomplete rank at $\lambda = 0$ (see Phillips and Ouliaris 1988). In the case of $b = 0$, $(X_{1,t}, X_{2,t})'$ is noncointegrated, $D(\lambda) \to C$ as $\lambda \longrightarrow 0^+$, and $\mathbf{F}_{\Delta^d}(\lambda)$ has full rank at $\lambda = 0$.

Thus, considering the results above, a new method to estimate parameter b and test the null hypothesis of noncointegration are proposed here by analyzing the slope of the function $D(\lambda)$ at the neighborhood of the zero frequency. This is discussed in the next section.

3. THE LOGGED DETERMINANT REGRESSION

Standard estimation methods for the memory parameter d, well discussed in the literature of long memory processes, can be used as alternative procedures to obtain estimates of b in Equation (2.10). Similar to the estimator of d proposed by Geweke and Porter-Hudak (1983) (GPH), an estimate of b can be computed from an approximated regression equation by taking the log in Equation (2.10):

$$\ln D(\lambda) \sim \ln C + \ln\frac{G(\lambda)}{C} + b\ln|1-e^{-\iota\lambda}|^2 \text{ as } \lambda \longrightarrow 0^+. \tag{3.1}$$

For a pair of series $(\Delta^d X_{1,t}, \Delta^d X_{2,t})'$ with a sample of size n, that is, $t = 1,\ldots,n$, the first step to implement the above regression model is to estimate the spectral density matrix, $\mathbf{F}_{\Delta^d}(\lambda)$, in Equation (2.7). Let the Fourier frequency $\lambda_j = 2\pi j/n$, $j = l, l+(2r+1), l+2(2r+1),\ldots,m-(2r+1), m$, where $l = (r+1)$ and $r \in \mathbb{N}^*$, with $r < m$ and $m < n$ (l and m are the trimming and the truncation numbers, respectively). Hence, for a fixed j, the estimate of $\mathbf{F}_{\Delta^d}(\lambda_j)$ is given by

$$\widehat{\mathbf{F}}_{\Delta^d,r}(\lambda_j) = \frac{1}{(2r+1)} \times \begin{bmatrix} \sum_{\upsilon=j-r}^{j+r} I_{n,\Delta^d X_1}(\lambda_\upsilon) & \sum_{\upsilon=j-r}^{j+r} I_{n,\Delta^d X_1\Delta^d X_2}(\lambda_\upsilon) \\ \sum_{\upsilon=j-r}^{j+r} I_{n,\Delta^d X_2\Delta^d X_1}(\lambda_\upsilon) & \sum_{\upsilon=j-r}^{j+r} I_{n,\Delta^d X_2}(\lambda_\upsilon) \end{bmatrix}, \tag{3.2}$$

where each diagonal term of $\widehat{\mathbf{F}}_{\Delta^d,r}(\lambda_j)$ is the average of $2r+1$ distinct periodograms centered at frequency j given by $I_{n,\Delta^d X_i}(\lambda_j) = \frac{1}{2\pi n}|\sum_{t=1}^n \Delta^d X_{i,t}e^{-\iota\lambda_j t}|^2$, for $i = 1, 2$. The off-diagonal terms of $\widehat{\mathbf{F}}_{\Delta^d,r}(\lambda_j)$ are also an average of $2r+1$ distinct cross-periodograms centered at frequency j that can be computed by $I_{n,\Delta^d X_s\Delta^d X_p}(\lambda_j) = \frac{1}{2\pi n}(\sum_{t=1}^n \Delta^d X_{s,t}e^{-\iota\lambda_j t}\sum_{t=1}^n \Delta^d X_{p,t}e^{\iota\lambda_j t})$, where $p, s = 1, 2$, $p \neq s$. The natural estimate of $D(\lambda_j)$ in Equation (3.1) is the determinant of $\widehat{\mathbf{F}}_{\Delta^d,r}(\lambda_j)$ denoted here by $\widehat{D}_r(\lambda_j)$. Using the fact that $\ln[\frac{\widehat{D}_r(\lambda_j)}{D(\lambda_j)}] - \ln\widehat{D}_r(\lambda_j) = -\ln D(\lambda_j)$ and replacing $D(\lambda_j)$ by the approximation in Equation (3.1), the following regression equation is obtained:

$$\ln\widehat{D}_r(\lambda_j) = \ln C + \ln\frac{G(\lambda_j)}{C} + c(r,j) + b\ln|1-e^{-\iota\lambda_j}|^2 + \zeta_j, \tag{3.3}$$

where $c(r,j) = \mathbf{E}\{\ln[\frac{\widehat{D}_r(\lambda_j)}{D(\lambda_j)}]\}$ and $\zeta_j = \{\ln[\frac{\widehat{D}_r(\lambda_j)}{D(\lambda_j)}] - c(r,j)\}$.

Therefore, the ordinary least-square estimator of b, $\hat{b}_r$, is given by

$$\hat{b}_r = \left(\sum_{j=(r+1)}^m \tilde{Z}_j^2\right)^{-1}\sum_{j=(r+1)}^m \tilde{Z}_j(\ln\widehat{D}_r(\lambda_j)), \tag{3.4}$$

where $Z_j = \ln(2-2\cos\lambda_j)$ and $\tilde{Z}_j = Z_j - \bar{Z}$, $\bar{Z}$ is the mean of Z_j, $j = l,\ldots,m$ with $l = r+1$. To state some asymptotic results for $\hat{b}$, under the null hypothesis of noncointegration ($H_0: b = 0$), the following assumption is introduced:

Assumption 1. Let l, m such that $\frac{l}{m} + \frac{m}{n} + \frac{1}{m} + \frac{\ln m}{m} \to 0$ as $n \to \infty$.

Based on the previous results and assumptions, the following theorem is now stated, which is proved in the Appendix.

Theorem 1. Let the series $(X_{1,t}, X_{2,t})'$ given in Equation (2.4a) and ε_t, defined in Equation (2.4b), $t = 1, 2, \ldots, n$, be $I(d)$ processes with $0 < d \leq 1$, that is, $b = 0$. Let Assumption 1 holds. Then, for a fixed positive integer r,

(a) $\mathbf{E}[\hat{b}_r] = \frac{\sum_{j=(r+1)}^m \tilde{Z}_j c(r,j)}{\sum_{j=(r+1)}^m \tilde{Z}_j^2} + \frac{\sum_{j=(r+1)}^m \tilde{Z}_j \ln G(\lambda_j)}{\sum_{j=(r+1)}^m \tilde{Z}_j^2} \to 0$, as $n \to \infty$,

(b) $\mathbf{V}[\hat{b}_r] = \frac{\sum_{j=(r+1)}^m \tilde{Z}_j^2\mathbf{V}[\zeta_j]}{(\sum_{j=(r+1)}^m \tilde{Z}_j^2)^2} + \frac{2\sum_{k=(r+1)}^m\sum_{j=k+(r+1)}^m \tilde{Z}_k\tilde{Z}_j\mathbf{cov}[\zeta_j,\zeta_k]}{(\sum_{j=(r+1)}^m \tilde{Z}_j^2)^2} \to 0$, as $n \to \infty$,

(c) $[m^{-1}(\psi^{(1)}(2r+1)+\psi^{(1)}(2r))]^{-1/2}\hat{b}_r \xrightarrow{d} N(0,1)$, as $n \to \infty$,

where $\psi^{(1)}(u)$ the Polygamma function of order 1, that is, $\psi^{(1)}(u) = \frac{d^2\ln\Gamma(u)}{du^2}$.

