Weiss–Weinstein Bound for Data-Aided Carrier Estimation

Alexandre Renaux, *Student Member, IEEE*

***Abstract*—This letter investigates Bayesian bounds on the mean-square error (MSE) applied to a data-aided carrier estimation problem. The presented bounds are derived from a covariance inequality principle: the so-called Weiss and Weinstein family. These bounds are of utmost interest to find the fundamental MSE limits of an estimator, even for critical scenarios (low signal-to-noise ratio and/or low number of observations). In a data-aided carrier estimation problem, a closed-form expression of the Weiss–Weinstein bound (WWB) that is known to be the tightest bound of the Weiss and Weinstein family is given. A comparison with the maximum likelihood estimator and the other bounds of the Weiss and Weinstein family is given. The WWB is shown to be an efficient tool to approximate this estimator's MSE and to predict the well-known threshold effect.**

***Index Terms*—Carrier frequency estimation, estimators performance, Weiss and Weinstein family bounds.**

I. Introduction

MINIMAL bounds are generally used to find the ultimate estimation performance in term of mean-square error (MSE). Consequently, they are a useful benchmark in order to check the accuracy of an estimator. This topic is of interest in many signal processing fields such as spectral analysis, array processing, or digital communications.

Two kinds of lower bounds on the MSE have been derived depending on the parameters assumptions. When the unknown parameters are assumed to be deterministic, bounds such as the well-known Cramér–Rao bound , the Bhattacharyya bound [1], the Barankin bound [2], and the Abel bound [3] have been proposed. Some of these bounds have been widely used to predict the threshold phenomena of an estimator (see, e.g., [4]–[6]). Indeed, in an estimation problem, when the parameters have a finite support, it appears as three distinct MSE areas [7]. At a high number of observations and/or high signal-to-noise ratio (SNR), the estimator MSE is small and the area is called asymptotic. When the scenario becomes critical, i.e., when the number of observations and/or the SNR decrease, the estimator MSE increases dramatically due to the outliers effect and the area is called threshold area. Finally, when the number of observations and/or the SNR is weak, the estimator is hugely corrupted by the noise and becomes a quasi-uniform random variable on the parameters support. This last area is called the no information region. The Cramér–Rao bound is only used in the asymptotic area and is not able to handle the threshold phenomena (i.e., when the performance breakdown appears). On the other hand, bounds such as Barankin or Abel bound are able to predict this threshold. However, the drawback of the deterministic bounds is that they do not take into account the support of the parameters, and consequently, they cannot give the fundamental limits of an estimator in terms of MSE over all the three areas.

To fill this lack, when the parameters are assumed to be random, other bounds have been derived: the so-called Bayesian bounds. These bounds take into account the support of the parameters throughout an *a priori* probability density function. Consequently, these bounds can give the fundamental limits in terms of MSE over all the three aforementioned areas and have also been widely used in the literature (see, e.g., [8]–[10]).

The Bayesian bounds can be decomposed into two subfamilies: the Ziv and Zakaï family, where the MSE is connected to the error probability of a binary hypothesis testing problem, and the Weiss and Weinstein family, which derives from a ”covariance inequality principle.” The Ziv and Zakaï family includes the Ziv–Zakaï bound [11], the Bellini–Tartara bound [12], the Chazan–Zakaï–Ziv bound [13], and the Bell–Steinberg–Ephraim–VanTrees bound [14]. The Weiss and Weinstein family includes the Bayesian Cramér–Rao bound (BCRB) [7], the Bayesian Bhattacharyya bound [7], the Bobrovsky–Zakaï bound (BZB) [15], the Bobrovsky–MayerWolf–Zakaï bound [16], the Bayesian Abel bound (BAB) [17], and the Weiss–Weinstein bound (WWB) [18]. This letter focuses on this latest family and more particularly on the Weiss–Weinstein bound, which is known to be the tightest bound of the Weiss and Weinstein family [19].

The goal here is to derive a closed-form expression of the WWB bound in the context of data-aided carrier estimation problem. Indeed, in digital communications, one of the key points is to estimate the carrier frequency in order to accurately recover the transmitted symbols. In practice, this estimation is performed by using a short training sequence of symbols that is known (data-aided) by the receiver. Some deterministic bounds have already been applied to this problem: e.g., the Barankin bound [20], and the Abel bound [21]. To the best of our knowledge, the bounds of the Weiss and Weinstein family have never been applied in this context. A comparison with the maximum likelihood (ML) estimator and the other bounds of the Weiss and Weinstein family is given. The WWB is shown to be an efficient tool to approximate these estimation MSEs and to predict the threshold effect.