Under the null hypothesis of $b = 0$, the estimator $\hat{b}_r$ is consistent and has an asymptotic normal distribution with zero mean and variance $\sigma_{r,n}^2 = [m^{-1}(\psi^{(1)}(2r+1)+\psi^{(1)}(2r))]$ (see (A.5)). The choice of the trimming l (or r) can deliver some influence on the statistical properties of the estimator $\hat{b}_r$. For a

Table 1. Estimates, size, and power for the LDR method at 5% significance level for Gaussian innovations, $r = 1$

		$b = 0$	$b = 0.1$	$b = 0.2$	$b = 0.5$	$b = 0.7$	$b = 1$
$n = 100$	mean	0.0018	0.1020	0.2104	0.4764	0.6487	0.8689
$\sigma_{1,n} = 0.2433$	sd	0.2424	0.2401	0.2467	0.2505	0.2536	0.2736
	mse	0.0588	0.0576	0.0610	0.0633	0.0669	0.0920
	rejection	4.83	9.91	20.17	59.51	82.97	95.71
$n = 500$	mean	−0.0008	0.1056	0.1978	0.4830	0.6562	0.8545
$\sigma_{1,n} = 0.1134$	sd	0.1134	0.1129	0.1141	0.1145	0.1205	0.1583
	mse	0.0129	0.0128	0.0130	0.0134	0.0164	0.0462
	rejection	5.31	22.74	51.69	99.63	100	100
$n = 1000$	mean	0.0023	0.1013	0.1994	0.4854	0.6639	0.8560
$\sigma_{1,n} = 0.0859$	sd	0.0879	0.0871	0.0847	0.0871	0.0970	0.1385
	mse	0.0077	0.0076	0.0072	0.0078	0.0107	0.0399
	rejection	5.71	30.86	73.97	100	100	100

fixed truncation number m, this estimator has smaller asymptotical bias for a small r, say $r < r'$. However, $\hat{b}_r$ has larger variance than $\hat{b}_{r'}$ due to the amount of smoothing in the computation of the average of the periodogram in Equation (3.2). This issue will be also discussed in the empirical study.

Based on Theorem 1 it is possible to construct a test for noncointegration. Therefore, a natural test statistic to test $H_0 : b = 0$ against $H_1 : b > 0$ can be constructed as

$$\text{LDR} = \frac{\hat{b}_r}{[\sigma^2_{r,n}]^{1/2}}. \tag{3.5}$$

Note that the test in Equation (3.5) uses the asymptotic variance of the residual ζ_j in the regression Equation (3.3). The standard variance of the estimator $\hat{b}_r$, obtained using ordinary least squares, could also be employed to obtain a variant test. However, in the small sample size investigation, this second choice gave less power than (3.5).

4. MONTE CARLO STUDY

In this section, the performance of the logged determinant regression (LDR) method is investigated, for finite sample sizes, by means of a Monte Carlo study. For all simulations, the empirical mean, the standard deviation (sd), and the mean squared error (mse) for the estimator of b, and the size and power of the noncointegration tests are calculated. Standard methods for noncointegration tests, found in the literature, are also considered for comparison purpose.

Let the series $X_{1,t}$, $t = 1, \ldots, n$, be generated by $X_{1,t} = X_{2,t} + \varepsilon_t$ and the vector $(X_{2,t}, \varepsilon_t)'$ be generated by

$$\begin{bmatrix} (1-B) & 0 \\ 0 & (1-B)^{1-b} \end{bmatrix} \begin{bmatrix} X_{2,t} \\ \varepsilon_t \end{bmatrix} = \begin{bmatrix} e_{1,t} \\ e_{2,t} \end{bmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

The sample sizes considered were $n = 100, 500, 1000$ and the results are based on 3500 replications. The parameter b assumed the values $\{0, 0.1, 0.2, 0.5, 0.7, 1\}$ and the bandwidth was set at $m = n^{0.7}$. For testing $H_0 : b = 0$ against the alternative $H_1 : b > 0$, the nominal size, α, was fixed at 5%. $\sigma_{r,n}$ refers to the standard deviation computed by Equation (A.5) and $r = 1, 3$ which means that 3 and 7 different frequencies were included to compute $\widehat{\mathbf{F}}_{\Delta^d,r}(\lambda_j)$, respectively (see Equation (3.2)). $r = 2$ was also considered and has shown similar conclusions. The results are available upon request.

Table 1 displays the performance of the LDR method with $r = 1$. The results indicated that the estimates of b are, in general, centered in the true value. Under H_0, that is, when $b = 0$, the empirical standard deviations and the sizes are very close to the true values, $\sigma_{1,n}$ and α, respectively. The power increases substantially across the b-values. In addition, the empirical performance of the test and the estimation of b improved significantly as n increases, that is, the mse reduces and the power increases.

Table 2. Estimates, size, and power for the LDR method at 5% significance level for Gaussian innovations, $r = 3$

		$b = 0$	$b = 0.1$	$b = 0.2$	$b = 0.5$	$b = 0.7$	$b = 1$
$n = 100$	mean	−0.0035	0.0805	0.1477	0.3559	0.4781	0.6378
$\sigma_{3,n} = 0.1534$	sd	0.1572	0.1613	0.1625	0.1639	0.1718	0.1776
	mse	0.0247	0.0264	0.0291	0.0476	0.0787	0.1627
	rejection	5.71	14.06	25.74	73.94	91.09	98.57
$n = 500$	mean	0.0010	0.0865	0.1682	0.4041	0.5499	0.7206
$\sigma_{3,n} = 0.0830$	sd	0.0882	0.0881	0.0877	0.0903	0.0957	0.1185
	mse	0.0078	0.0079	0.0087	0.0173	0.0317	0.0921
	rejection	6.69	27.71	63.74	99.89	100	100
$n = 1000$	mean	−0.0013	0.0894	0.1741	0.4255	0.5738	0.7520
$\sigma_{3,n} = 0.0640$	sd	0.0682	0.0670	0.0677	0.0683	0.0740	0.1081
	mse	0.0047	0.0046	0.0052	0.0102	0.0214	0.0732
	rejection	5.97	41.46	85.71	100	100	100

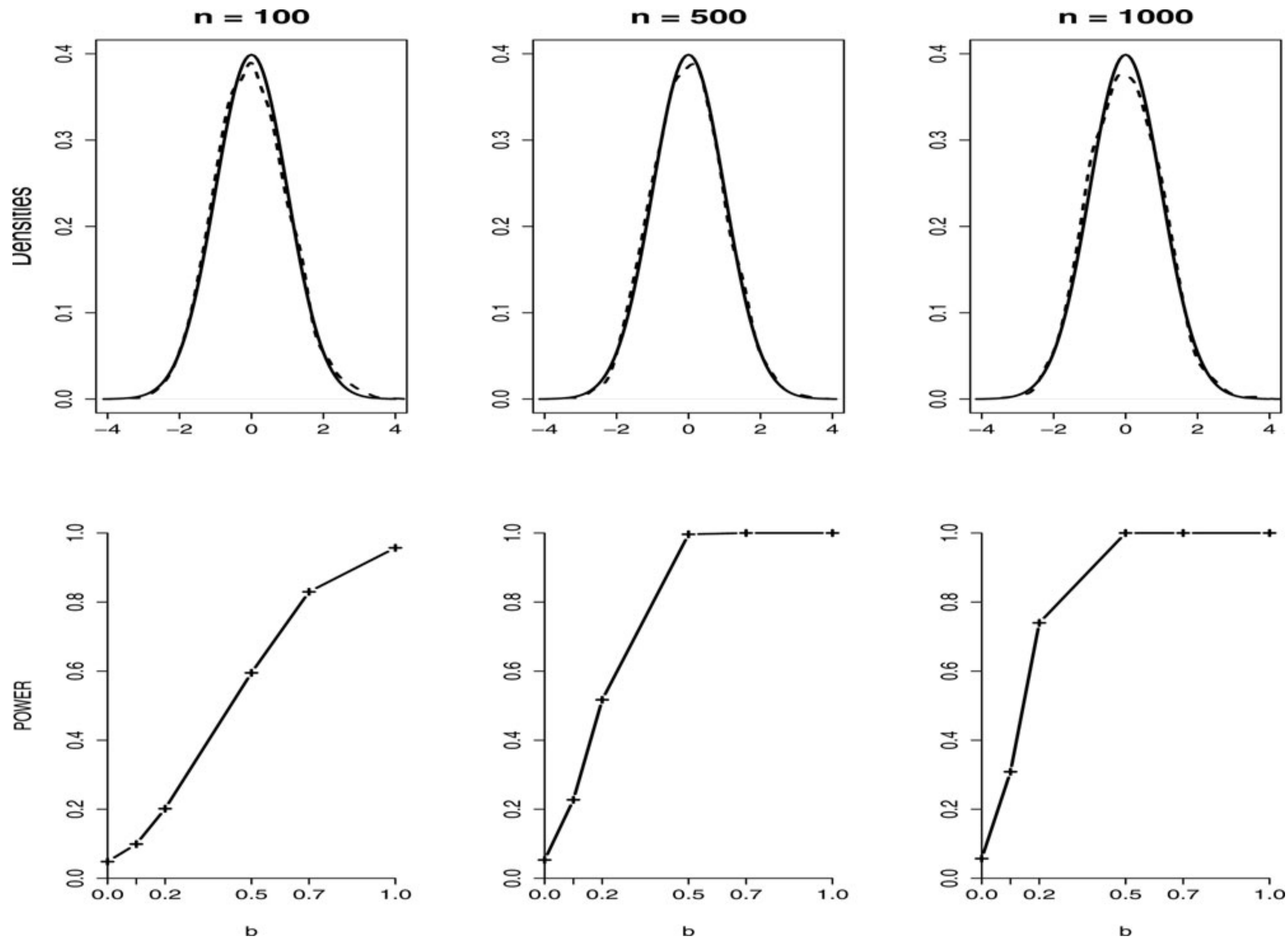

Figure 1. Empirical densities of standardized statistics and power of LDR at 5% significance level and $r = 1$.

As previously discussed, increasing r will produce a smoother estimate of the matrix in Equation (3.2) and this delivers estimates with larger negative bias and substantial smaller standard deviation (sd). This can be seen in Table 2 when $r = 3$. The mse also decreases for small b-values, while the situation is reversed when b is close and equal to one (when $b = 1$ the series are cointegrated). In this range, the mse is more pronounced by the bias, which increases substantially. The power increases up to 42% ($b = 0.1$ and $n = 100$) compared with the results given in Table 1. These empirical findings corroborate the asymptotic results given in Theorem 1.

Figure 1 plots the "*t-like*" densities (dashed lines) for the LDR ($r = 1$) together with the standard Gaussian density function (solid lines). As expected, the density of the LDR statistic is very close to the standard Gaussian density, even for $n = 100$. Figure 1 also shows the empirical power of the noncointegration test for the three sample sizes, where it can be seen that the sizes of the test are always close to the 5% level.

The above investigation was also carried out in the context of non-Gaussian innovations, that is: (1) Student-t distribution with 3 degrees of freedom to handle heavy-tailed noises, (2) chi-squared with 1 degree of freedom, subtracted by 1 to obtain a zero-mean and asymmetrical innovation, and (3) uniform distribution in the interval $[-3, 3]$. The performance of the method, for $r = 1$, presented similar conclusions to the Gaussian case. Table 3 displays the results for $n = 100$. Other simulations are available upon request. Even for a small sample size, it can be seen that the performance of the method is very accurate. This empirical evidence also supports the asymptotic results discussed previously.

The LDR test is now compared with the residual-based tests given in Dittmann (2000) and this is displayed in Table 4. The standard tests are: GPH, Lagrange multiplier (LM) (see Lobato and Robinson 1998 for details), modified rescaled range (MRR) (see Lo 1991 for details), Phillips-Perron ρ-test (PP_ρ), Phillips-Perron t-test (PP_t), and augmented Dickey-Fuller (ADF) (see Hamilton 1994 for details of Phillips-Perron ρ-test, Phillips-Perron t-test, and augmented Dickey-Fuller test). To make this comparison, the same experiments conducted by Dittmann (2000) were also used here to obtain the empirical size and the power of the LDR test.

In general, it can be observed that LDR tests play an intermediate role when compared to other tests. When sizes are evaluated, that is, when the simulated series are ARIMA(1,1,0) and ARIMA(0,1,1) models, LDR completely dominates the frequency domain GPH and LM tests by presenting less oversized significance. Only for few cases, for example, $r = 3$ and $n = 100$, the LDR method is outperformed by the GPH.

Compared to the time domain tests, in most of the cases, LDR displayed better sizes than PP_ρ and PP_t, but it is outperformed by the ADF and MRR tests, although the latter is more conservative and frequently underestimates the real size.

It can also be concluded that the LDR with $r = 1$ presents better results than $r = 3$. Similar to what happens to the methods discussed in Dittmann (2000), the LDR also presents large sizes for MA(1) innovations compared to the AR(1) case. The ADF and MRR tests display the best size performance among the evaluated tests. However, in general, they have less power against fractionally alternative, especially when d is close to the null hypothesis of noncointegration ($d = 1$). In this range and small n, LDR($r = 3$) test is more powerful than the other ones.

To end the empirical investigation, the proposed LDR method is now compared with the log coherence regression (LCR) suggested by Velasco (2003) intending to evaluate their performance across different values of β. Based on a regression of logged squared coherence between the pair of series $(X_{1,t}, X_{2,t})'$ on logged Fourier frequencies, Velasco (2003) proposed a method to estimate the parameter b and to test the null of cointegration.

Note that the LCR method is built for a nondifferentiated vector $(X_{1,t}, X_{2,t})'$ where $X_{i,t} \sim I(d)$, $i \in \{1, 2\}$, with $d \in (0, 1.5)$. In the context of nonstationarity, Velasco (2003) suggested to use data tapering for controlling the periodogram bias. He showed the consistency of his estimator when $0 < b \leq d$, that is, when the vector are cointegrated. Despite the fact that the LCR differs from LDR in terms of the null hypothesis, both estimates the same parameter b, which makes the comparison meaningful.

To evaluate the empirical performance of the LDR and LCR methods over different β, a Monte Carlo experiment was conducted with CI(1, 1) and β varying in the range $\{1, 0.5, 0.125, 0.065\}$. Equation (2.1) was simulated with innovations following a Gaussian white-noise process with $\sigma_\eta^2 = 1$, $\sigma_{w_1}^2 = 1$, $\sigma_{w_2}^2 = 0$, and $\beta_2 = 1$ (which makes $X_{2,t} = T_t$ and $\mathrm{cov}(\varepsilon_t, X_{2,t}) = 0$). In all replications the sample size was fixed at $n = 256$ to keep comparability to the models simulated by Velasco (2003).