II. Problem Setup

We consider a linearly modulated signal, obtained by applying a known (data-aided context) complex-valued data symbol sequence taken from a unit energy constellation to a square-root Nyquist transmit filter. This signal is transmitted over an additive white Gaussian noise (AGWN) channel. The resulting noisy signal is applied to a receiver filter, matched to the transmit filter. The receiver filter output signal is sampled

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The author is with the Ecole Normale Superieure of Cachan, SATIE Laboratory, 94235 Cachan, France (e-mail: renaux@satie.ens-cachan.fr).

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at the correct decision instants, which yields to the following observation model:

$$x_k = a_k e^{j(2\pi k\theta+\phi)} + n_k, \quad k = 0,\dots,N-1 \tag{1}$$

where x_k is the kth noisy observation. The observations are assumed to be independent. $\mathbf{a} = [a_0 \quad a_1 \quad \cdots \quad a_{N-1}]^T$ is the known data symbol sequence. $\{n_k\}$ is a sequence of i.i.d, circular, zero-mean complex Gaussian noise variable with a known variance σ^2. The SNR is equal to $1/\sigma^2$. ϕ is the carrier phase and is assumed to be known and compensated for (or that $\phi = 0$). The unknown real parameter $\theta \in \Theta = [-(1/2),(1/2)]$ corresponds to the carrier frequency offset. The carrier frequency offset is typically due to the transmitter and receiver oscillators drift and is a random variable with an *a priori* probability density function $p(\theta)$ assumed to be Gaussian with mean m and variance σ_θ^2

$$p(\theta) = \frac{1}{\sqrt{2\pi}\sigma_\theta} e^{-1/2\sigma_\theta^2(\theta-m)^2}. \tag{2}$$

Assuming independent observations, the likelihood of the vector $\mathbf{x} = [x_0 \quad x_1 \quad \cdots \quad x_{N-1}]^T$, with $\mathbf{x}\in\Omega$, is given by

$$\begin{aligned} p(\mathbf{x}|\theta) &= \prod_{k=0}^{N-1} p(x_k|\theta) \\ &= \frac{1}{(\pi\sigma^2)^N} e^{-1/\sigma^2\left(\|\mathbf{x}\|^2+\|\mathbf{a}\|^2-2\mathrm{Re}\left\{\sum_{k=0}^{N-1} x_k^* a_k e^{j2\pi k\theta}\right\}\right)}. \end{aligned} \tag{3}$$

Finally, the joint density probability function of the observations and of the parameter is denoted $p(\mathbf{x},\theta) = p(\mathbf{x}|\theta)p(\theta)$.

III. Weiss–Weinstein Bound

In this section, the WWB is derived for the data-aided carrier estimation problem, and an optimization in terms of computational cost is provided.

A. Weiss–Weinstein Bound Derivation

The WWB checks the following inequality:

$$\mathbb{E}_{\mathbf{x},\theta}\left[\left(\hat{\theta}(\mathbf{x})-\theta\right)^2\right] \geq \mathrm{WWB} \tag{4}$$

where $\mathbb{E}_{\mathbf{x},\theta}\left[\left(\hat{\theta}(\mathbf{x})-\theta\right)^2\right] = \int_\Theta\int_\Omega \left(\hat{\theta}(\mathbf{x})-\theta\right)^2 p(\mathbf{x},\theta)\, d\mathbf{x}d\theta$ is the global MSE. Note that, contrary to the deterministic bounds, no assumptions are made on the estimator $\hat{\theta}(\mathbf{x})$, e.g., $\hat{\theta}(\mathbf{x})$ can be biased.

The general form of the WWB is given by [18]

$$\mathrm{WWB} = \sup_{h,s} \frac{h^2 e^{2\eta(s,h)}}{e^{\eta(2s,h)} + e^{\eta(2-2s,-h)} - 2e^{\eta(s,2h)}} \tag{5}$$

where $s \in [0,1]$, where h is chosen on the parameter support that will be approximated by $[-3\sigma_\theta, 3\sigma_\theta]$ in our case, and where $\eta(\alpha,\beta)$ is the semi-invariant moment generating function generally used to bound the probability of error in binary hypothesis testing problems and given by

$$\eta(\alpha,\beta) = \ln \int_\Theta\int_\Omega \frac{p^\alpha(\mathbf{x},\theta+\beta)}{p^{\alpha-1}(\mathbf{x},\theta)} d\mathbf{x}d\theta. \tag{6}$$

The double integral in (6) can be written as follows:

$$\int_\Theta\int_\Omega \frac{p^\alpha(\mathbf{x},\theta+\beta)}{p^{\alpha-1}(\mathbf{x},\theta)} d\mathbf{x}d\theta = \int_\Theta \frac{p^\alpha(\theta+\beta)}{p^{\alpha-1}(\theta)} \int_\Omega \frac{p^\alpha(\mathbf{x}|\theta+\beta)}{p^{\alpha-1}(\mathbf{x}|\theta)} d\mathbf{x}d\theta. \tag{7}$$

Let us set $I = \int_\Omega (p^\alpha(\mathbf{x}|\theta+\beta))/(p^{\alpha-1}(\mathbf{x}|\theta))d\mathbf{x}$. With (3), we have (8), shown at the bottom of the page.

Let us set

$$y_k = x_k - a_k e^{j2\pi k\theta}\mu_k \tag{9}$$

$$\text{with } \mu_k = \alpha\left(e^{j2\pi k\beta}-1\right)+1. \tag{10}$$

The term $|y_k|^2$ is then equal to

$$|y_k|^2 = |x_k|^2 + |a_k|^2|\mu_k|^2 - 2\mathrm{Re}\left\{x_k^* a_k \mu_k e^{j2\pi k\theta}\right\}. \tag{11}$$

Note that $|\mu_k|^2 = 1 + 2\alpha(\alpha-1)(1-\cos(2\pi k\beta))$. Then

$$\begin{aligned} \|\mathbf{y}\|^2 &- 2\alpha(\alpha-1)\sum_{k=0}^{N-1}|a_k|^2(1-\cos(2\pi k\beta)) \\ &= \|\mathbf{x}\|^2 + \|\mathbf{a}\|^2 - 2\sum_{k=0}^{N-1}\mathrm{Re}\left\{x_k^* a_k e^{j2\pi k\theta}\mu_k\right\}. \end{aligned} \tag{12}$$

The right-hand side of (12) is equal to the term in the exponential of (8). Consequently, we have

$$\frac{p^\alpha(\mathbf{x}|\theta+\beta)}{p^{\alpha-1}(\mathbf{x}|\theta)} = \frac{1}{(\pi\sigma^2)^N} \times e^{-1/\sigma^2\left(\|\mathbf{y}\|^2-2\alpha(\alpha-1)\sum_{k=0}^{N-1}|a_k|^2(1-\cos(2\pi k\beta))\right)} \tag{13}$$

where $\mathbf{y} = [y_0 \quad y_1 \quad \cdots \quad y_{N-1}]^T$. Then

$$\begin{aligned} I &= \int_\Omega \frac{p^\alpha(\mathbf{x}|\theta+\beta)}{p^{\alpha-1}(\mathbf{x}|\theta)} d\mathbf{x} \\ &= \frac{1}{(\pi\sigma^2)^N} \times e^{2\alpha(\alpha-1)/\sigma^2\sum_{k=0}^{N-1}|a_k|^2(1-\cos(2\pi k\beta))} \times \underbrace{\int_\Omega e^{-1/\sigma^2\|\mathbf{y}\|^2}d\mathbf{x}}_{=(\pi\sigma^2)^N} \\ &= e^{2\alpha(\alpha-1)/\sigma^2\sum_{k=0}^{N-1}|a_k|^2(1-\cos(2\pi k\beta))} \end{aligned} \tag{14}$$

which is independent of θ. Consequently, we have

$$\int_\Theta\int_\Omega \frac{p^\alpha(\mathbf{x},\theta+\beta)}{p^{\alpha-1}(\mathbf{x},\theta)} d\mathbf{x}d\theta = I\int_\Theta \frac{p^\alpha(\theta+\beta)}{p^{\alpha-1}(\theta)}d\theta. \tag{15}$$

$$\frac{p^\alpha(\mathbf{x}|\theta+\beta)}{p^{\alpha-1}(\mathbf{x}|\theta)} = \frac{1}{(\pi\sigma^2)^N} e^{-1/\sigma^2\left(\|\mathbf{x}\|^2+\|\mathbf{a}\|^2-2\,\mathrm{Re}\left\{\sum_{k=0}^{N-1} x_k^* a_k e^{j2\pi k\theta}\left[\alpha\left(e^{j2\pi k\beta}-1\right)+1\right]\right\}\right)} \tag{8}$$