The bandwidth for LCR was fixed at $m = 36$ since this value has displayed the best results for the empirical mean squared error, for most models considered in Velasco (2003), while, for the LDR method, the bandwidth was fixed at $m = n^{0.7}$ and $r = 1$. Table 5 displays the results. Although LCR asymptotically does not depend on β (see Velasco 2003), this parameter can be influential on small samples. The bias and the mean squared errors of the LCR estimates increase with decreasing β. On the other hand, the LDR is unaffected by the choice of β and showed a better performance in terms of the mean squared error.

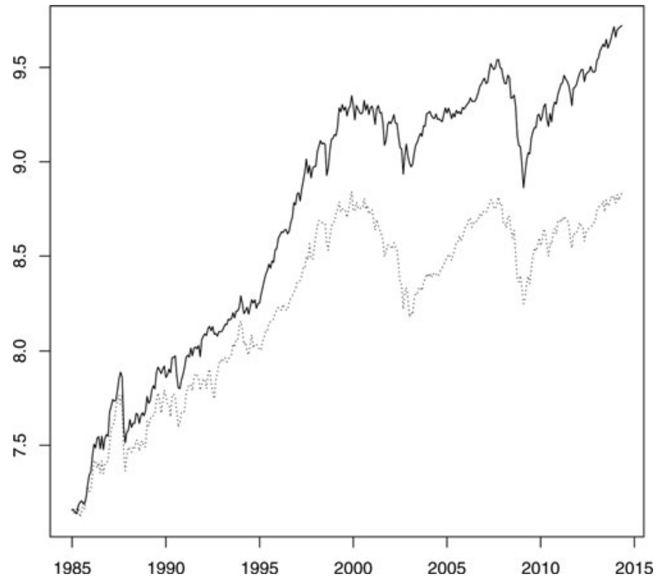

Figure 2. Logarithm of stock values, Dow Jones (solid) and FTSE 100 (dotted).

5. APPLICATION

The LDR estimator was applied to monthly observations of logged stock values for United States markets (Dow Jones Industrial Average Index - DJ) and United Kingdom markets (Financial Times Stock Exchange 100 - FTSE). Data ranges from January 1985 to May 2014. Looking at Figure 2 it can be seen that stock values in US and UK markets seem to share some long-run relationship.

First, GPH was employed in both series to estimate d using two different values of bandwidth $n^{0.5}$ and $n^{0.7}$. For the DJ series, the estimated values of d were 0.9281 (sd = 0.1942) and 0.9878 (sd = 0.0941) for $n^{0.5}$ and $n^{0.7}$, respectively, while, for the FTSE series, estimated values of d were 0.9490 (sd =

Table 3. Estimates, size, and power of LDR for non-Gaussian innovations, $r = 1$

		$b = 0$	$b = 0.1$	$b = 0.2$	$b = 0.5$	$b = 0.7$	$b = 1$
$t_{(3)}$	mean	−0.0001	0.1035	0.1918	0.4799	0.6451	0.8795
	sd	0.2389	0.2350	0.2325	0.2396	0.2540	0.2699
	mse	0.0570	0.0552	0.0541	0.0578	0.0675	0.0873
	rejection	5.06	10.37	17.49	61.86	84.00	96.26
$\chi^2_{(1)} - 1$	mean	0.0020	0.0992	0.2019	0.4753	0.6474	0.8846
	sd	0.2365	0.2315	0.2400	0.2482	0.2500	0.2745
	mse	0.0559	0.0536	0.0576	0.0622	0.0653	0.0887
	rejection	4.89	9.37	19.34	61.29	83.94	95.74
Unif	mean	0.0014	0.1000	0.2011	0.4790	0.6459	0.8469
	sd	0.2487	0.2413	0.2440	0.2495	0.2526	0.2596
	mse	0.0618	0.0582	0.0595	0.0627	0.0667	0.0908
	rejection	5.57	11.49	19.63	61.43	83.74	96.17

Table 4. Size and power comparison at 5% significance level

		Power					Size									
		I(d) processes with $d =$					ARIMA(1,1,0) with $\phi =$					ARIMA(0,1,1) with $\theta =$				
Sample size	Test	0.1	0.3	0.5	0.7	0.9	−0.474	−0.412	−0.333	−0.231	−0.091	0.718	0.525	0.382	0.245	0.092
100	GPH	98.93	97.05	84.02	45.15	11.11	10.15	10.53	10.12	8.56	6.85	86.37	46.98	24.98	13.18	6.71
	LM	99.89	99.25	92.96	60.92	13.35	16.8	17.48	16.06	13.65	8.13	96.53	71.84	44.08	22.43	8.68
	MRR	15.32	26.15	30.82	24.19	9.58	2.20	3.16	4.27	5.33	5.87	5.34	5.98	6.08	6.60	5.78
	PP_ρ	100	100	96.26	52.57	11.22	23.85	20.06	14.70	10.00	6.54	89.32	51.25	25.96	13.84	6.86
	PP_t	100	100	95.28	48.83	10.45	23.82	20.01	14.78	9.79	6.42	89.32	50.77	25.39	13.17	6.68
	ADF	98.33	93.44	70.88	36.33	9.46	4.73	5.82	6.72	6.83	5.73	41.85	17.77	11.06	9.06	5.74
	$LDR_{r=1}$	93.59	82.47	60.53	30.44	10.38	9.82	8.98	8.59	7.64	6.13	72.94	36.53	19.88	10.96	6.46
	$LDR_{r=3}$	98.54	95.87	84.08	55.39	16.75	15.38	13.75	13.25	10.56	7.70	90.43	60.74	34.36	17.22	8.14
250	GPH	99.99	99.99	99.78	85.07	18.63	10.43	10.55	10.88	9.88	7.57	96.77	61.09	31.11	16.39	8.01
	LM	100	100	99.96	94.4	25.25	14.83	15.14	15.07	13.88	8.97	99.26	82.4	51.22	26.86	10.02
	MRR	61.49	72.15	68.09	46.55	13.91	2.73	3.32	4.51	5.21	6.10	8.83	5.46	5.85	6.04	5.75
	PP_ρ	100	100	99.98	78.65	15.36	20.66	16.18	11.95	9.49	6.58	89.39	47.19	22.2	11.4	6.88
	PP_t	100	100	99.94	75.98	13.84	20.66	15.79	11.72	9.16	6.18	89.22	46.66	21.55	10.87	6.52
	ADF	99.97	99.48	89.75	50.78	12.30	4.59	4.54	5.20	5.76	5.89	23.96	9.89	7.32	6.00	5.93
	$LDR_{r=1}$	99.90	99.36	93.28	58.81	15.06	8.67	8.16	7.93	7.10	5.84	88.43	43.18	20.75	10.8	6.46
	$LDR_{r=3}$	99.84	99.71	97.91	76.93	20.89	10.70	9.99	9.73	8.84	6.86	96.40	56.18	27.74	14.24	7.32
500	GPH	100	100	100	98.38	30.49	10.18	9.69	9.81	9.41	7.62	93.98	68.8	34.95	17.64	8.55
	LM	100	100	100	99.77	41.43	13.6	13.04	13.8	12.98	10.2	99.62	86.5	55.26	28.7	11.21
	MRR	95.61	97.82	94.46	68.75	17.77	3.41	4.22	4.56	5.27	5.8	13.58	6.50	5.75	5.98	5.58
	PP_ρ	100	100	100	90.16	19.13	17.95	13.96	10.21	7.73	5.75	87.16	41.21	18.9	9.8	6.47
	PP_t	100	100	100	88.3	16.82	17.17	13.23	9.91	7.14	5.41	86.81	40.59	18.15	9.32	6.22
	ADF	100	99.97	97.81	65.72	14.45	4.44	4.75	5.02	4.64	5.03	16.34	8.09	6.38	5.94	5.6
	$LDR_{r=1}$	100	99.99	99.6	82.56	21.1	7.56	7.64	7.34	6.91	5.67	92.81	42.08	18.71	10.32	6.00
	$LDR_{r=3}$	100	100	99.89	92.43	27.62	9.17	8.41	8.9	7.44	6.68	97.03	50.79	22.92	11.79	6.86
1000	GPH	100	100	100	99.99	50.39	9.79	9.65	9.98	9.72	7.98	98.91	75.47	38.43	19.31	8.90
	LM	100	100	100	100	66.16	13.00	12.78	13.2	12.93	11.12	99.66	88.94	59.45	60.51	12.79
	MRR	99.91	99.98	99.79	88.64	23.93	4.11	4.89	5.19	5.70	5.74	16.92	6.82	6.07	6.11	5.88
	PP_ρ	100	100	100	96.29	24.24	15.12	11.23	8.97	7.32	6.03	84.51	35.51	17.93	8.47	6.00
	PP_t	100	100	100	95.46	21.54	14.55	10.66	8.55	7.00	6.02	84	34.66	14.21	8.29	5.96
	ADF	100	100	99.76	79.21	17.95	5.05	4.79	4.56	4.85	5.43	12.88	6.85	5.82	5.67	5.40
	$LDR_{r=1}$	100	100	100	96.4	29.95	7.29	7.03	6.01	6.43	5.90	94.71	39.67	16.69	9.38	5.98
	$LDR_{r=3}$	100	100	100	99.17	38.86	7.71	7.60	7.89	7.26	6.32	97.90	47.69	19.97	10.83	6.26

NOTE: Values from GPH, LM, MRR, PP_ρ, PP_t, and ADF are from Table A3, p. 638 by Dittmann (2000).

0.1941) and 1.0181 (sd = 0.0941) for $n^{0.5}$ and $n^{0.7}$, respectively. These results were used to calculate the *t-like* statistic for the GPH unit root test (see Santander, Reisen, and Abraham 2003) in which the null hypothesis is H_0 : *the series is I(1)* versus H_1 : *the series is I(d)*, $d < 1$. In addition, the ADF and PP tests were also implemented. From Table 6 it can be seen that these tests suggest that both series have a unit root.

Table 5. Robustness properties of the methods to β variations

β	Statistic	$\hat{b}_{\mathrm{LCR}}$	$\hat{b}_1$
	mean	0.8399	0.8505
1	sd	0.2195	0.1882
	mse	0.0738	0.0578
	mean	0.6718	0.8531
0.5	sd	0.1991	0.1938
	mse	0.1474	0.0591
	mean	0.2190	0.8570
0.125	sd	0.1550	0.1867
	mse	0.6341	0.0553
	mean	0.0814	0.8529
0.065	sd	0.1256	0.1844
	mse	0.8596	0.0556

The next step was to test the null of noncointegration using LDR, GPH, and EG. (Critical values for GPH test can be found

Table 6. Values of unit root test statistic and critical values

	GPH[a] ($n^{0.5}$)	GPH[a] ($n^{0.7}$)	ADF	PP
DJ	−0.3702	−0.1297	2.2928	−1.8714
FTSE	−0.2628	0.1923	1.5746	−2.0234
Critical values ($\alpha = 5\%$)	−1.61	−1.58	−1.95	−2.87
Critical values ($\alpha = 10\%$)	−1.16	−1.18	−1.62	−2.57

NOTE: Critical values for GPH test can be found in Santander, Reisen, and Abraham (2003).

Table 7. Estimates for b and test statistic for noncointegration between DJ and FTSE

	LDR($r=1$)		GPH[a b]		EG
Bandwidth	$n^{0.5}$	$n^{0.7}$	$n^{0.5}$	$n^{0.7}$	—
$\hat{b}$	−0.14	0.02	−0.14	0.03	—
Standard deviations	0.29	0.13	0.19	0.09	—
t-like statistic	−0.49	0.18	−0.74	0.32	−2.27
Critical values ($\alpha = 5\%$)	1.68	1.68	2.24	2.11	−2.87
Critical values ($\alpha = 10\%$)	1.29	1.29	1.78	1.67	−2.57

NOTE: [a]To keep comparability with LDR, critical values were adjusted to test $H_0 : 1 - d_{\text{res}} = 0$ versus $H_1 : 1 - d_{\text{res}} > 0$.
[b]Critical values for GPH test can be found in Santander, Reisen, and Abraham (2003).

in Santander, Reisen, and Abraham (2003). For EG test see MacKinnon (1991).) The last two tests were performed on the residuals of the regression equation for DJ and FTSE. Estimated values for parameter b using LDR ($r = 1$) and GPH (which was obtained using $1 - \hat{d}_{\text{res}}$, where $\hat{d}_{\text{res}}$ is the estimated d for regression residuals) can be seen in Table 7. LDR ($r = 3$) gave similar conclusion to the case of $r = 1$ and this is available upon request. Note that the EG test does not estimate the parameter b.

The results in Table 7 indicated that the series are not cointegrated, which is in accordance with the majority of the empirical evidence discussed in the literature, that is, most of the international stock prices analyzed are not pairwise cointegrated (see Aloy et al. 2013 or Kanas 1998).

6. CONCLUSION

The present work investigates the properties of the determinant of the spectral density matrix close to the origin for bivariate cointegrated series and proposes a method to test the null hypothesis of noncointegration. Under the null hypothesis of noncointegration, expression of the bias and the variance of the estimator were derived and consistency was obtained. It was also established the asymptotic normality of the estimator under Gaussian and non-Gaussian innovations.

Monte Carlo simulations showed that the method presented here possesses, in general, the correct size and good power for moderate sample sizes, compared with some standard noncointegration tests given in the literature. In addition, the simulations also showed that the method is insensitive to the choice of the cointegration relationship (the β parameter). An application is given to show the usefulness of the proposed test. Future work encompasses the proposal of an M-robust noncointegration test, which is robust to outliers and heavy-tailed distribution.

APPENDIX: TECHNICAL LEMMAS AND PROOF OF THEOREM 1

Before proving the theorem, one should consider the following Lemmas:

Lemma A.1. The random matrix $\widehat{\mathbf{F}}_{\Delta^d,r}(\lambda_j)$ defined in Equation (3.2) is such that, $\widehat{\mathbf{F}}_{\Delta^d,r}(\lambda_j) \xrightarrow{d} W_2^c(2r+1, \mathbf{F}_{\Delta^d,r}(\lambda_j))$, as $n \to \infty$.