With (2), we have

$$\int_{\Theta} \frac{p^{\alpha}(\theta+\beta)}{p^{\alpha-1}(\theta)} d\theta = \frac{1}{\sqrt{2\pi}\sigma_\theta} e^{-1/2\sigma_\theta^2[\alpha\beta^2-2\alpha\beta m+m^2]} \times \int_{\Theta} e^{-1/2\sigma_\theta^2[\theta^2+2\theta(\alpha\beta-m)]} d\theta$$
$$= e^{-1/2\sigma_\theta^2[\alpha\beta^2-2\alpha\beta m+m^2]+(\alpha\beta-m)^2/2\sigma_\theta^2}$$
$$= e^{(-\alpha\beta^2/2\sigma_\theta^2)(1-\alpha)} \quad (16)$$

where the integral $\int_{\Theta} e^{-1/2\sigma_\theta^2\left[\theta^2+2\theta(\alpha\beta-m)\right]} d\theta$ is obtained by [22, p. 355, Eqn. (BI((28))](1).

Finally, we have (17), shown at the bottom of the page.

Then, the closed-from expression of $\eta(\alpha,\beta)$ for the observation model (1) is given by

$$\eta(\alpha,\beta) = \alpha(\alpha-1) \times \left[\frac{\beta^2}{2\sigma_\theta^2} + \frac{2}{\sigma^2}\sum_{k=0}^{N-1} |a_k|^2 (1-\cos(2\pi k\beta))\right]. \quad (18)$$

Finally, let us set $\xi(h) = 4\mathrm{SNR}\sum_{k=0}^{N-1} |a_k|^2 (1 - \cos(2\pi k h))$, where $\mathrm{SNR} = (1/\sigma^2)$. By using (18) into (5), the WWB becomes as in (19), shown at the bottom of the page.

B. Weiss-Weinstein Bound Computational Cost Optimization

This bound needs to be optimized over h and s. This leads to a more important computational cost than the other bounds of the Weiss and Weinstein family—the BCRB, the BZB, or the BAB—for which we remind the closed-form expressions [19]

$$\begin{cases} \mathrm{BCRB} = \dfrac{\sigma_\theta^2}{8\mathrm{SNR}\pi^2\sigma_\theta^2 \sum_{k=0}^{N-1} |a_k|^2 k^2 + 1} \\ \mathrm{BZB} = \sup_h \dfrac{h^2}{e^{4\mathrm{SNR}\sum_{k=0}^{N-1}|a_k|^2(1-\cos(2\pi k h)) + \frac{h^2}{\sigma_\theta^2}} - 1} \\ \mathrm{BAB} = \sup_h \dfrac{\mathrm{BCRB}^{-1}+\mathrm{BZB}^{-1}-2\phi}{\mathrm{BCRB}^{-1}\mathrm{BZB}^{-1}-\phi^2} \\ \text{with } \phi = \frac{4\mathrm{SNR}\pi}{h}\sum_{k=0}^{N-1} k|a_k|^2 \sin(2\pi k h) + \frac{h}{\sigma_\theta^2} \end{cases} \quad (20)$$

No optimization is required for the BCRB computation, and an optimization over only one point is required for the BZB and the BAB. Consequently, the WWB will have a higher computational cost than these bounds. Here, three ways concerning the WWB computational cost reduction are presented.

1) As previously stated, h is chosen on the parameter support that is approximated by $[-3\sigma_\theta, 3\sigma_\theta]$. This support can be reduced to $[0, 3\sigma_\theta]$ since the function (20) is even with respect to h.

2) As proposed by Weiss and Weinstein in [18], it is sometimes a good choice to put $s = 1/2$. This approximation is intuitively justified by the fact that the WWB tends to the BZB (which is known to be weaker than the WWB [19]) when s tends to zero or one. If we set $s = 1/2$, the WWB is modified as in (21), shown at the bottom of the page. The resulting bound has approximately the same computational cost as the BZB and the BAB.

3) The tightest WWB is given by an optimization over the test point h on the parameter support. For lower bounds obtained by an optimization over test points, as the Barankin bound or the WWB, it is known that the optimum choices of h are related to the ambiguity function of the problem (and consequently to the carrier signal properties). Indeed, the value of the test points that maximizes the bound are those for which the ambiguity function takes local maxima (see, e.g., [5], [10], [21], and).