Proof. Mohanty and Pourahmadi (1996, p. 295) stated that if a bivariate process $\mathbf{\Upsilon}_t$ with representation $\mathbf{\Upsilon}_t = \sum_{k=0}^{\infty} \mathbf{\Xi}_k \mathbf{Z}_{t-k}$, where $\mathbf{Z}_t$'s are independent and identically distributed random vectors with zero mean, finite fourth moment and the entries of the matrices $\mathbf{\Xi}_k(i,j)$, $i,j = 1,2$ satisfies the condition $\sum_{k=0}^{\infty} \sqrt{k}|\mathbf{\Xi}_k(i,j)| < \infty$, then the matrix $\mathbf{I}_{n,\Upsilon}(\lambda_j) = \frac{1}{2\pi n}|\sum_{t=1}^{n} \mathbf{\Upsilon}_t e^{-\iota\lambda_j t}|^2 \xrightarrow{d} \mathbf{Y}\overline{\mathbf{Y}}$, where $\mathbf{Y} \sim N_2^c(0, \mathbf{F}_\Upsilon(\lambda))$, $\overline{\mathbf{Y}}$ is the transposed and conjugated matrix of $\mathbf{Y}$ and $\mathbf{F}_\Upsilon(\lambda)$ is the spectral density matrix of the process $\mathbf{\Upsilon}_t$. The series defined in Equation (2.5) satisfies the above conditions. First, one should note that if $b = 0$ the vector $(\Delta^d X_{1,t}, \Delta^d X_{2,t})'$ will follow an ARMA(p,q) process and the above conditions are completely fulfilled. In this context, each random variable is defined by $\mathbf{I}_{n,\Delta^d}(\lambda_j) = \frac{1}{2\pi n}|\sum_{t=1}^{n} \Delta^d \mathbf{X}_t e^{-\iota\lambda_j t}|^2 \xrightarrow{d} \mathbf{Y}\overline{\mathbf{Y}}$. Following Brillinger (1981, pp. 305), the sum $\widehat{\mathbf{F}}_{\Delta^d,r}(\lambda_j) = \frac{1}{(2r+1)}\sum_{\upsilon=j-r}^{j+r} \mathbf{I}_{n,\Delta^d}(\lambda_\upsilon) \xrightarrow{d} W_2^c(2r+1, \mathbf{F}_{\Delta^d,r}(\lambda_j))$. □

Lemma A.2. Following Lemma A.1, as $n \to \infty$, the random variable $\mathscr{D}_j = \ln[4(2r+1)^2 \widehat{D}_r(\lambda_j)/D_r(\lambda_j)]$ is such that $\mathscr{D}_j \xrightarrow{d} \mathscr{W}$, where $\mathscr{W} \stackrel{d}{=} ln(\chi^2_{(4r+2)}\chi^2_{(4r)})$. $\chi^2_{(4r+2)}$ and $\chi^2_{(4r)}$ are chi-squared random variables with $(4r+2)$ and $4r$ degrees of freedom, respectively. Therefore, the characteristic function of $\mathscr{D}_j$, say $\phi_{\mathscr{D}_j}(t)$, is such that $\phi_{\mathscr{D}_j}(t) \to \phi_{\mathscr{W}}(t) = 4^{\iota t}\frac{\Gamma(2r+1+\iota t)\Gamma(2r+\iota t)}{\Gamma(2r+1)\Gamma(2r)}$, as $n \to \infty$.

Proof. See Goodman (1963) □

Remark A.1. Since $\mathbf{E}[\mathscr{D}_j] \to \mathbf{E}[\mathscr{W}]$, $\mathbf{V}[\mathscr{D}_j] \to \mathbf{V}[\mathscr{W}]$ and $\mathbf{E}[\mathscr{W}^p] = (-\iota)^p \phi_{\mathscr{W}}^{(p)}(0)$, then $\mathbf{E}[\mathscr{D}_j] \to (\psi^{(0)}(2r+1) + \psi^{(0)}(2r) + \ln 4)$ and $\mathbf{V}[\mathscr{D}_j] \to \psi^{(1)}(2r+1) + \psi^{(1)}(2r)$, as $n \to \infty$. To accomplish the above results, one just needs to note that

$$\phi_{\mathscr{W}}^{(1)}(0) = (\psi^{(0)}(2r+1) + \psi^{(0)}(2r) + \ln 4)\iota$$

and

$$\phi_{\mathscr{W}}^{(2)}(0) = -\big[2\ln 4 + \psi^{(0)}(2r)^2 + 2\psi^{(0)}(2r+1)\big(\ln 4 + \psi^{(0)}(2r+1)\big) + \psi^{(0)}(2r+1)\big(\ln 16 + \psi^{(0)}(2r+1)\big) + \psi^{(1)}(2r) + \psi^{(1)}(2r+1)\big].$$

Lemma A.3. $\text{cov}\{\ln \widehat{D}_r(\lambda_j)/D_r(\lambda_j), \ln \widehat{D}_r(\lambda_k)/D_r(\lambda_k)\} = O(1/n)$, $\lambda_j \neq \lambda_k$.

Proof. Under the assumptions of Equations (2.2), (2.3a), and (2.3b), the periodogram matrix of $(X_{1,t}, X_{2,t})'$ has the property $\text{cov}\{I_{n,\Delta^d X_{s_1}\Delta^d X_{p_1}}(\lambda_j), I_{n,\Delta^d X_{s_2}\Delta^d X_{p_2}}(\lambda_k)\} = O(1/n)$, $\lambda_j \neq \lambda_k$, as $n \to \infty$, where $s_1, p_1, s_2, p_2 = 1, 2$ (see Theorem 11.7.1 from Brockwell and Davis 2013). Since the quantities in $\widehat{\mathbf{F}}_{\Delta^d,r}$ are calculated with nonoverlapping Fourier frequencies then $\text{cov}\{\ln \widehat{D}_r(\lambda_j)/D_r(\lambda_j), \ln \widehat{D}_r(\lambda_k)/D_r(\lambda_k)\} = O(1/n)$. □

Lemma A.4. Under Assumption 1, as $n \to \infty$, $\sum_{j=(r+1)}^{m} \tilde{Z}_j^2 = m + o(m)$, where $Z_j = \ln(2 - 2\cos\lambda_j)$, $\tilde{Z}_j = Z_j - \bar{Z}$, $j = 1, \ldots, m$ and $\bar{Z}$ is the mean of Z_j.

Proof. See (Hurvich and Beltrao 1994, p. 301). □

Lemma A.5. Under Assumption 1, $\tilde{Z}_j = O(\ln m)$, as $n \to \infty$.

Proof. Hurvich and Beltrao (1994) stated that $\tilde{Z}_j = \ln j - \frac{1}{m}\ln m! + o(1)$, where the first term, $\ln j = O(\ln m)\ \forall j = 1, \ldots, m$. The following limit,

$$\lim_{m\to\infty} \frac{1}{m} \ln m!$$

is an indeterminate form of the type $\frac{\infty}{\infty}$. By the L'Hopital's rule, the previous limit can be rewritten as: $\lim_{m\to\infty} \psi^{(0)}(m+1)$. Equation (6.3.18) from Abramowitz and Stegun (1972, p. 259) states that: $\psi^{(0)}(z) \sim$

$\ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{256z^6} + \cdots$ as $z \to \infty$. As a result, $\psi^{(0)}(z) = O(\ln z)$ and, consequently, $\tilde{Z}_j = O(\ln m)$. □

Lemma A.6. Let $\psi^{(1)}(u)$ be the Polygamma function of order 1, that is, $\psi^{(1)}(u) = \frac{d^2 \ln \Gamma(u)}{du^2}$. Then, for $z \in \mathbb{N}^*$, $\lim_{z\to\infty} \psi^{(1)}(z) = 0$.

Proof. Jeffrey and Zwillinger (2007, p. 905) stated that for any $z \in \mathbb{N}^*$, $\psi^{(1)}(z) = \frac{\pi^2}{6} - \sum_{k=1}^{z-1} \frac{1}{k^2}$. Thus, $\lim_{z\to\infty} \psi^{(1)}(z) = \frac{\pi^2}{6} - \lim_{z\to\infty} \sum_{k=1}^{z-1} \frac{1}{k^2}$. In addition, Jeffrey and Zwillinger (2007, p. 8) stated that $\lim_{z\to\infty} \sum_{k=1}^{z-1} \frac{1}{k^2} = \frac{\pi^2}{6}$. As a result, $\lim_{z\to\infty} \psi^{(1)}(z) = 0$. □

Proof of Theorem 1. The proof of the theorem is split into three parts. Before proving each part, one should note that, by replacing Equation (3.3) into Equation (3.4), the LDR estimator can be rewritten as

$$\hat{b} - b = \frac{\sum_{j=(r+1)}^{m} \tilde{Z}_j c(r, j)}{\sum_{j=(r+1)}^{m} \tilde{Z}_j^{\,2}} + \frac{\sum_{j=(r+1)}^{m} \tilde{Z}_j \ln G(\lambda_j)}{\sum_{j=(r+1)}^{m} \tilde{Z}_j^{\,2}} + \frac{\sum_{j=(r+1)}^{m} \tilde{Z}_j \zeta_j}{\sum_{j=(r+1)}^{m} \tilde{Z}_j^{\,2}}, \tag{A.1}$$

□

Proof of part (a): By Assumption 1, the bias of $\hat{b}$ is given by

$$\mathbf{E}\left[\hat{b} - b\right] = \frac{\sum_{j=l}^{m} \tilde{Z}_j c(r, j)}{\sum_{j=l}^{m} \tilde{Z}_j^{\,2}} + \frac{\sum_{j=l}^{m} \tilde{Z}_j \ln G(\lambda_j)}{\sum_{j=l}^{m} \tilde{Z}_j^{\,2}} + \frac{\sum_{j=l}^{m} \tilde{Z}_j \mathbf{E}[\zeta_j]}{\sum_{j=l}^{m} \tilde{Z}_j^{\,2}}. \tag{A.2}$$

From Lemma A.2, $c(r, j) \to \psi^{(0)}(2r+1) + \psi^{(0)}(2r) - \ln(2r+1)^2$, as $n \to \infty$. Since the series $\sum_{j=l}^{m} \tilde{Z}_j - \bar{Z}_j = 0$, $\forall m$, $|\ln c(r, j)| < \infty$ and $|\ln G(\lambda_j)| < \infty$, $\forall j$, then both series $\sum_{j=l}^{m} \tilde{Z}_j c(r, j)$ and $\sum_{j=l}^{m} \tilde{Z}_j \ln G(\lambda_j)$ converges (see Jeffrey and Zwillinger 2007, p. 7). By Lemma A.4, $\sum_{j=l}^{m} \tilde{Z}_j^{\,2} = m + o(m)$. Then,

$$\frac{\sum_{j=l}^{m} \tilde{Z}_j c(r, j)}{\sum_{j=l}^{m} \tilde{Z}_j^{\,2}} \to 0 \text{ and } \frac{\sum_{j=l}^{m} \tilde{Z}_j \ln G(\lambda_j)}{\sum_{j=l}^{m} \tilde{Z}_j^{\,2}} \to 0, \text{ as } n \to \infty. \tag{A.3}$$

By construction, $\mathbf{E}[\zeta_j] = 0$, $\forall j$. Therefore, $\frac{\sum_{j=l}^{m} \tilde{Z}_j \mathbf{E}_{H_0}[\zeta_j]}{\sum_{j=l}^{m} \tilde{Z}_j^{\,2}} = 0$, $\forall j$. □

Proof of part (b): By Equation (A.1), the variance of $\hat{b}$ is given by

$$\mathbf{V}\left[\hat{b}\right] = \frac{\sum_{j=l}^{m} \tilde{Z}_j^2 \mathbf{V}[\zeta_j]}{\left(\sum_{j=l}^{m} \tilde{Z}_j^{\,2}\right)^2} + \frac{2\sum_{k=l}^{m} \sum_{j=k+l}^{m} \tilde{Z}_k \tilde{Z}_j \mathbf{cov}\left[\zeta_j, \zeta_k\right]}{\left(\sum_{j=l}^{m} \tilde{Z}_j^{\,2}\right)^2}. \tag{A.4}$$

By Remark A.1, $\mathbf{V}[\zeta_j] \to \psi^{(1)}(2r+1) + \psi^{(1)}(2r)$, as $n \to \infty$. Then, by Lemma A.4, $\sum_{j=l}^{m} \tilde{Z}_j^2 \mathbf{V}[\zeta_j] = [m + o(m)][\psi^{(1)}(2r+1) + \psi^{(1)}(2r)]$. Therefore,

$$\frac{\sum_{j=l}^{m} \tilde{Z}_j^2 \mathbf{V}[\zeta_j]}{\left(\sum_{j=l}^{m} \tilde{Z}_j^{\,2}\right)^2} = \frac{\psi^{(1)}(2r+1) + \psi^{(1)}(2r)}{[m + o(m)]} \to 0, \text{ as } n \to \infty. \tag{A.5}$$

In addition, by Lemmas A.3 and A.5, the term $2\sum_{k=l}^{m} \sum_{j=k+l}^{m} \tilde{Z}_k \tilde{Z}_j \mathbf{cov}[\zeta_j, \zeta_k] \le O(\frac{m^2 \ln^2 m}{n})$. Therefore,

$$\frac{2\sum_{k=l}^{m} \sum_{j=k+l}^{m} \tilde{Z}_k \tilde{Z}_j \mathbf{cov}\left[\zeta_j, \zeta_k\right]}{\left(\sum_{j=l}^{m} \tilde{Z}_j^{\,2}\right)^2} \le O\left(\frac{\ln^2 m}{n}\right) \to 0, \text{ as } n \to \infty. \tag{A.6}$$

□

Proof of part (c): From parts (a) and (b) of Theorem 1, the quantity $\frac{\sum_{j=l}^{m} \tilde{Z}_j \zeta_j}{\sum_{j=l}^{m} \tilde{Z}_j^{\,2}} (\frac{m}{\psi^{(1)}(2r+1) + \psi^{(1)}(2r)})^{1/2}$ is a zero-mean and unit-variance random variable.

Consider now the sequence of random variables U_j, $j \le m$ that forms the following triangular array:

$$\begin{aligned} &U_1; \\ &U_1, U_2; \\ &\cdots\cdots\cdots; \\ &U_1, U_2 \cdots, U_m, \end{aligned} \tag{A.7}$$

where the random variables in each row of (A.8) are asymptotically independent and for each j,

$$\mathbf{E}\left\{U_j\right\} = 0, \tag{A.8}$$

and $\mathbf{V}\{U_j\} = \sigma_j^2$ such that $\sigma_j^2 < \infty$ and

$$s_m^2 = \sum_{j=1}^{m} \sigma_j^2 = 1. \tag{A.9}$$

Theorem 7.2.1 given in Chung (2001) states that, for a triangular array where the conditions established by Equations (A.8) and (A.9) hold, the sum $S_m = \sum_{j=1}^{m} U_j \xrightarrow{d} N(0, 1)$, as $m \to \infty$.