IV. Simulation Results

This section examines the relevance of the derived bounds for predicting the MSE and the SNR threshold in the data-aided frequency estimation problem. For that purpose, the empirical global MSE of the ML estimator is considered. The ML estimator is given by

$$\hat{\theta}_{ML}(\mathbf{x}) = \arg\max_{\theta} p(\mathbf{x}|\theta) = \arg\min_{\theta} \frac{1}{\sigma^2} \times \left(\|\mathbf{x}\|^2 + \|\mathbf{a}\|^2 - 2\,\mathrm{Re}\left\{\sum_{k=0}^{N-1} x_k^* a_k e^{j2\pi k\theta}\right\}\right) \quad (22)$$

and the global MSE of the ML estimator is given by

$$\mathrm{GMSE} = \int_{\Theta} \mathrm{LMSE}(\theta) p(\theta)\, d\theta \quad (23)$$

where $\mathrm{LMSE}(\theta)$ corresponds to the local MSE

$$\mathrm{LMSE}(\theta) = \int_{\Omega} \left(\hat{\theta}_{ML}(\mathbf{x}) - \theta\right)^2 p(\mathbf{x}|\theta)\, d\mathbf{x}. \quad (24)$$

The simulation is performed with a QPSK pilot sequence that contains $N = 20$ symbols.

$$\int_{\Theta}\int_{\Omega} \frac{p^{\alpha}(\mathbf{x},\theta+\beta)}{p^{\alpha-1}(\mathbf{x},\theta)} d\mathbf{x} d\theta = e^{(\alpha(\alpha-1)\beta^2/2\sigma_\theta^2)+(2\alpha(\alpha-1)/\sigma^2)\sum_{k=0}^{N-1}|a_k|^2(1-\cos(2\pi k\beta))} \quad (17)$$

$$\mathrm{WWB} = \sup_{h,s} \frac{h^2 e^{s(s-1)\left[h^2/\sigma_\theta^2+\xi(h)\right]}}{e^{s(2s-1)\left[h^2/\sigma_\theta^2+\xi(h)\right]} + e^{(1-s)(1-2s)\left[h^2/\sigma_\theta^2+\xi(h)\right]} - 2e^{2s(s-1)\left[h^2/\sigma_\theta^2+1/4\xi(2h)\right]}} \quad (19)$$

$$\mathrm{WWB} = \sup_h \frac{h^2}{2} \frac{e^{(-h^2/4\sigma_\theta^2)-\mathrm{SNR}\sum_{k=0}^{N-1}|a_k|^2(1-\cos(2\pi k h))}}{1 - e^{(-h^2/2\sigma_\theta^2)-(\mathrm{SNR}/2)\sum_{k=0}^{N-1}|a_k|^2(1-\cos(4\pi k h))}} \quad (21)$$

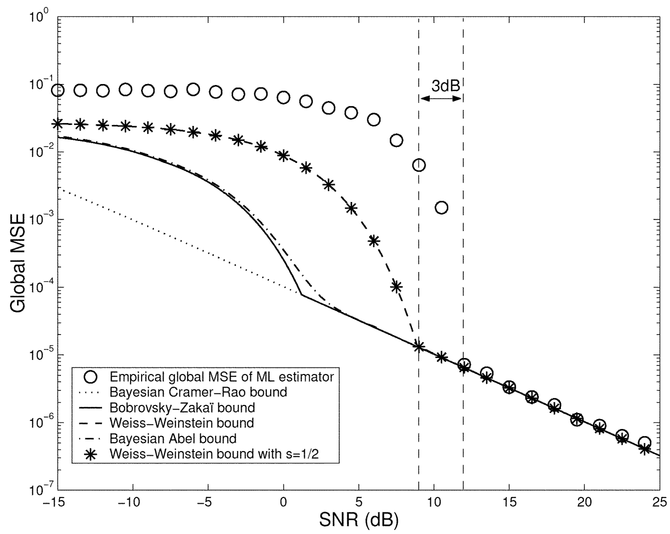

Fig. 1. ML estimator empirical global MSE, BCRB, BZB, and WWB versus SNR. QPSK modulation with $N = 20$ observations.

Fig. 1 superimposes the global MSE of the ML estimator evaluated over 1000 Monte Carlo trials, the BCRB, the BZB, the BAB (20), the WWB obtained by (19), and the WWB obtained by (21). The BAB is close to the BZB. This figure shows the threshold behavior of the ML estimator when the SNR decreases. The WWBs obtained by numerical evaluation of (19) and (21) are the same; therefore, $s = 1/2$ is the optimum value in this problem. Unfortunately, no sound proof that this result is true in general is available in the literature. The WWB bounds provide a better prediction of the MSE in comparison with the BZB. The WWB threshold value provides a good approximation of the effective SNR at which the ML estimator experiences the threshold behavior.

V. Conclusion

In this letter, a closed-form expression of the WWB is proposed in the data-aided carrier frequency framework. This bound is shown to be a good tool in order to predict the ultimate performance of an estimator in terms of MSE for any SNR and exhibits tight results in comparison with the other bounds of the Weiss and Weinstein family.

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