One should observe that term $\frac{\sum_{j=l}^{m} \tilde{Z}_j \zeta_j}{\sum_{j=l}^{m} \tilde{Z}_j^{\,2}} (\frac{m}{\psi^{(1)}(2r+1) + \psi^{(1)}(2r)})^{1/2}$, satisfies Equations (A.8) and (A.9). For all $j \le m$, let

$$U_j = \frac{\sqrt{m} \tilde{Z}_j \zeta_j}{\sum_{j=l}^{m} \tilde{Z}_j^{\,2} \sqrt{\left(\psi^{(1)}(2r+1) + \psi^{(1)}(2r)\right)}}. \tag{A.10}$$

By the definition of U_j one can see directly that $\mathbf{E}\{S_m\} = 0$, as $n \to \infty$. By Lemmas A.4 and A.5, $\mathbf{V}\{U_j\} = (\frac{\sqrt{m}\tilde{Z}_j}{\sum_{k=l}^{m} \tilde{Z}_k^{\,2}})^2 = O(\frac{\ln^2 m}{m})$. Therefore, $\mathbf{V}\{S_m\} = \mathbf{V}\{\sum_{j=1}^{m} U_j\} = \sum_{j=l}^{m} (\frac{\sqrt{m}\tilde{Z}_j}{\sum_{k=l}^{m} \tilde{Z}_k^{\,2}})^2 = 1$. As a result, $\sqrt{\frac{m}{\psi^{(1)}(2r+1) + \psi^{(1)}(2r)}}\hat{b} \xrightarrow{d} N(0, 1)$ as $n \to \infty$. □

ACKNOWLEDGMENTS

The work of Prof. Valderio Anselmo Reisen was partially supported by CNRS and CentraleSupélec, CNPq and FAPES. G.C. Franco was partially supported by CNPq-Brazil (grant number: 305646/2012-5), and also by FAPEMIG Foundation (grant number: PPM-00021-14). The results in this paper is part of the PhD thesis of the first author at UFMG under supervision of the second and third authors. The authors acknowledge the editor, the associate editor and the referees for their valuables discussions and comments.

[Received October 2014. Revised May 2016.]

REFERENCES

Abramowitz, M., and Stegun, I. A. (eds.) (1972), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (*National Bureau of Standards Applied Mathematics Series*, Vol. 55), Washington, DC: United States Government Printing Office. [8]

Aloy, M., Boutahar, M., Gente, K., and Péguin-Feissolle, A. (2013), "Long-Run Relationships Between International Stock Prices: Further Evidence From Fractional Cointegration Tests," *Applied Economics*, 45, 817–828. [8]

Baillie, R. T., and Bollerslev, T. (1994), "Cointegration, Fractional Cointegration, and Exchange Rate Dynamics," *The Journal of Finance*, 49, 737–745. [1]

Brillinger, D. R. (1981), *Time Series: Data Analysis and Theory* (*Classics in Applied Mathematics*, Vol. 36), Philadelphia, PA: SIAM. [8]

Brockwell, P. J., and Davis, R. A. (2013), *Time Series: Theory and Methods*, New York: Springer. [8]

Cheung, Y. W., and Lai, K. S. (1993), "A Fractional Cointegration Analysis of Purchasing Power Parity," *Journal of Business & Economic Statistics*, 11, 103–112. [1]

Chung, K. L. (2001), *A Course in Probability Theory*, San Diego: Academic Press. [9]

Cuestas, J. C., Gil-Alana, L. A., and Staehr, K. (2014), "Government Debt Dynamics and the Global Financial Crisis: Has Anything Changed in the EA12?" *Economics Letters*, 124, 64–66. [1]

Dittmann, I. (2000), "Residual-Based Tests For Fractional Cointegration: A Monte Carlo Study," *Journal of Time Series Analysis*, 21, 615–647. [2]

——— (2001), "Fractional Cointegration of Voting and Non-Voting Shares," *Applied Financial Economics*, 11, 321–332. [1]

Engle, R. F., and Granger, C. W. J. (1987), "Co-Integration and Error Correction: Representation, Estimation, and Testing," *Econometrica: Journal of the Econometric Society*, 55, 251–276. [1]

Geweke, J., and Porter-Hudak, S. (1983), "The Estimation and Application of Long Memory Time Series Models," *Journal of Time Series Analysis*, 4, 221–238. [3]

Goodman, N. (1963), "The Distribution of the Determinant of a Complex Wishart Distributed Matrix," *The Annals of Mathematical Statistics*, 34, 178–180. [8]

Granger, C. W. J. (1981), "Some Properties of Time Series Data and Their Use in Econometric Model Specification," *Journal of Econometrics*, 16, 121–130. [1]

Hamilton, J. D. (1994), *Time Series Analysis* (Vol. 2), Princeton: Princeton University Press. [5]

Hosking, J. R. (1981), "Fractional Differencing," *Biometrika*, 68, 165–176. [2]

Hurvich, C. M., and Beltrao, K. I. (1994), "Automatic Semiparametric Estimation of the Memory Parameter of a Long-Memory Time Series," *Journal of Time Series Analysis*, 15, 285–302. [8]

Jeffrey, A., and Zwillinger, D. (eds.) (2007), *Table of Integrals, Series, and Products*, San Diego: Academic Press. [9]

Johansen, S. (1991), "Estimation and Hypothesis Testing of Cointegration Vectors in Gaussian Vector Autoregressive Models," *Econometrica: Journal of the Econometric Society*, 59, 1551–1580. [1]

Kanas, A. (1998), "Linkages Between the US and European Equity Markets: Further Evidence From Cointegration Tests," *Applied Financial Economics*, 8, 607–614. [8]

Lo, A. W. (1991), "Long-Term Memory in Stock Market Prices," *Econometrica*, 59, 1279–1313. [5]

Lobato, I. N., and Robinson, P. M. (1998), "A Nonparametric Test for I(0)," *The Review of Economic Studies*, 65, 475–495. [5]

MacKinnon, J. G. (1991), "Critical Values for Cointegration Tests," in *Long-Run Economic Relationships: Readings in Cointegration*, eds. R. F. Engle, and C. W. J. Granger, Oxford: Oxford University Press, pp. 267–276. [8]

Marinucci, D., and Robinson, P. M. (2001), "Semiparametric Fractional Cointegration Analysis," *Journal of Econometrics*, 105, 225–247. [1]

McHale, I., and Peel, D. (2010), "Habit and Long Memory in UK Lottery Sales," *Economics Letters*, 109, 7–10. [1]

Mohanty, R., and Pourahmadi, M. (1996), "Estimation of the Generalized Prediction Error Variance of a Multiple Time Series," *Journal of the American Statistical Association*, 91, 294–299. [8]

Nielsen, M. Ø. (2004), "Spectral Analysis of Fractionally Cointegrated Systems," *Economics Letters*, 83, 225–231. [3]

Phillips, P. C., and Ouliaris, S. (1988), "Testing for Cointegration Using Principal Components Methods," *Journal of Economic Dynamics and Control*, 12, 205–230. [1,3]

Priestley, M. B. (1981), *Spectral Analysis and Time Series*, London: Academic Press. [2]

Robinson, P. M. (1994), "Semiparametric Analysis of Long-Memory Time Series," *The Annals of Statistics*, 22, 515–539. [1]

Robinson, P. M., and Marinucci, D. (1998), "Semiparametric Frequency Domain Analysis of Fractional Cointegration," STICERD Econometrics Discussion Paper 348, London School of Economics. [2]

——— (2001), "Narrow-Band Analysis of Nonstationary Processes," *Annals of Statistics*, 29, 947–986. [1]

Robinson, P. M., and Yajima, Y. (2002), "Determination of Cointegrating Rank in Fractional Systems," *Journal of Econometrics*, 106, 217–241. [1]

Santander, L. A. M., Reisen, V. A., and Abraham, B. (2003), "Non-Cointegration Tests and Fractional ARFIMA Process," *Statistical Methods*, 5, 1–22. [7,8]

Velasco, C. (2003), "Nonparametric Frequency Domain Analysis of Nonstationary Multivariate Time Series," *Journal of Statistical Planning and Inference*, 116, 209–247. [1,2,6]